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FREDHOLM AND PROPERNESS PROPERTIES OF QUASILINEAR ELLIPTIC SYSTEMS OF SECOND ORDER

HICHAM G. GEBRAN AND CHARLES A. STUART

IACS-FSB, Ecole Polytechnique Fédérale Lausanne, CH-1015 Lausanne, Switzerland
(hicham.gebran@epfl.ch; charles.stuart@epfl.ch)

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Abstract For a large class of subsets $\Omega \subset \mathbb{R}^N$ (including unbounded domains), we discuss the Fredholm and properness properties of second-order quasilinear elliptic operators viewed as mappings from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega;\mathbb{R}^m)$ with $N < p < \infty$ and $m \geqslant 1$. These operators arise in the study of elliptic systems of m equations on Ω. A study in the case of a single equation ($m = 1$) on $\mathbb{R}^N$ was carried out by Rabier and Stuart.

Keywords: Fredholm operator; properness; quasilinear elliptic operator

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1. Introduction

In a recent paper [**8**], Rabier and Stuart studied the Fredholm and properness properties of quasilinear elliptic operators viewed as mappings from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ for $N < p < \infty$. The motivation for this work was to prepare the way for the use of the topological degree for C^1-Fredholm maps of index zero that are proper on closed bounded sets (see [**7**]), and they subsequently showed how this degree can be used to obtain new results about the global bifurcation in $W^{2,p}(\mathbb{R}^N)$ of solutions of quasilinear elliptic equations [**9**].

The purpose of the present article is to extend the approach developed in [**8**] to cover systems of quasilinear elliptic operators viewed as mappings from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega;\mathbb{R}^m)$, where m is an integer greater than or equal to 1, Ω is an open (possibly unbounded) subset of $\mathbb{R}^N$ and $N < p < \infty$. The operators under consideration arise in the study of a system of m partial differential equations of second order for m unknown functions $u^1, \dots, u^m$ on a domain* Ω,

$$\begin{aligned} L^{11}u^1 + \cdots + L^{1m}u^m + b^1 &= 0, \\ &\vdots \\ L^{m1}u^1 + \cdots + L^{mm}u^m + b^m &= 0, \end{aligned}$$

* For us, a domain is just an open subset, which need not be connected nor bounded.

where

$$L^{ij}u^j = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^{ij}(x, u^1, \dots, u^m, \nabla u^1, \dots, \nabla u^m)\partial_{\alpha\beta}^2 u^j$$

and

$$b^i = b^i(x, u^1, \dots, u^m, \nabla u^1, \dots, \nabla u^m) \quad \text{for } i, j = 1, \dots, m.$$

To deal with such systems, we adopt a matrix-vector notation, which we believe to be best suited for our purposes. So let $u = (u^1, u^2, \dots, u^m) : \Omega \to \mathbb{R}^m$ be a vector-valued function, $\nabla u = (\partial_1 u^1, \dots, \partial_N u^1, \partial_1 u^2, \dots\dots, \partial_1 u^m, \dots, \partial_N u^m)$, $b : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}^m$ a vector-valued map and $a_{\alpha\beta} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}^{m\times m}$ a family of matrix-valued maps $(\alpha, \beta = 1, \dots, N)$. Then we consider the differential operator

$$F(u) = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(\cdot, u, \nabla u) \cdot \partial_{\alpha\beta}^2 u + b(\cdot, u, \nabla u), \tag{1.1}$$

where

$$[a_{\alpha\beta}(\cdot, u, \nabla u) \cdot \partial_{\alpha\beta}^2 u]_i = \sum_{j=1}^{N} a_{\alpha\beta}^{ij}(\cdot, u, \nabla u)\partial_{\alpha\beta}^2 u^j,$$

as a mapping from $W^{2,p}(\Omega; \mathbb{R}^m)$ to $L^p(\Omega; \mathbb{R}^m)$, and we investigate conditions for the Fredholmness and the properness of F on the closed bounded subsets of $W^{2,p}(\Omega; \mathbb{R}^m) \cap W_0^{1,p}(\Omega; \mathbb{R}^m)$.

Our approach is the same as in [**8**], and many arguments and proofs remain valid after some modifications. In the present notation, the work in [**8**] deals with the situation $m = 1$ and $\Omega = \mathbb{R}^N$, so in addition to extending the treatment to systems, we are also generalizing the approach to more general domains by allowing Ω to be a domain in $\mathbb{R}^N$ whose boundary is a bounded set. This covers the following cases: $\Omega = \mathbb{R}^N$; Ω is the exterior of a bounded domain; and Ω is a bounded domain. In some places, the modifications required to the analogous arguments in [**8**] are little more than notational, but in others they are more substantial. Therefore, for the reader's convenience, we have included fairly complete proofs of all the results and tried to highlight the arguments that make the generalization possible.

For problems on bounded domains, there are strong connections between ellipticity and Fredholmness [**4**, **6**]. Let us mention in particular the theorems on complete collections of isomorphisms (see, for example, [**2**, **10**, **14**]). In our work, the ellipticity condition intervenes first in proving Lemma 3.5, and this is done through the L^p- *a priori* estimates of Koshelev [**5**], available for linear systems with continuous coefficients that are elliptic in the sense of Petrovskii. We deal with quasilinear systems that are elliptic in a similar sense (see (3.1)), since this is sufficiently general to cover the applications we have in mind such as reaction-diffusions systems or systems that satisfy the strict Legendre–Hadamard condition. Furthermore, it allows us to use the same function spaces for all the components of the vector u, which would not be the case if we adopted a more general notion of ellipticity such as that due to Agmon *et al*. Note that we use standard Sobolev spaces. For work on analogous issues in local Sobolev spaces, weighted Sobolev spaces and Hölder spaces, one may consult the recent papers [**11–13**] and the references therein.

Section 2 begins with basic notation and properties of the domain Ω and then establishes the regularity properties of the operator (1.1). The Fredholm property is treated in § 3 following a discussion of ellipticity. The main result here is Theorem 3.9. The study of properness is begun is § 4, where we relate properness to Fredholm properties and to a notion of uniform decay of sequences of functions as $|x| \to \infty$. More explicit conclusions are obtained in § 5 under the assumption that the system is asymptotically periodic as $|x| \to \infty$, the main results being Theorems 5.5 and 5.7.

2. Definitions and smoothness of some Nemytskii operators

Our first task is to make sure that the operator in (1.1) is well defined and has enough smoothness for the subsequent discussion. Therefore, it is necessary to study the smoothness of the Nemytskii operators $u \mapsto b(\cdot, u, \nabla u)$ and $u \mapsto a_{\alpha\beta}(\cdot, u, \nabla u)$, entering in F. This leads us to consider maps of the type $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^d$.

We make use of the following notation: if $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^d$ and $u : \Omega \to \mathbb{R}^m$, then the Nemytskii operator generated by $f : u \mapsto f(\cdot, u, \nabla u)$ will be denoted by $\boldsymbol{f}$ (i.e. $\boldsymbol{f}(u) = f(\cdot, u, \nabla u)^*$). Note that if $f = (f^1, \dots, f^d)$ and each component f^j gives rise to a Nemytskii operator $\boldsymbol{f}^j$, then the Nemytskii operator associated with f is $\boldsymbol{f} = (\boldsymbol{f}^1, \dots, \boldsymbol{f}^d)$, and any smoothness property of $\boldsymbol{f}$ is equivalent to the same property of each component. So it is sufficient to study scalar-valued maps. But before going further, let us continue to fix the notation we shall use below.

The integer N will always denote the dimension of the space of the independent variable, i.e. $\mathbb{R}^N$, and m the dimension of the system (m equations with m unknown functions). The real number p will always satisfy $N < p < \infty$.

If $z_1, z_2 \in \mathbb{R}^m$ and A is an $m \times m$ matrix, $z_1 \cdot z_2$ will denote the scalar product of z_1 and z_2, and $A \cdot z_1$ will denote the usual matrix-vector multiplication. Also, $|z|$ and $|A|$ denote, respectively, the Euclidean norms of $z \in \mathbb{R}^d$ and $A \in \mathbb{R}^{m\times m}$.

Whenever we need to display the components of $\xi \in \mathbb{R}^{m(N+1)}$, we shall write

$$\xi = (\xi_0, \xi_1, \dots, \xi_N), \quad \text{with } \xi_k = (\xi_k^1, \dots, \xi_k^m) \in \mathbb{R}^m,\ k = 0, \dots, N.$$

For $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$, $x \in \Omega$, $\xi \in \mathbb{R}^{m(N+1)}$, we denote by

$$\nabla_{\xi_k} f(x, \xi) = (\partial_{\xi_k^1}, \dots, \partial_{\xi_k^m}) f(x, \xi)$$

the partial gradient with respect to the ξ_k block variable. $\mathrm{D}_\xi f$ is the gradient of f with respect to ξ.

As in [**8**], we use the standard notation for the Lebesgue and Sobolev spaces. Let $O \subset \mathbb{R}^N$ be an open set, $l \in \mathbb{N}$, $q \in [1, \infty]$, the norm in $(W^{l,q}(O))^m = W^{l,q}(O; \mathbb{R}^m)$ is the norm in a Cartesian product of Banach spaces and will be denoted by $\|u\|_{l,q,O}$ (i.e. if

* $\nabla u = (\partial_1 u, \dots, \partial_N u)$ and $\partial_k u = (\partial_k u^1, \dots, \partial_k u^m)$, $k = 1, \dots, N$, $\partial_0 u = u$.

$u = (u^1, \dots, u^m) \in W^{l,q}(O; \mathbb{R}^m)$, then $\|u\|_{l,q,O} = \|u^1\|_{W^{l,q}(O)} + \cdots + \|u^m\|_{W^{l,q}(O)})$. To simplify the writing, we often use $Y_p(\Omega) = (L^p(\Omega))^m$ and $X_p(\Omega) = (W^{2,p}(\Omega))^m$; when $\Omega = \mathbb{R}^N$, we write X_p and Y_p.

To deal with the Dirichlet problem, we introduce the space

$$D_p(\Omega) = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^m.$$

Note that $D_p(\Omega)$ is a closed subspace of $X_p(\Omega)$ and so it is also reflexive. Finally, $D_p(\mathbb{R}^N) = X_p(\mathbb{R}^N) = X_p$.

$C^1(\bar{\Omega})$ is the subspace of $C^0(\bar{\Omega}) \cap C^1(\Omega)$ of the functions v for which ∇v has a continuous extension to $\bar{\Omega}$. We also use the space $C_d^1(\bar{\Omega})$ introduced in [**8**],

$$C_d^1(\bar{\Omega}) = \Big\{ v \in C^1(\bar{\Omega}) : \lim_{x \in \Omega,\, |x| \to \infty} |v(x)| = \lim_{x \in \Omega,\, |x| \to \infty} |\nabla v(x)| = 0 \Big\}.$$

This is a Banach space for the norm $\max(\max_{x \in \bar{\Omega}} |v(x)|, \max_{x \in \bar{\Omega}} |\nabla v(x)|)$. Note also that $C_d^1(\bar{\Omega}) \subset W^{1,\infty}(\Omega)$ and $C_d^1(\bar{\Omega}) = C^1(\bar{\Omega})$ when Ω is bounded. Some important properties of the spaces used here are recalled in the appendix.

2.1. Remarks on the domain Ω

Ω will always have a bounded and Lipschitz boundary $\partial\Omega$ (possibly empty), so that Ω can be a bounded domain, an exterior domain or $\mathbb{R}^N$ itself. In the main results of §§ 3–5, it is furthermore assumed that Ω has a C^2 boundary, and it is explicitly mentioned. This implies some remarks that will be useful in § 3.

Remark 2.1. We have two cases: either Ω is bounded or not. If Ω is unbounded, then necessarily $\complement^{\Omega}$ is bounded. This is due to the boundedness of the boundary. Indeed, let B_r be a ball containing $\partial\Omega$. We claim that B_r contains $\complement^{\Omega}$. If not, there is a point $x \in \complement^{\Omega} \cap \complement^{B_r}$. Since Ω is also unbounded, there is a $y \in \Omega \cap \complement^{B_r}$. Now recall that $\complement^{B_r}$ is path connected, so we can join x to y by a path in $\complement^{B_r}$. This path, joining an exterior point to an interior point of Ω, should meet the boundary, but it does not, since the boundary lies inside the ball B_r. Therefore, $K = \complement^{\Omega}$ is bounded (compact).

Remark 2.2. Let Ω be unbounded. For every ball B_r containing $\partial\Omega$, we have

$$\partial(\Omega \cap B_r) = \partial\Omega \cup \partial B_r.$$

Proof. We clearly have $\overline{\Omega \cap B_r} \subset \bar{\Omega} \cap \bar{B}_r$. Let us prove the reverse inclusion. Let $x \in \bar{\Omega} \cap \bar{B}_r$. Then either (i) $x \in B_r$ or (ii) $x \in \partial B_r$. First let $x \in B_r$ and V be an open neighbourhood of x. If $V \cap (\Omega \cap B_r) = \varnothing$, then $x \in V \cap B_r \subset K$, which means that x is an interior point of K. But this is impossible, since

$$x \in \bar{\Omega} = \overline{\complement^{K}} = \complement^{\dot{K}}.$$

Therefore, $x \in \overline{\Omega \cap B_r}$. Next, if $x \in \partial B_r \subset \complement^{B_r} \subset \Omega$, then, for all $\varepsilon > 0$ sufficiently small, $B(x, \varepsilon) \subset \Omega$. Clearly, $B(x, \varepsilon) \cap B_r \neq \varnothing$, and so $B(x, \varepsilon) \cap (\Omega \cap B_r) \neq \varnothing$, and once again $x \in \overline{\Omega \cap B_r}$. Finally, $\overline{\Omega \cap B_r} = \bar{\Omega} \cap \bar{B}_r$.

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On the other hand,

$$\overline{\complement^{\Omega\cap B_r}} = \overline{\complement^{\Omega}} \cup \overline{\complement^{B_r}}.$$

So

$$\partial(\Omega\cap B_r) = \bar{\Omega}\cap\overline{\complement^{\Omega}}\cap\bar{B}_r \cup \bar{B}_r\cap\overline{\complement^{B_r}}\cap\bar{\Omega} = \partial\Omega\cap\bar{B}_r\cup\partial B_r\cap\bar{\Omega} = \partial\Omega\cup\partial B_r,$$

since $\partial\Omega\subset B_r\subset\bar{B}_r$ and $\partial B_r\subset\complement^{B_r}\subset\Omega\subset\bar{\Omega}$. □

Remark 2.3. The connection between the Dirichlet problem and the space $D_p(\Omega) = (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^m$ introduced above is given by the following theorem (see, for example, Brezis [**3**, Théorème IX.17]). Let Ω have a C^1 boundary and let $u\in W^{1,q}(\Omega)\cap C(\bar{\Omega})$ with $1\leqslant q<\infty$. Then the following conditions are equivalent.

(i) $u=0$ on $\partial\Omega$.

(ii) $u\in W_0^{1,q}(\Omega)$.

2.2. A preliminary study of Nemytskii operators

It is clear that the smoothness of a Nemytskii operator generated by $f:\Omega\times(\mathbb{R}^m\times\mathbb{R}^{mN})\to\mathbb{R}$ should be derived from smoothness assumptions on f. In this paper, as in [**8**], the following property of equicontinuity plays an important role.

Definition 2.4. We say that $f:\Omega\times\mathbb{R}^M\to\mathbb{R}^d$ is an equicontinuous C^0 bundle map if f is continuous and the collection $(f(x,\cdot))_{x\in\Omega}$ is equicontinuous at every point of $\mathbb{R}^M$. If $k\geqslant 0$ is an integer, we say that f is an equicontinuous C^k_ξ bundle map if the partial derivatives D^γ_ξ, $|\gamma|\leqslant k$, exist and are equicontinuous C^0 bundle maps.

Note that $f=(f^1,\dots,f^d)$ is a C^k_ξ bundle map if only if each component f^j is a C^k_ξ bundle map. Note also that a sum of C^k_ξ bundle maps is a C^k_ξ bundle map.

Now we give some important properties and examples of C^k_ξ bundle maps.

Lemma 2.5. *Let $f:\Omega\times\mathbb{R}^M\to\mathbb{R}^d$ be an equicontinuous C^0 bundle map. Then we have the following.*

(i) *The collection $(f(x,\cdot))_{x\in\Omega}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}^M$.*

(ii) *If A is a measurable subset of Ω and $f(\cdot,0)\in L^\infty(A)$, the collection $(f(x,\cdot))_{x\in A}$ is equibounded on the bounded subsets of $\mathbb{R}^M$.*

Proof. (i) If not, there exist a compact set $K\subset\mathbb{R}^m$, $\varepsilon_0>0$ and three sequences $(x_n)\subset\Omega$, $(\xi_n),(\eta_n)\subset\mathbb{R}^M$ such that, for all $n\in\mathbb{N}$,

$$|\xi_n-\eta_n|\leqslant\frac{1}{n}\quad\text{and}\quad|f(x_n,\xi_n)-f(x_n,\eta_n)|\geqslant\varepsilon_0.$$

But (ξ_n) belongs to a compact set, so it contains a subsequence $(\xi_{\varphi(n)})$ converging to some ξ, which also implies that $\eta_{\varphi(n)}\to\xi$. By the equicontinuity of $(f(x,\cdot))_x$ at ξ, we have, for all n large enough,

$$|f(x_{\varphi(n)},\xi_{\varphi(n)})-f(x_{\varphi(n)},\xi)|<\tfrac{1}{4}\varepsilon_0\quad\text{and}\quad|f(x_{\varphi(n)},\eta_{\varphi(n)})-f(x_{\varphi(n)},\xi)|<\tfrac{1}{4}\varepsilon_0,$$

and therefore

$$|f(x_{\varphi(n)}, \xi_{\varphi(n)}) - f(x_{\varphi(n)}, \eta_{\varphi(n)})| < \tfrac{1}{2}\varepsilon_0,$$

a contradiction.

(ii) Let K be a bounded subset of $\mathbb{R}^m$ and B be a closed ball in $\mathbb{R}^m$ containing 0 and K. By (i), there is $\delta > 0$ such that, for all $x \in \Omega$, $|f(x,\xi) - f(x,\eta)| < 1$ whenever $|\xi - \eta| \leqslant \delta$ and $\xi, \eta \in B$. For any $\xi \in K$, one can divide the segment joining 0 to ξ into $[|\xi|/\delta] + 1$ segments of length not greater than δ. Thus, for $x \in A$,

$$|f(x,\xi)| \leqslant |f(x,0)| + |f(x,\xi) - f(x,0)| < \|f(\cdot,0)\|_{L^\infty(A)} + \left[\frac{|\xi|}{\delta}\right] + 1.$$

But $|\xi|$ is bounded by the diameter of B, so the proof is complete. □

Remark 2.6. Let $g : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^m$ be a C^0 bundle map. Then, for $i = 0, \dots, N$, $(x, \xi) \mapsto g(x,\xi) \cdot \xi_i$ is a scalar-valued C^0 bundle map.

Proof. Fix $\eta \in \mathbb{R}^m \times \mathbb{R}^{mN}$. Then

$$g(x,\xi) \cdot \xi_i - g(x,\eta) \cdot \eta_i = g(x,\xi) \cdot (\xi_i - \eta_i) + (g(x,\xi) - g(x,\eta)) \cdot \eta_i.$$

The result follows from the equicontinuity of $(g(x,\cdot))_x$ at η and its equiboundedness on bounded subsets of $\mathbb{R}^m \times \mathbb{R}^{mN}$ (Lemma 2.5 (ii)). □

Remark 2.7. Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ be a C^1_ξ bundle map. Define $g : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^m$ by

$$g(x,\xi) = \int_0^1 \nabla_{\xi_i} f(x, t\xi)\, \mathrm{d}t.$$

Then g is a C^0 bundle map.

Proof. Fix $\eta \in \mathbb{R}^m \times \mathbb{R}^{mN}$. Then

$$g(x,\xi) - g(x,\eta) = \int_0^1 (\nabla_{\xi_i} f(x,t\xi) - \nabla_{\xi_i} f(x,t\eta))\, \mathrm{d}t.$$

If $|\xi - \eta| \leqslant 1$, then $t\xi$ and $t\eta$ belongs to the closed ball with centre 0 and radius $|\eta| + 1$. Thus the conclusion follows from Lemma 2.5 (i) applied to $\nabla_{\xi_i} f$. □

Remark 2.8. If f is of class C^k and $f(\cdot,\xi)$ is N-periodic in x with period $T = (T_1, \dots, T_N)$ for every $\xi \in \mathbb{R}^{m(N+1)}$, then f is a C^k_ξ bundle map. This follows from the uniform continuity of $D^\gamma_\xi f$ on $[0,T_1] \times \cdots \times [0,T_N] \times K$ for every compact $K \subset \mathbb{R}^m \times \mathbb{R}^{mN}$ and $|\gamma| \leqslant k$ (see § 5).

Lemma 2.9. *Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ be an equicontinuous C^0 bundle map. Suppose that $f(\cdot,0) \in L^\infty(\Omega)$. Then the Nemytskii operator $\boldsymbol{f}$ has the following properties.*

(i) *It is well defined and continuous from $(C^1_d(\bar{\Omega}))^m$ to $L^\infty(\Omega)$.*

(ii) *It is well defined and continuous from $(W^{2,p}(\Omega))^m$ to $L^\infty(\Omega)$ and maps bounded subsets onto bounded subsets.*

(iii) *If Ω is bounded, it is completely continuous from $(W^{2,p}(\Omega))^m$ to $L^\infty(\Omega)$ (hence also to $L^q(\Omega)$, $1 \leqslant q \leqslant \infty$).*

(iv) *The multiplication $(u,v) \in (W^{2,p}(\Omega))^m \times L^p(\Omega) \mapsto f(\cdot,u,\nabla u)v \in L^p(\Omega)$ is weakly sequentially continuous.*

Proof. (i) If $u \in C^1_d(\bar{\Omega};\mathbb{R}^m)$, the function $x \in \bar{\Omega} \to f(x,u(x),\nabla u(x))$ is continuous and hence measurable. From the boundedness of u and ∇u on $\bar{\Omega}$, there is a bounded subset $K \subset \mathbb{R}^{m(N+1)}$ containing $(u(x),\nabla u(x))$ for all $x \in \bar{\Omega}$. Therefore, by Lemma 2.5 (ii), there is a constant $M_K > 0$ such that $|f(x,u(x),\nabla u(x))| \leqslant M_K$ $\forall x \in \bar{\Omega}$. This means that $f(\cdot,u,\nabla u) \in L^\infty(\Omega)$.

To prove the continuity, let $u_n, u \in C^1_d(\bar{\Omega};\mathbb{R}^m)$ and $u_n \to u$ in $C^1_d(\bar{\Omega};\mathbb{R}^m)$. Then, since $\{u\} \cup \{u_n, n \in \mathbb{N}\}$ is compact and hence bounded in $C^1_d(\bar{\Omega};\mathbb{R}^m)$, there is a compact $K \subset \mathbb{R}^{m(N+1)}$ containing $(u(x),\nabla u(x))$ and $(u_n(x),\nabla u_n(x))$ for all $x \in \bar{\Omega}$ and $n \in \mathbb{N}$. But $|(u_n(x),\nabla u_n(x)) - (u(x),\nabla u(x))|$ can be made arbitrary small uniformly in $x \in \bar{\Omega}$, for n large enough. So, by Lemma 2.5 (i), given $\varepsilon > 0$, we have

$$|f(x,u_n(x),\nabla u_n(x)) - f(x,u(x),\nabla u(x))| \leqslant \varepsilon \quad \forall x \in \bar{\Omega}.$$

Lastly, if $\mathcal{B} \subset C^1_d(\bar{\Omega};\mathbb{R}^m)$ is bounded, then there is a bounded subset $K \subset \mathbb{R}^{m(N+1)}$ containing $(u(x),\nabla u(x))$ for all $x \in \bar{\Omega}$ and $u \in \mathcal{B}$. The boundedness of $\boldsymbol{f}(\mathcal{B})$ follows from Lemma 2.5 (ii).

(ii) Follows from the imbedding $W^{2,p}(\Omega;\mathbb{R}^m) \hookrightarrow C^1_d(\bar{\Omega};\mathbb{R}^m)$.

(iii) The above imbedding is compact when Ω is bounded.

(iv) Let $u_n \rightharpoonup u$ in $W^{2,p}(\Omega,\mathbb{R}^m)$ and $v_n \rightharpoonup v$ in $L^p(\Omega)$. From (ii), the sequence $(\boldsymbol{f}(u_n))$ is bounded in $L^\infty(\Omega)$, and hence $(\boldsymbol{f}(u_n)v_n)$ is bounded in $L^p(\Omega)$. Let $\Omega' \subset \Omega$ be any open ball. By (iii), $\boldsymbol{f}(u_n)|_{\Omega'} \to \boldsymbol{f}(u)|_{\Omega'}$ in $L^\infty(\Omega')$, which implies that $\boldsymbol{f}(u_n)v_n|_{\Omega'} \rightharpoonup \boldsymbol{f}(u)v|_{\Omega'}$ in $L^p(\Omega')$. Now, if a subsequence of $(\boldsymbol{f}(u_n)v_n)$ converges weakly to w in $L^p(\Omega)$, and hence in $L^p(\Omega')$, we have $w|_{\Omega'} = \boldsymbol{f}(u)v|_{\Omega'}$, and therefore $w = \boldsymbol{f}(u)v$, since the ball is arbitrary. This means that $(\boldsymbol{f}(u_n)v_n)$ has a unique weak cluster point, which yields (see the appendix) $\boldsymbol{f}(u_n)v_n \rightharpoonup \boldsymbol{f}(u)v$ in $L^p(\Omega)$. □

Lemma 2.10. *Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ have the form*

$$f(x,\xi) = f_0(x) + \sum_{i=0}^{N} g_i(x,\xi) \cdot \xi_i, \tag{2.1}$$

where g_i is a C^0 bundle map, with $g_i(\cdot,0) \in L^\infty(\Omega;\mathbb{R}^m)$, $0 \leqslant i \leqslant N$. Suppose that $f_0 \in L^p(\Omega)$. In particular, the above conditions hold if f is a C^1_ξ bundle map with $f(\cdot,0) \in L^p(\Omega)$ and $D_\xi f(\cdot,0)$ bounded in Ω. Then the Nemytskii operator has the following properties.

(i) *It is well defined and continuous from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega)$ and maps bounded subsets onto bounded subsets.*

(ii) *It is weakly sequentially continuous from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega)$.*

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Proof. To see the 'in particular', note that, for a C^1_ξ bundle map f, one can write

$$\begin{aligned} f(x,\xi) - f(x,0) &= \int_0^1 \frac{\partial}{\partial t} f(x,t\xi)\,\mathrm{d}t \\ &= \int_0^1 \sum_{i=0}^{N} \nabla_{\xi_i} f(x,t\xi)\cdot\xi_i\,\mathrm{d}t \\ &= \sum_{i=0}^{N}\biggl(\int_0^1 \nabla_{\xi_i} f(x,t\xi)\,\mathrm{d}t\biggr)\cdot\xi_i. \end{aligned}$$

Take

$$g_i(x,\xi) = \int_0^1 \nabla_{\xi_i} f(x,t\xi)\,\mathrm{d}t.$$

Then, by Remark 2.7, g_i is a C^0 bundle map. Furthermore, $g_i(\cdot,0) = \nabla_{\xi_i} f(\cdot,0) \in L^\infty(\Omega;\mathbb{R}^m)$.

(i) Applying Lemma 2.9 (ii) to each component of g_i, we have that $\boldsymbol{g}_i : W^{2,p}(\Omega,\mathbb{R}^m) \to L^\infty(\Omega;\mathbb{R}^m)$ is continuous and maps bounded subsets onto bounded ones. As a result, the operator

$$u \mapsto \sum_{i=0}^{N} \boldsymbol{g}_i(u)\cdot\partial_i u \in L^p(\Omega) \tag{2.2}$$

is continuous and maps bounded subsets onto bounded ones. By (2.1), this is $\boldsymbol{f} - f_0$, and the conclusion follows from the assumption $f_0 \in L^p(\Omega)$.

(ii) Let $u_n \rightharpoonup u$ in $W^{2,p}(\Omega;\mathbb{R}^m)$. By part (i), $(\boldsymbol{f}(u_n))$ is bounded in $L^p(\Omega)$. Let $\Omega' \subset \Omega$ be an open ball. Since $f - f_0$ is an equicontinuous C^0 bundle map (see Remark 2.6) and vanishes when $\xi = 0$, Lemma 2.9 (iii) applies and yields $\boldsymbol{f}(u_n)|_{\Omega'} \to \boldsymbol{f}(u)|_{\Omega'}$ in $L^p(\Omega')$. Now, if a subsequence of $(\boldsymbol{f}(u_n))$ converges weakly to some w in $L^p(\Omega)$, and hence in $L^p(\Omega')$, we have $w|_{\Omega'} = \boldsymbol{f}(u)|_{\Omega'}$, and therefore $w = \boldsymbol{f}(u)$, since the ball is arbitrary. This means that $(\boldsymbol{f}(u_n))$ has a unique weak cluster point, and thus $\boldsymbol{f}(u_n) \rightharpoonup \boldsymbol{f}(u)$ in $L^p(\Omega)$. □

Theorem 2.11. *Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ be an equicontinuous C^1_ξ bundle map. Suppose that $f(\cdot,0) \in L^\infty(\Omega)$ (respectively, $f(\cdot,0) \in L^p(\Omega)$) and that $D_\xi f(\cdot,0)$ is bounded on Ω. Then the Nemytskii operator $\boldsymbol{f}$ is of class C^1 from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^\infty(\Omega)$ (respectively, $L^p(\Omega)$), with derivative*

$$\mathrm{D}\boldsymbol{f}(u)v = \sum_{i=0}^{N} \nabla_{\xi_i} f(\cdot,u,\nabla u)\cdot\partial_i v. \tag{2.3}$$

Furthermore, $\mathrm{D}\boldsymbol{f}$ is bounded on the bounded subsets of $W^{2,p}(\Omega;\mathbb{R}^m)$, and hence $\boldsymbol{f}$ is uniformly continuous on these subsets.

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Proof. Define, for $u \in W^{2,p}(\Omega;\mathbb{R}^m)$,

$$T_u v = \sum_{i=0}^{N} \nabla_{\xi_i} f(\cdot, u, \nabla u) \cdot \partial_i v.$$

By Lemma 2.9 (ii) applied to each component of $\nabla_{\xi_i} f$, we have that $\nabla_{\xi_i} f(\cdot, u, \nabla u)$ is bounded on Ω. Thus

$$\|T_u v\|_{0,p,\Omega} \leqslant \sum_{i=0}^{N} \|\nabla_{\xi_i} f(\cdot, u, \nabla u)\|_{0,\infty,\Omega} \|\partial_i v\|_{0,p,\Omega} \leqslant \text{const.} \times \|v\|_{2,p,\Omega}$$

and

$$\|T_u v\|_{0,\infty,\Omega} \leqslant \text{const.} \times \|v\|_{1,\infty,\Omega} \leqslant \text{const.} \times \|v\|_{2,p,\Omega}.$$

Therefore, T_u is linear and bounded from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega)$ and to $L^\infty(\Omega)$.

Note that

$$\begin{aligned} f(\cdot, u+v, \nabla(u+v)) - f(\cdot, u, \nabla u) &= \int_0^1 \frac{\partial}{\partial t} f(\cdot, u + tv, \nabla u + t\nabla v)\,\mathrm{d}t \\ &= \sum_{i=0}^{N} \biggl(\int_0^1 \nabla_{\xi_i} f(\cdot, u + tv, \nabla u + t\nabla v)\,\mathrm{d}t \biggr) \cdot \partial_i v. \end{aligned}$$

So

$$\begin{aligned} &f(\cdot, u+v, \nabla(u+v)) - f(\cdot, u, \nabla u) - T_u v \\ &\qquad = \sum_{i=0}^{N} \biggl(\int_0^1 \nabla_{\xi_i} f(\cdot, u + tv, \nabla u + t\nabla v) - \nabla_{\xi_i} f(\cdot, u, \nabla u)\,\mathrm{d}t \biggr) \cdot \partial_i v. \end{aligned}$$

Thus, if we define

$$k_{u,i}(x,\xi) := \int_0^1 (\nabla_{\xi_i} f(x, u(x) + t\xi_0, \nabla u(x) + t\xi') - \nabla_{\xi_i} f(x, u(x), \nabla u(x)))\,\mathrm{d}t,$$

where $\xi' = (\xi_1, \dots, \xi_N)$, we get

$$f(\cdot, u+v, \nabla(u+v)) - f(\cdot, u, \nabla u) - T_u v = \sum_{i=0}^{N} k_{u,i}(\cdot, v, \nabla v) \cdot \partial_i v. \tag{2.4}$$

Now, one can check, as in Remark 2.7, that $k_{u,i}$ is an equicontinuous C^0 bundle map satisfying $k_{u,i}(\cdot, 0) = 0 \in L^\infty(\Omega;\mathbb{R}^m)$. Therefore, by Lemma 2.9 (ii) applied to each component of $k_{u,i}$, we have that $\boldsymbol{k}_{u,i}$ is continuous from $W^{2,p}(\Omega,\mathbb{R}^m)$ to $L^\infty(\Omega;\mathbb{R}^m)$. So, given $\varepsilon > 0$, we have that $\|\boldsymbol{k}_{u,i}(v)\|_{0,\infty,\Omega} \leqslant \varepsilon$, provided $\|v\|_{2,p,\Omega}$ is small enough.

Now, if $f(\cdot, 0) \in L^\infty(\Omega)$, then $\boldsymbol{f}$ maps $X_p(\Omega)$ to $L^\infty(\Omega)$ (see Lemma 2.9 (ii)). By (2.4), we obtain

$$\|\boldsymbol{f}(u+v) - \boldsymbol{f}(u) - T_u v\|_{0,\infty,\Omega} \leqslant \text{const.} \times \varepsilon \|v\|_{1,\infty,\Omega} \leqslant \text{const.} \times \varepsilon \|v\|_{2,p,\Omega},$$

which means that $\boldsymbol{f}$ is differentiable and $\mathrm{D}\boldsymbol{f}(u) = T_u$.

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If $f(\cdot,0) \in L^p(\Omega)$, then $\boldsymbol{f}$ maps $X_p(\Omega)$ to $L^p(\Omega)$ (Lemma 2.10 (i)). By (2.4), we obtain

$$\|\boldsymbol{f}(u+v) - \boldsymbol{f}(u) - T_u v\|_{0,p,\Omega} \leqslant \text{const.} \times \varepsilon \|v\|_{1,p,\Omega} \leqslant \text{const.} \times \varepsilon \|v\|_{2,p,\Omega},$$

which means that $\boldsymbol{f}$ is differentiable and $\mathrm{D}\boldsymbol{f}(u) = T_u$.

To prove the continuity of $\mathrm{D}\boldsymbol{f}$, note that

$$\|\mathrm{D}\boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u^0)\| \leqslant \|D_\xi f(\cdot,u,\nabla u) - D_\xi f(\cdot,u^0,\nabla u^0)\|_{0,\infty,\Omega},$$

so the result follows from Lemma 2.9 (ii), which also ensures that $\mathrm{D}\boldsymbol{f}$ is bounded on the bounded subsets of $X_p(\Omega)$. □

Remark 2.12. If f takes values in $\mathbb{R}^m$, the derivative of the Nemytskii operator generated by f is just

$$\mathrm{D}\boldsymbol{f}(u)v = (\mathrm{D}\boldsymbol{f}^1(u)v, \dots, \mathrm{D}\boldsymbol{f}^m(u)v),$$
$$\text{with, for } k=1,\dots,m,\ \mathrm{D}\boldsymbol{f}^k(u)v = \sum_{i=0}^{N} \nabla_{\xi_i} f^k(\cdot,u,\nabla u)\cdot \partial_i v. \quad (2.5)$$

Similarly, if $f = (f^{k,j})_{k,j=1,\dots,m}$ is an $m \times m$ matrix, the derivative of the Nemytskii operator $\boldsymbol{f}$ is the matrix

$$\mathrm{D}\boldsymbol{f}(u)v = (\mathrm{D}\boldsymbol{f}^{k,j}(u)v)_{k,j=1,\dots,m}, \quad \text{with } \mathrm{D}\boldsymbol{f}^{k,j}(u)v = \sum_{i=0}^{N} \nabla_{\xi_i} f^{k,j}(\cdot,u,\nabla u)\cdot \partial_i v. \quad (2.6)$$

Lemma 2.13 (cf. Lemma 2.9 of [8]). *Let X, Y and Z be normed spaces with $X \hookrightarrow Y$ and let $\boldsymbol{f} : X \to Z$ be uniformly continuous on the bounded subsets of X. Suppose that there is a dense subset $D \subset X$ such that whenever $u \in D$ and $(u_n) \subset X$ is a bounded sequence with $u_n \to u$ in Y, we have $\boldsymbol{f}(u_n) \to \boldsymbol{f}(u)$ in Z. Then the restriction of $\boldsymbol{f}$ to the bounded subsets of X remains continuous for the topology induced by Y.*

Lemma 2.14. *Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ be an equicontinuous C^1_ξ bundle map. Suppose that $f(\cdot,0) \in L^p(\Omega)$ and that $\nabla_{\xi_i} f(\cdot,0) \in (L^p(\Omega) \cap L^\infty(\Omega))^m$, $0 \leqslant i \leqslant N$. Then the restriction of the Nemytskii operator to any bounded subset of $W^{2,p}(\Omega;\mathbb{R}^m)$ is continuous into $L^p(\Omega)$ for the topology of $C^1_d(\bar{\Omega};\mathbb{R}^m)$.*

Proof. Recall that $\boldsymbol{f}$ is uniformly continuous on the bounded subsets of $W^{2,p}(\Omega;\mathbb{R}^m)$ by Theorem 2.11. Note also that if

$$D = \{u \in C^\infty(\Omega;\mathbb{R}^m);\ \exists v \in C^\infty_0(\mathbb{R}^N;\mathbb{R}^m) \text{ such that } v|_\Omega = u\},$$

then D is dense in $W^{2,p}(\Omega,\mathbb{R}^m)$ (see [**1**, Theorem 3.18]). We show that if $u \in D$ and (u_n) is a bounded sequence from $X_p(\Omega)$ converging to u in $C^1_d(\bar{\Omega};\mathbb{R}^m)$, then $\boldsymbol{f}(u_n) \to \boldsymbol{f}(u)$ in $L^p(\Omega)$. The result will follow from Lemma 2.13 with $X = W^{2,p}(\Omega;\mathbb{R}^m)$, $Y = C^1_d(\bar{\Omega};\mathbb{R}^m)$, $Z = L^p(\Omega)$ and D defined above.

In Lemma 2.10, we have already established that

$$\boldsymbol{f}(v) = f(\cdot, 0) + \sum_{i=0}^{N} \boldsymbol{g}_i(v) \cdot \partial_i v,$$

where

$$g_i(x, \xi) = \int_0^1 \nabla_{\xi_i} f(x, t\xi)\, \mathrm{d}t,$$

and, by Remark 2.7, the g_i are C^0 bundle maps. Hence, by Lemma 2.9 (i) applied to each component of g_i, $\boldsymbol{g}_i : C^1_d(\bar{\Omega}; \mathbb{R}^m) \to L^\infty(\Omega; \mathbb{R}^m)$ is continuous.

Clearly, the problem reduces to showing that $\boldsymbol{g}_i(u_n) \cdot \partial_i u_n \to \boldsymbol{g}_i(u) \cdot \partial_i u$ in $L^p(\Omega)$, $0 \leqslant i \leqslant N$. To see this, we write

$$\boldsymbol{g}_i(u_n) \cdot \partial_i u_n - \boldsymbol{g}_i(u) \cdot u = (\boldsymbol{g}_i(u_n) - \boldsymbol{g}_i(u)) \cdot \partial_i u_n + \boldsymbol{g}_i(u) \cdot \partial_i(u_n - u). \tag{2.7}$$

The first term tends to zero in $L^p(\Omega)$ because $\boldsymbol{g}_i(u_n) \to \boldsymbol{g}_i(u)$ in $L^\infty(\Omega; \mathbb{R}^m)$, and $(\partial_i u_n)$ is bounded in $L^p(\Omega; \mathbb{R}^m)$. On the other hand, $(u_n - u) \to 0$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$, which is continuously imbedded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. The last term of (2.7) tends to zero if we show that $\boldsymbol{g}_i(u) \in L^p(\Omega; \mathbb{R}^m)$, and this is true for the following reason. Let $\Omega' \subset \Omega$ be the support of $u \in D$. Then first $\boldsymbol{g}_i(u) \in L^\infty(\Omega; \mathbb{R}^m) \subset L^\infty(\Omega'; \mathbb{R}^m) \subset L^p(\Omega'; \mathbb{R}^m)$, and secondly $\boldsymbol{g}_i(u)(x) = g_i(x, u(x), \nabla u(x)) = g_i(x, 0) = \nabla_{\xi_i} f(x, 0)$ when $x \in \Omega \setminus \Omega'$. But $\nabla_{\xi_i} f(\cdot, 0) \in L^p(\Omega; \mathbb{R}^m)$. Therefore, $\boldsymbol{g}_i(u) \in L^p(\Omega \setminus \Omega'; \mathbb{R}^m)$, and thus $\boldsymbol{g}_i(u) \in L^p(\Omega; \mathbb{R}^m)$, as claimed. □

Lemma 2.15. *Let $f : \Omega \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}$ be an equicontinuous C^1_ξ bundle map. Suppose that $f(\cdot, 0) \in L^p(\Omega)$ and that $D_\xi f(\cdot, 0)$ is bounded on Ω (so that the Nemytskii operator $\boldsymbol{f}$ is of class C^1 from $W^{2,p}(\Omega; \mathbb{R}^m)$ to $L^p(\Omega)$ by Theorem 2.11). If $(u_n) \subset W^{2,p}(\Omega; \mathbb{R}^m)$ is a bounded sequence and $u \in W^{2,p}(\Omega; \mathbb{R}^m)$ is such that $u_n \to u$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$ (hence $u_n \rightharpoonup u$ in $W^{2,p}(\Omega; \mathbb{R}^m)$ (see the appendix)), we have*

$$\boldsymbol{f}(u_n) - \boldsymbol{f}(u) - D\boldsymbol{f}(u)(u_n - u) \to 0 \quad \text{in } L^p(\Omega). \tag{2.8}$$

Proof. Let $v_n = u_n - u$, so that $v_n \rightharpoonup 0$ in $W^{2,p}(\Omega, \mathbb{R}^m)$ and $v_n \to 0$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$. Then the left-hand side of (2.8) is $\boldsymbol{f}(u + v_n) - \boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u)v_n = \boldsymbol{g}(v_n) - \boldsymbol{g}(0)$, where $\boldsymbol{g}(v) := \boldsymbol{f}(u + v) - \mathrm{D}\boldsymbol{f}(u)v$ for $v \in W^{2,p}(\Omega, \mathbb{R}^m)$. Note that $\boldsymbol{g}$ is the Nemytskii operator associated with (see (2.3))

$$g(x, \xi) := f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \sum_{i=0}^{N} \nabla_{\xi_i} f(x, u(x), \nabla u(x)) \cdot \xi_i,$$

where $\nabla_{\xi_i} f(\cdot, u, \nabla u)$ is continuous and bounded (see Lemma 2.9 (ii)), so one can check, using Lemma 2.5 (i), that g is an equicontinuous C^1_ξ bundle map with

$$\nabla_{\xi_i} g(x, \xi) = \nabla_{\xi_i} f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \nabla_{\xi_i} f(x, u(x), \nabla u(x)).$$

Furthermore, $g(\cdot, 0) = f(\cdot, u, \nabla u) \in L^p(\Omega)$ (see Lemma 2.10 (i)) and $\nabla_{\xi_i} g(\cdot, 0) = 0 \in L^p(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$. Thus g verifies the conditions of Lemma 2.14, and therefore $\boldsymbol{g}(v_n) \to \boldsymbol{g}(0)$ in $L^p(\Omega)$, which completes the proof. □

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2.3. Smoothness of F

Let the coefficients of F in (1.1) satisfy the following assumptions:

$$a_{\alpha\beta} \text{ are equicontinuous } C^1_\xi \text{ bundle maps, } 1 \leqslant \alpha,\, \beta \leqslant N; \tag{2.9}$$

$$a_{\alpha\beta}(\cdot,0) \text{ and } D_\xi a_{\alpha\beta}(\cdot,0) \text{ are bounded on } \Omega,\ 1 \leqslant \alpha,\, \beta \leqslant N; \tag{2.10}$$

$$b \text{ is an equicontinuous } C^1_\xi \text{ bundle map;} \tag{2.11}$$

$$b(\cdot,0) \in L^p(\Omega;\mathbb{R}^m), D_\xi b(\cdot,0) \text{ is bounded on } \Omega. \tag{2.12}$$

Lemma 2.16. *The operator F in (1.1) is both continuous and weakly sequentially continuous from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega;\mathbb{R}^m)$ and it maps bounded subsets onto bounded subsets.*

Proof. By Lemma 2.9 (ii) applied to each component of $a_{\alpha\beta}$, the Nemytskii operators $\boldsymbol{a}_{\alpha\beta}$ are continuous from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^\infty(\Omega;\mathbb{R}^{m\times m})$ and they map bounded subsets onto bounded subsets. By Lemma 2.10 (i) (applied to each component of b), $\boldsymbol{b}$ is continuous from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega;\mathbb{R}^m)$ and maps bounded subsets onto bounded ones. This proves the continuity and the boundedness properties.

Now if $(u_n) \subset W^{2,p}(\Omega,\mathbb{R}^m)$ converges weakly to u, we have that $\partial^2_{\alpha\beta} u_n \rightharpoonup \partial^2_{\alpha\beta} u$ in $L^p(\Omega,\mathbb{R}^m)^*$. By Lemma 2.9 (iv), $\boldsymbol{a}_{\alpha\beta}(u_n)\cdot\partial^2_{\alpha\beta} u_n \rightharpoonup \boldsymbol{a}_{\alpha\beta}(u)\cdot\partial^2_{\alpha\beta} u$ in $L^p(\Omega;\mathbb{R}^m)$. Next, by Lemma 2.10 (ii), $\boldsymbol{b}(u_n) \rightharpoonup \boldsymbol{b}(u)$ in $L^p(\Omega;\mathbb{R}^m)$. This proves the weak continuity of F. □

Remark 2.17. Note that the proof of the above lemma requires only the following weaker assumptions: $a_{\alpha\beta}$ are C^0 bundle maps, with $a_{\alpha\beta}(\cdot,0)$ bounded, $b(x,\xi) = b_0(x) + \sum_{i=1}^N c_i(x,\xi)\cdot\xi_i$, where $b_0 \in L^p(\Omega;\mathbb{R}^m)$, and the c_i are C^0 equicontinuous bundle maps with $c_i(\cdot,0)$ bounded. This will be used in § 5.

Theorem 2.18. *The operator F in (1.1) is of class C^1 from $W^{2,p}(\Omega;\mathbb{R}^m)$ to $L^p(\Omega;\mathbb{R}^m)$, with derivative*

$$\mathrm{D}F(u)v = -\sum_{\alpha,\beta=1}^N \boldsymbol{a}_{\alpha\beta}(u)\cdot\partial^2_{\alpha\beta} v + \mathrm{D}\boldsymbol{b}(u)v - \sum_{\alpha,\beta=1}^N (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\cdot\partial^2_{\alpha\beta} u, \tag{2.13}$$

where $\mathrm{D}\boldsymbol{b}(u)$ and $\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)$ are given by (2.5) and (2.6), respectively.

In particular, the restriction of F to the subspace $D_p(\Omega) = (W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega))^m$ is C^1 from $D_p(\Omega)$ to $L^p(\Omega;\mathbb{R}^m)$.

Proof. Recall that

$$F(u) = -\sum_{\alpha,\beta=1}^N \boldsymbol{a}_{\alpha\beta}(u)\cdot\partial^2_{\alpha\beta} u + \boldsymbol{b}(u).$$

By Theorem 2.11, $\boldsymbol{b} \in C^1(X_p(\Omega), Y_p(\Omega))$ and $\boldsymbol{a}_{\alpha\beta} \in C^1(X_p(\Omega), L^\infty(\Omega;\mathbb{R}^{m\times m}))$. Now let $G(u) := \boldsymbol{a}_{\alpha\beta}(u)\cdot\partial^2_{\alpha\beta} u$ and $B : L^\infty(\Omega;\mathbb{R}^{m\times m})\times Y_p(\Omega) \to Y_p(\Omega)$ be the bounded bilinear operator defined by $B(M,x) = M\cdot x$. Then $G = B\circ(\boldsymbol{a}_{\alpha\beta}, \partial^2_{\alpha\beta})$, and the result follows from the chain rule. □

* $\partial^2_{\alpha\beta} : X_p(\Omega) \to Y_p(\Omega)$ is linear and bounded and therefore weakly continuous.

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3. Fredholmness

We now begin the investigation of the Fredholmness of the second-order differential operator (1.1). To the hypotheses (2.9)–(2.12), we add an ellipticity condition, which implies that the linearization $\mathrm{D}F(u)$ is a compact perturbation of a linear elliptic operator (of second order),

$$\det\biggl(\sum_{\alpha,\beta=1}^{N}(\eta_\alpha\eta_\beta a_{\alpha\beta}(x,\xi))\biggr) \geqslant \gamma(x,\xi)|\eta|^{2m} \quad \forall\eta\in\mathbb{R}^N,\ (x,\xi)\in\Omega\times(\mathbb{R}^m\times\mathbb{R}^{mN}), \tag{3.1}$$

where $\gamma:\Omega\times(\mathbb{R}^m\times\mathbb{R}^{mN})\to(0,\infty)$ is bounded from below by a positive constant on every compact subset of $\Omega\times(\mathbb{R}^m\times\mathbb{R}^{mN})$.

Note that, in the case of a single equation ($m=1$), this condition reduces to

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,\xi)\eta_\alpha\eta_\beta \geqslant \gamma(x,\xi)|\eta|^2 \quad \forall\eta\in\mathbb{R}^N,\ (x,\xi)\in\Omega\times\mathbb{R}^{N+1},$$

and this is just the ellipticity condition used in [**8**] with $\Omega=\mathbb{R}^N$. In the remainder of the paper, the coefficients of the operator F in (1.1) will satisfy hypotheses (2.9)–(2.12) and (3.1).

Note that

$$\mathrm{D}F(u)v = L(u)v - \sum_{\alpha,\beta=1}^{N}(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\cdot\partial^2_{\alpha\beta}u,$$

where

$$L(u)v := -\sum_{\alpha,\beta=1}^{N}\boldsymbol{a}_{\alpha\beta}(u)\cdot\partial^2_{\alpha\beta}v + \mathrm{D}\boldsymbol{b}(u)v, \tag{3.2}$$

and clearly $L(u)\in\mathcal{L}(X_p(\Omega),Y_p(\Omega))$.

Lemma 3.1. *Let $u\in X_p(\Omega)$. Then the difference $\mathrm{D}F(u)-L(u)$ is compact between $X_p(\Omega)$ and $Y_p(\Omega)$. Therefore, given $\mu\in\boldsymbol{Z}\cup\{\pm\infty\}$, we have the following.*

(i) $\mathrm{D}F(u)\in\Phi_\mu(X_p(\Omega),Y_p(\Omega)) \Leftrightarrow L(u)\in\Phi_\mu(X_p(\Omega),Y_p(\Omega))$.

(ii) $\mathrm{D}F(u)\in\Phi_\mu(D_p(\Omega),Y_p(\Omega)) \Leftrightarrow L(u)\in\Phi_\mu(D_p(\Omega),Y_p(\Omega))$.

Proof. If we show that the difference is compact, then (i) and (ii) will follow from the stability of $\Phi_\mu(X_p(\Omega),Y_p(\Omega))$ and $\Phi_\mu(D_p(\Omega),Y_p(\Omega))$ under compact perturbations (see Note A in the appendix).

By (3.2), we have

$$\mathrm{D}F(u)v - L(u)v = -\sum_{\alpha,\beta=1}^{N}(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\cdot\partial^2_{\alpha\beta}u.$$

Clearly, it suffices to show that each $(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\cdot\partial^2_{\alpha\beta}u$ is compact, and again, for that, it suffices to show that each component $\sum_{j=1}^m(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)_{k,j}\partial^2_{\alpha\beta}u^j$ ($k=1,\dots,m$) is compact. Now, each term of this sum is (by (2.6))

$$\biggl(\sum_{i=0}^N \nabla_{\xi_i}a^{k,j}_{\alpha\beta}(\cdot,u,\nabla u)\cdot\partial_i v\biggr)\partial^2_{\alpha\beta}u^j.$$

Once again, writing the components, we deal with the terms $(\partial_{\xi_i^l}a^{k,j}_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u^j)\partial_i v^l$, $k,j,l=1,\dots,m$. Now, indeed, $\partial_{\xi_i^l}a^{k,j}_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u^j\in L^p(\Omega)$. But since $N<p<\infty$, the multiplication by a fixed function of $L^p(\Omega)$ is a compact operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ (see the appendix). Thus $w\mapsto Tw:=(\partial_{\xi_i^l}a^{k,j}_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u^j)\partial_i w$ is compact from $W^{2,p}(\Omega)$ to $L^p(\Omega)$*. Now, going back through the steps, we see that $\mathrm{D}F(u)-L(u)$ is a compact operator from $X_p(\Omega)$ to $Y_p(\Omega)$, and therefore also from $D_p(\Omega)$ to $Y_p(\Omega)$. □

Fix u and let $A_{\alpha\beta}(x)=\boldsymbol{a}_{\alpha\beta}(u)(x)$. Furthermore, let $B_\alpha(x)$ be the matrix with lines $\nabla_{\xi_\alpha}b^k(x,u(x),\nabla u(x))$, $k=1,\dots,m$, and let $C(x)$ be the matrix with lines $\nabla_{\xi_0}b^k(x,u(x),\nabla u(x))$, $k=1,\dots,m$. Then

$$L(u)v=-\sum_{\alpha,\beta=1}^N A_{\alpha\beta}(x)\cdot\partial^2_{\alpha\beta}v+\sum_{\alpha=1}^N B_\alpha(x)\cdot\partial_\alpha v+C(x)\cdot v$$

is a linear second-order differential operator, with continuous and bounded coefficients. Condition (3.1) implies that

$$\det\biggl(\sum_{\alpha,\beta=1}^N(A_{\alpha\beta}(x)\eta_\alpha\eta_\beta)\biggr)\geqslant\gamma(x,u(x),\nabla u(x))|\eta|^{2m}.$$

As x varies over a compact set, the continuity of u and ∇u ensures that $(x,u(x),\nabla u(x))$ remain in a compact set K. Thus, by (3.1), there exists $\gamma_K>0$ such that

$$\det\biggl(\sum_{\alpha,\beta=1}^N(A_{\alpha\beta}(x)\eta_\alpha\eta_\beta)\biggr)\geqslant\gamma_K|\eta|^{2m}.$$

This leads us to introduce the following definition for linear differential operators.

Definition 3.2. Let $A_{\alpha\beta}$, B_α, C ($\alpha,\beta=1,\dots,N$) be (matrix-valued) functions from Ω to $\mathbb{R}^{m\times m}$. Define a second-order linear differential operator L by

$$Lv:=-\sum_{\alpha,\beta=1}^N A_{\alpha\beta}(x)\cdot\partial^2_{\alpha\beta}v+\sum_{\alpha=1}^N B_\alpha(x)\cdot\partial_\alpha v+C(x)\cdot v.$$

L is said to be elliptic at x in the sense of Petrovskii if there exists a positive constant γ ($=\gamma(x)$) such that

$$\det\biggl(\sum_{\alpha,\beta=1}^N(A_{\alpha\beta}(x)\eta_\alpha\eta_\beta)\biggr)\geqslant\gamma|\eta|^{2m}\quad\forall\eta\in\mathbb{R}^N.$$

* $w_n\rightharpoonup w$ in $W^{2,p}\Rightarrow\partial_i w_n\rightharpoonup\partial_i w$ in $W^{1,p}\Rightarrow Tw_n\to Tw$ in L^p.

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We say that L is strictly elliptic on a subset $K \subset \Omega$ if, in the above definition, one can choose the same γ for all $x \in K$.

Remark 3.3. From the above, we can say that, for each fixed $u \in X_p(\Omega)$, the differential operator $L(u)$ is strictly Petrovskii-elliptic on the compact subsets of Ω.

Lemma 3.4 (cf. Theorem 17 of [5]). *Let $\Omega' \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, and $1 < q < \infty$. Let L be as in the above definition, strictly elliptic in Ω', with continuous coefficients. If $v \in W^{2,q}(\Omega';\mathbb{R}^m) \cap W_0^{1,q}(\Omega';\mathbb{R}^m)$, then v satisfies the* a priori *estimate*

$$\|v\|_{2,q,\Omega'} \leqslant c(\|Lv\|_{0,q,\Omega'} + \|v\|_{0,1,\Omega'}), \tag{3.3}$$

where c is a positive constant.

Lemma 3.5. *Assume that $\partial\Omega$ is of class C^2. Let L be a second-order linear differential operator, strictly elliptic on the compact subsets of Ω, with continuous bounded coefficients*. If $(u_n) \subset D_p(\Omega)$ is a sequence converging weakly to zero in $D_p(\Omega)$ and $Lu_n \to 0$ in $Y_p(\Omega)$, then $u_n \to 0$ in $X_p(\Omega')$ for all open and bounded subsets $\Omega' \subset \Omega$.*

Proof. We distinguish between two cases.

Case 1 (Ω is bounded). Since $u_n \rightharpoonup 0$ in $W^{2,p}(\Omega;\mathbb{R}^m) \hookrightarrow_{\text{comp}} L^1(\Omega;\mathbb{R}^m)$, we have $u_n \to 0$ in $L^1(\Omega;\mathbb{R}^m)$. On the other hand, $Lu_n \to 0$ in $L^p(\Omega,\mathbb{R}^m)$. Note that (u_n), L and Ω satisfy the conditions of Lemma 3.4. So, by letting $q = p$ and $v = u_n$ in (3.3), we get $u_n \to 0$ in $W^{2,p}(\Omega,\mathbb{R}^m)$.

Case 2 (Ω is unbounded). For every $r > 0$, set $B_r = \{x \in \mathbb{R}^N;\ |x| < r\}$ and $\Omega_r = \Omega \cap B_r$. Clearly, it is equivalent to show that the result holds when $\Omega' = \Omega_r$ for $r > 0$ large enough†.

Let B_r be a ball containing $\partial\Omega$, and $R > r$. It follows from the remarks made about Ω in § 2 that $\partial\Omega_R = \partial\Omega \cup \partial B_R$, so that $\partial\Omega_R$ is C^2 since $\partial\Omega \cap \partial B_R = \varnothing$. Define $\varphi : \mathbb{R}^N \to \mathbb{R}$ to be a C^∞ function with compact support such that $\varphi = 1$ on B_r, $\varphi = 0$ outside B_R and $\|\varphi\|_{0,\infty} \leqslant 1$. Define a new sequence (v_n) by $v_n = \varphi u_n$, so that $u_n = v_n$ on $\Omega \cap B_r$, and $v_n \in D_p(\Omega_R)$.

Recall that $u_n \rightharpoonup 0$ in $D_p(\Omega) \hookrightarrow_{\text{comp}} W^{1,p}(\Omega_R;\mathbb{R}^m) \hookrightarrow L^1(\Omega_R;\mathbb{R}^m)$, so that $u_n \to 0$ in $W^{1,p}(\Omega_R;\mathbb{R}^m)$ as well as in $L^1(\Omega_R;\mathbb{R}^m)$, and therefore also $v_n \to 0$ in $L^1(\Omega_R;\mathbb{R}^m)$, since φ is bounded.

On the other hand, a direct calculation leads to

$$Lv_n = \varphi Lu_n + \sum_{\alpha,\beta}(\partial^2_{\alpha\beta}\varphi A_{\alpha\beta}) \cdot u_n + \sum_{\alpha,\beta}(\partial_\alpha\varphi A_{\alpha\beta}) \cdot \partial_\beta u_n + \sum_{\alpha,\beta}(\partial_\beta\varphi A_{\alpha\beta}) \cdot \partial_\alpha u_n + \sum_{\alpha}(\partial_\alpha\varphi B_\alpha) \cdot u_n.$$

* The boundedness of the coefficients ensures that L maps continuously $W^{2,p}(\Omega;\mathbb{R}^m)$ into $L^p(\Omega;\mathbb{R}^m)$.
† Since Ω_r is open and bounded and every open and bounded subset of Ω is contained in a subset of the form Ω_r.

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Thus, due to the boundedness of φ, of its derivatives and of the coefficients of L, we get $Lv_n \to L^p(\Omega_R;\mathbb{R}^m)$.

Estimate (3.3) now gives that $v_n \to 0$ in $X_p(\Omega_R)$, and therefore also in $X_p(\Omega_r)$. This finally implies that $u_n \to X_p(\Omega_r)$. □

For the next result, we need the following concept introduced in [**8**].

Definition 3.6. Let X and Y be real Banach spaces with X reflexive and let $T, L \in \mathcal{L}(X,Y)$ be given. We say that T is compact modulo L if, for every sequence $(u_n) \subset X$, we have $\{u_n \rightharpoonup 0 \text{ in } X,\ Lu_n \to 0 \text{ in } Y\} \Rightarrow Tu_n \to 0$ in Y.

Lemma 3.7 (cf. Lemma 3.7 of [8]). *Let X and Y be real Banach spaces with X reflexive and let $L_0, L_1 \in \mathcal{L}(X,Y)$ be given. Suppose that $L_0 - L_1$ is compact modulo both L_0 and L_1. Then we have the following.*

(i) *If $(u_n) \subset X$ is a sequence converging weakly to zero, we have $L_0u_n \to 0$ in Y if and only if $L_1u_n \to 0$.*

(ii) *$L_0 \in \Phi_+(X,Y)$ if and only if $L_1 \in \Phi_+(X,Y)$.*

For $t \in [0,1]$, define $L_t := tL_1 + (1-t)L_0$. If $L_0 - L_1$ is compact modulo L_t $\forall t \in [0,1]$, then the following holds.

(iii) *$L_t \in \Phi_+(X,Y)$ for all $t \in [0,1]$ if and only if this holds for some $t_0 \in [0,1]$, and, in this case, the index of L_t is independent of t.*

Lemma 3.8. *Assume that $\partial\Omega$ is C^2. For $L(u)$ defined by (3.2), the relation $L(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$ holds for every $u \in D_p(\Omega)$ if and only if it holds for some $u^0 \in D_p(\Omega)$.*

Proof. We prove that $L(u) - L(u^0)$ is compact modulo $L(u)$. By exchanging the roles of u and u^0, this shows that $L(u) - L(u^0)$ is compact modulo both $L(u)$ and $L(u^0)$. The conclusion follows from Lemma 3.7 (ii).

Let $(v_n) \subset D_p(\Omega)$ be such that $v_n \rightharpoonup 0$ in $D_p(\Omega)$ and $L(u)v_n \to 0$ in $Y_p(\Omega)$. From the equicontinuity of $a_{\alpha\beta}$ and $D_\xi b$ at $\xi = 0$, we have that, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|a_{\alpha\beta}(x,\xi) - a_{\alpha\beta}(x,0)| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |D_\xi b(x,\xi) - D_\xi b(x,0)| < \tfrac{1}{2}\varepsilon$$

for $|\xi| < \delta$ and all $x \in \Omega$. Due to the embedding $W^{2,p}(\Omega,\mathbb{R}^m) \hookrightarrow C^1_d(\bar{\Omega};\mathbb{R}^m)$ and the definition of $C^1_d(\bar{\Omega})$, there is $r > 0$ such that $|(u(x), \nabla u(x))| < \delta$ and $|(u^0(x), \nabla u^0(x))| < \delta$ for $|x| \geqslant r$ (we can choose r such that $\partial\Omega \subset B_r$). Therefore,

$$|a_{\alpha\beta}(x, u(x), \nabla u(x)) - a_{\alpha\beta}(x, u^0(x), \nabla u^0(x))| < \varepsilon$$

and

$$|D_\xi b(x, u(x), \nabla u(x)) - D_\xi b(x, u^0(x), \nabla u^0(x))| < \varepsilon$$

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whenever $|x| \geqslant r$. Now let $\Omega_r = \{x \in \Omega : |x| < r\}$, $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$ and recall that

$$(L(u) - L(u^0))v = -\sum_{\alpha,\beta=1}^{N} (\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}_{\alpha\beta}(u^0)) \cdot \partial^2_{\alpha\beta} v + \sum_{i=0}^{N} (\nabla_{\xi_i} b(\cdot, u, \nabla u) - \nabla_{\xi_i} b(\cdot, u^0, \nabla u^0)) \cdot \partial_i v.$$

Therefore,

$$\|(L(u) - L(u^0))v\|_{0,p,\tilde{\Omega}_r} \leqslant m^2(N^2 + N + 1)\varepsilon \|v\|_{2,p,\tilde{\Omega}_r} \quad \text{for } v \in D_p(\Omega).$$

Hence

$$\|(L(u) - L(u^0))v_n\|_{0,p,\tilde{\Omega}_r} \leqslant m^2 M(N^2 + N + 1)\varepsilon, \tag{3.4}$$

where M is a bound for $\|v_n\|_{2,p,\Omega}$.

As already observed, $L(u)$ verifies the conditions required in Lemma 3.5, thus $v_n \to 0$ in $X_p(\Omega_r)$, so $L(u^0)v_n$ and $L(u)v_n$ converge to zero in $Y_p(\Omega_r)$*, which means that, for any $\varepsilon > 0$ and n large enough,

$$\|(L(u) - L(u^0))v_n\|_{0,p,\Omega_r} \leqslant \varepsilon. \tag{3.5}$$

Equations (3.4) and (3.5) together yield that $\|(L(u) - L(u^0))v_n\|_{0,p,\Omega}$ can be made arbitrary small for n large enough. This completes the proof. □

Theorem 3.9. *Let $\partial\Omega$ be of class C^2. The operator F in (1.1) is semi-Fredholm of index $\mu \in \boldsymbol{Z} \cup \{-\infty\}$ (i.e. $\mathrm{D}F(u) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$ for every $u \in D_p(\Omega)$) if and only if there is some $u^0 \in D_p(\Omega)$ such that $\mathrm{D}F(u^0) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$.*

Proof. By Lemmas 3.1 and 3.8,

$$\begin{aligned} \mathrm{D}F(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) &\Leftrightarrow L(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \\ &\Leftrightarrow L(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \qquad \forall u \in D_p(\Omega) \\ &\Leftrightarrow \mathrm{D}F(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \quad \forall u \in D_p(\Omega). \end{aligned}$$

By Theorem 2.18, $\mathrm{D}F$ is continuous as a map from $D_p(\Omega)$ into $\mathcal{L}(D_p(\Omega), Y_p(\Omega))$. Recall also that the index of a semi-Fredholm operator is locally constant, whence $u \mapsto \operatorname{index} \mathrm{D}F(u)$ is locally constant and therefore constant since $D_p(\Omega)$ is connected. □

4. Properness

Lemma 4.1. *Let $u \in W^{2,p}(\Omega; \mathbb{R}^m)$ and $(u_n) \subset W^{2,p}(\Omega; \mathbb{R}^m)$ be a bounded sequence converging to u in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$. Then $F(u_n) - F(u) - \mathrm{D}F(u)(u_n - u) \to 0$ in $L^p(\Omega; \mathbb{R}^m)$.*

* $L(u) \in \mathcal{L}(X_p(\Omega_r), Y_p(\Omega_r))$ by Theorem 2.18, with Ω replaced by Ω_r.

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Proof. Note first that $u_n \rightharpoonup u$ in $W^{2,p}(\Omega;\mathbb{R}^m)$ (see Note E2 of the appendix),

$$F(u_n) - F(u) - \mathrm{D}F(u)(u_n - u)$$
$$= -\sum_{\alpha,\beta=1}^{N} (\boldsymbol{a}_{\alpha\beta}(u_n) - \boldsymbol{a}_{\alpha\beta}(u)) \cdot \partial^2_{\alpha\beta} u_n$$
$$+ \sum_{\alpha,\beta=1}^{N} (D\boldsymbol{a}_{\alpha\beta}(u)(u_n - u)) \cdot \partial^2_{\alpha\beta} u + \boldsymbol{b}(u_n) - \boldsymbol{b}(u) - D\boldsymbol{b}(u)(u_n - u). \quad (4.1)$$

As already observed in the proof of Lemma 3.1, $v \mapsto (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v) \cdot \partial^2_{\alpha\beta} u$ is a compact linear operator from $X_p(\Omega)$ to $Y_p(\Omega)$. Therefore,

$$\sum_{\alpha,\beta=1}^{N} (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)(u_n - u)) \cdot \partial^2_{\alpha\beta} u \to 0 \quad \text{in } L^p(\Omega;\mathbb{R}^m). \quad (4.2)$$

By Lemma 2.15,

$$\boldsymbol{b}(u_n) - \boldsymbol{b}(u) - \mathrm{D}\boldsymbol{b}(u)(u_n - u) \to 0 \quad \text{in } L^p(\Omega;\mathbb{R}^m). \quad (4.3)$$

By Lemma 2.9 (i), $\boldsymbol{a}_{\alpha\beta}(u_n) \to \boldsymbol{a}_{\alpha\beta}(u)$ in $L^\infty(\Omega;\mathbb{R}^{m\times m})$, and since $\partial^2_{\alpha\beta} u_n$ is bounded in $L^p(\Omega;\mathbb{R}^m)$, we have

$$\sum_{\alpha,\beta=1}^{N} (\boldsymbol{a}_{\alpha\beta}(u_n) - \boldsymbol{a}_{\alpha\beta}(u)) \cdot \partial^2_{\alpha\beta} u_n \to 0 \quad \text{in } L^p(\Omega;\mathbb{R}^m) \quad (4.4)$$

□

Theorem 4.2. *Let Ω have a C^2 boundary. Suppose that there exists $u^0 \in D_p(\Omega)$ for which $DF(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. The following properties are equivalent.*

(i) *$F : D_p(\Omega) \to Y_p(\Omega)$ is proper on the closed bounded subsets of $D_p(\Omega)$.*

(ii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$ contains a subsequence converging in $C^1_d(\bar{\Omega};\mathbb{R}^m)$.*

Proof. (i) ⇒ (ii) is evident, since $D_p(\Omega) \hookrightarrow C^1_d(\bar{\Omega};\mathbb{R}^m)$.

(ii) ⇒ (i). Let $(u_n) \subset D_p(\Omega)$ be bounded and such that $(F(u_n))$ converges in $Y_p(\Omega)$. By assumption, there is a subsequence $(u_{\phi(n)})$ converging to some u in $C^1_d(\bar{\Omega};\mathbb{R}^m)$, and hence, by Lemma 4.1, $F(u_{\phi(n)}) - F(u) - \mathrm{D}F(u)(u_{\phi(n)} - u) \to 0$ in $Y_p(\Omega)$. By Note F2 in the appendix, $u_{\phi(n)} \rightharpoonup u$ in $X_p(\Omega)$, and F is weakly continuous (see Lemma 2.16), so $F(u_{\phi(n)}) \rightharpoonup F(u)$, and hence $F(u_{\phi(n)}) \to F(u)$ in $Y_p(\Omega)$. Thus $\mathrm{D}F(u)(u_{\phi(n)} - u) \to 0$ in $Y_p(\Omega)$. But we know that $\mathrm{D}F(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$ (see Theorem 3.9), and hence it is proper*. Therefore, $u_{\phi(n)} \to u$ in $D_p(\Omega)$. □

* Yood's criterion, see the appendix.

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As in [**8**], we can give an equivalent formulation of Theorem 4.2 in terms of sequences vanishing uniformly at infinity.

Definition 4.3. We say that the sequence $u_n \subset C^1_d(\bar{\Omega}; \mathbb{R}^m)$ vanishes uniformly at infinity in the sense of $C^1_d(\bar{\Omega})$ if the following condition holds: $\forall \varepsilon > 0$, $\exists R > 0$, $\exists n_0 \in \mathbb{N}$ such that $|u_n(x)| + |\nabla u_n(x)| \leqslant \varepsilon$ for all $|x| \geqslant R$ and $n \geqslant n_0$.

Lemma 4.4. *Let $(u_n) \subset X_p(\Omega)$ be a bounded sequence. For $u \in X_p(\Omega)$, the following conditions are equivalent.*

(i) *$u_n \to u$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$.*

(ii) *$u_n \rightharpoonup u$ in $X_p(\Omega)$ and (u_n) vanishes uniformly at infinity in the sense of $C^1_d(\bar{\Omega})$.*

Proof. Let $\Omega_r = \{x \in \Omega;\ |x| < r\}$ and $\tilde{\Omega}_r = \{x \in \Omega;\ |x| > r\}$ for every $r > 0$.

(i) $\Rightarrow$ (ii). It follows from Note F2 in the appendix that $u_n \rightharpoonup u$ in $X_p(\Omega)$. Let $\varepsilon > 0$ be given. There exists $r > 0$ for which $|u(x)| + |\nabla u(x)| \leqslant \frac{1}{2}\varepsilon$ whenever $x \in \tilde{\Omega}_r$. Let n_0 be such that $\|u_n - u\|_{1,\infty,\Omega} \leqslant \frac{1}{2}\varepsilon$ for $n \geqslant n_0$. Then, for every $x \in \tilde{\Omega}_r$ and $n \geqslant n_0$, we have $|u_n(x)| + |\nabla u_n(x)| \leqslant \varepsilon$.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be given and $r > 0$ and $n_0 \in \mathbb{N}$ be such that $|u_n(x)| + |\nabla u_n(x)| \leqslant \frac{1}{2}\varepsilon$ whenever $x \in \tilde{\Omega}_r$ and $n \geqslant n_0$. After increasing r if necessary, we may assume that $|u(x)| + |\nabla u(x)| \leqslant \frac{1}{2}\varepsilon$. Hence

$$|u(x) - u_n(x)| + |\nabla u(x) - \nabla u_n(x)| \leqslant \varepsilon \quad \forall x \in \tilde{\Omega}_r \quad \forall n \geqslant n_0.$$

Next, since $X_p(\Omega_r) \hookrightarrow_{\mathrm{comp}} C^1_d(\bar{\Omega}_r; \mathbb{R}^m)$, there exists $n_1 \in \mathbb{N}$ such that

$$|u(x) - u_n(x)| + |\nabla u(x) - \nabla u_n(x)| \leqslant \varepsilon \quad \forall x \in \bar{\Omega}_r \quad \forall n \geqslant n_1.$$

Thus, finally, we have $\|u_n - u\|_{1,\infty,\Omega} \leqslant \varepsilon$ for $n \geqslant \max(n_0, n_1)$, which shows that $u_n \to u$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$ as claimed. □

Corollary 4.5. *Let Ω have a C^2 boundary. Suppose there exists $u^0 \in D_p(\Omega)$ for which $\mathrm{D}F(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. The following conditions are equivalent.*

(i) *$F : D_p(\Omega) \to Y_p(\Omega)$ is proper on the closed bounded subsets of $D_p(\Omega)$.*

(ii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$ vanishes uniformly at infinity in the sense of $C^1_d(\bar{\Omega})$.*

(iii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$ contains a subsequence vanishing uniformly at infinity in the sense of $C^1_d(\bar{\Omega})$.*

Proof. (i) $\Rightarrow$ (ii). Let (u_n) be a bounded sequence from $D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$, and suppose that (u_n) does not vanish uniformly at infinity. Denote, for simplicity, $\theta_n(x) = |u_n(x)| + |\nabla u_n(x)|$. Then there exist $\varepsilon_0 > 0$, a subsequence $u_{\phi(n)}$ and a sequence $(x_n) \subset \Omega$ such that

$$|x_n| \geqslant n \quad \text{and} \quad \theta_{\phi(n)}(x_n) = |u_{\phi(n)}(x_n)| + |\nabla u_{\phi(n)}(x_n)| \geqslant \varepsilon_0.$$

Now $(u_{\phi(n)})$ is also bounded and its image by F convergent. So, by Theorem 4.2, it contains a subsequence $u_{\phi(\psi(n))}$ converging in C^1_d and therefore vanishing uniformly at infinity. Accordingly, there exists $n_0 \in \mathbb{N}$ and $r > 0$ such that $\theta_{\phi(\psi(n))}(x) < \frac{1}{2}\varepsilon_0$ whenever $|x| \geqslant r$ and $n \geqslant n_0$. So, for $n \geqslant \max(r, n_0)$ (since $\psi(n) \geqslant n$), we have $\varepsilon_0 \leqslant \theta_{\phi(\psi(n))}(x_{\psi(n)}) < \frac{1}{2}\varepsilon_0$, a contradiction.

(ii) ⇒ (iii) is evident.

(iii) ⇒ (i). Let (u_n) be a bounded sequence from $D_p(\Omega)$ such that $(F(u_n))$ converges. By assumption, it contains a subsequence $(u_{\phi(n)})$ vanishing uniformly at infinity in the sense of C^1_d. But this subsequence is also bounded in $D_p(\Omega)$, and therefore it contains a subsequence $(u_{\phi(\psi(n))})$ converging weakly to some u in $D_p(\Omega)$. So, by Lemma 4.4, $u_{\phi(\psi(n))} \to u$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$. Hence F is proper by Theorem 4.2 (ii). □

Lemma 4.6. *Let $(u_n) \subset W^{2,p}(\Omega)$ be a sequence converging to zero in $W^{2,p}(\Omega')$, for every bounded and open subset $\Omega' \subset \Omega$. Then, given $v \in W^{2,p}(\Omega), \varepsilon \in (0,1)$ and $n_0 \in \mathbb{N}$, there exists $n_1 \in \mathbb{N}$, $n_1 \geqslant n_0$, such that, for every $n \geqslant n_1$:*

(i) $\|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} - \varepsilon \leqslant \|v + u_n\|^p_{2,p,\Omega} \leqslant \|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} + \varepsilon$; *and*

(ii) $\|v + u_n\|_{1,\infty,\Omega} \leqslant \max(\|v\|_{1,\infty,\Omega}, \|u_n\|_{1,\infty,\Omega}) + \varepsilon$.

Proof. Let $\Omega_r = \{x \in \Omega : |x| < r\}$ and $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$. Since $v \in X_p(\Omega) \hookrightarrow C^1_d(\bar{\Omega}; \mathbb{R}^m)$, there exists $r > 0$ such that

$$\|v\|_{2,p,\tilde{\Omega}_r} \leqslant \varepsilon \quad \text{and} \quad \|v\|_{1,\infty,\tilde{\Omega}_r} \leqslant \varepsilon. \tag{4.5}$$

By assumption, we have $u_n \to 0$ in $W^{2,p}(\Omega_r)$ and hence also in $C^1(\bar{\Omega}_r)$. Thus, for n large enough,

$$\|u_n\|_{2,p,\Omega_r} \leqslant \varepsilon \quad \text{and} \quad \|u_n\|_{1,\infty,\Omega_r} \leqslant \varepsilon. \tag{4.6}$$

Using the preceding inequalities and $|a - b|^p \geqslant a^p - p(a + b)^{p-1} b$, we get

$$\begin{aligned}\|v + u_n\|^p_{2,p,\Omega} &\geqslant |\|v\|_{2,p,\Omega_r} - \|u_n\|_{2,p,\Omega_r}|^p + |\|v\|_{2,p,\tilde{\Omega}_r} - \|u_n\|_{2,p,\tilde{\Omega}_r}|^p \\ &\geqslant \|v\|^p_{2,p,\Omega_r} + \|u_n\|^p_{2,p,\tilde{\Omega}_r} - 2p(M + \varepsilon)^{p-1}\varepsilon,\end{aligned}$$

where M is a bound for $\|v\|_{2,p,\Omega}$ and $\|u_n\|_{2,p,\Omega}$. We also deduce from (4.5) and (4.6) that

$$\|v\|^p_{2,p,\Omega} \leqslant \|v\|^p_{2,p,\Omega_r} + \varepsilon^p \quad \text{and} \quad \|u_n\|^p_{2,p,\Omega} \leqslant \|u_n\|^p_{2,p,\tilde{\Omega}_r} + \varepsilon^p.$$

Thus

$$\|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} - 2(p(M + 1)^{p-1} + 1)\varepsilon \leqslant \|v + u_n\|^p_{2,p,\Omega}. \tag{4.7}$$

Analogously, using (4.5) and (4.6) and $(a + b)^p \leqslant a^p + p(a + b)^{p-1} b$, we prove that

$$\|v + u_n\|^p_{2,p,\Omega} \leqslant \|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} + 2p(M + 1)^{p-1}\varepsilon.$$

This proves (i), since ε is arbitrary.

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For (ii), we have

$$\begin{aligned}\|v+u_n\|_{1,\infty,\Omega} &= \max(\|v+u_n\|_{1,\infty,\Omega_r}, \|v+u_n\|_{1,\infty,\tilde{\Omega}_r}) \\ &\leqslant \max(\|v\|_{1,\infty,\Omega_r}+\varepsilon, \|u_n\|_{1,\infty,\tilde{\Omega}_r}+\varepsilon) \\ &= \max(\|v\|_{1,\infty,\Omega_r}, \|u_n\|_{1,\infty,\tilde{\Omega}_r})+\varepsilon \\ &\leqslant \max(\|v\|_{1,\infty,\Omega}, \|u_n\|_{1,\infty,\Omega})+\varepsilon.\end{aligned}$$

□

Lemma 4.7. *Assume that Ω has a C^2 boundary. Let*

$$L = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\cdot\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{N} B_{\alpha}(x)\cdot\partial_{\alpha} + C(x)\cdot$$

be a differential operator that is strictly elliptic on the compact subsets of Ω, with continuous and bounded coefficients. Suppose there is a sequence (u_n) in $D_p(\Omega)$ such that $u_n \rightharpoonup 0$ in $D_p(\Omega)$, $Lu_n \to 0$ in $Y_p(\Omega)$ and (u_n) contains no subsequence converging to 0 in $D_p(\Omega)$. Then there is a sequence $(w_n) \subset D_p(\Omega)$ such that $w_n \rightharpoonup 0$ in $D_p(\Omega)$ and $Lw_n \to 0$ in $Y_p(\Omega)$, (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$, but, furthermore, $w_n \to 0$ in $C^1_d(\bar{\Omega};\mathbb{R}^m)$.

Proof. For simplicity, we denote by $\|u\|_{k,p}$ the norm of u in $W^{k,p}(\Omega;\mathbb{R}^m)$.

Since (u_n) contains no subsequence converging to 0, there exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that $\|u_n\|_{2,p} \geqslant \gamma$ for $n \geqslant n_0$. Therefore, at least one component $(u_n^{l_n})$ of (u_n) verifies $\|u_n^{l_n}\|_{2,p} \geqslant \gamma/m = \delta$. Since $(l_n) \subset \{1,\dots,m\}$ is finite, it contains a constant subsequence $l_{\psi(n)} = l$, so that $\|u^l_{\psi(n)}\|_{2,p} \geqslant \delta$. In the remainder of the proof, l is fixed, and, for the sake of further simplicity, we denote by u_n the subsequence $u_{\psi(n)}$.

Note that the hypotheses made about (u_n) imply, by Lemma 3.5, that $u_n \to 0$ in $W^{2,p}(\Omega';\mathbb{R}^m)$ for every open and bounded subset $\Omega' \subset \Omega$. Therefore, $u_n^j \to 0$ in $W^{2,p}(\Omega')$ $\forall j = 1,\dots,m$.

Let ε_n be a sequence from $(0,1)$ such that $\sum_{n=0}^{\infty}\varepsilon_n = \delta^p$.

We construct a sequence (v_n) in $D_p(\Omega)$ and a subsequence $(u_{\varphi(n)})$ verifying $v_{n+1} = v_n + u_{\varphi(n+1)}$. Recall that $Lu_n \to 0$, so there is an integer $\varphi(0)$ for which $\|Lu_{\varphi(0)}\|_{0,p} \leqslant \varepsilon_0$. Set $v_0 = u_{\varphi(0)}$. In Lemma 4.6, let $v = u^j_{\varphi(0)}$, $\varepsilon = \varepsilon_1$ and $n_0 = \varphi(0)$. This produces an integer $n_1(j)$. Also, there exists an $n_2 \in \mathbb{N}$ such that $k \geqslant n_2 \Rightarrow \|Lu_k\|_{0,p} \leqslant \varepsilon_1$. Set $\varphi(1) = \max\{n_2, n_1(j), 1 \leqslant j \leqslant m\} + 1$ and $v_1 = v_0 + u_{\varphi(1)}$. By induction, suppose (v_n) and $\varphi(n)$ already constructed. Let $v = v_n^j$ and $\varepsilon = \varepsilon_{n+1}$ and $n_0 = \varphi(n)$ in Lemma 4.6. This produces an integer $n_1(j)$, from which the estimates of this lemma hold. Also, there exists an $n_2 \in \mathbb{N}$ such that $k \geqslant n_2 \Rightarrow \|Lu_k\|_{0,p} \leqslant \varepsilon_{n+1}$. Set $\varphi(n+1) = \max\{n_2, n_1(j) : 1 \leqslant j \leqslant m\} + 1$ and $v_{n+1} = v_n + u_{\varphi(n+1)}$. Note that, by construction, $\|Lu_{\varphi(n)}\|_{0,p} \leqslant \varepsilon_n$. Note also that the relation defining v_n shows, by induction, that $v_n \in D_p(\Omega)$.

By Lemma 4.6 (i), we have

$$\|v_k^j\|^p_{2,p} + \|u^j_{\varphi(k+1)}\|^p_{2,p} - \varepsilon_{k+1} \leqslant \|v^j_{k+1}\|^p_{2,p} \leqslant \|v_k^j\|^p_{2,p} + \|u^j_{\varphi(k+1)}\|^p_{2,p} + \varepsilon_{k+1} \quad \forall k \in \mathbb{N}.$$

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Thus, by summation for $n \geqslant 1$,

$$\sum_{k=0}^{n} \|u^j_{\varphi(k)}\|^p_{2,p} - \sum_{k=1}^{n} \varepsilon_k \leqslant \|v^j_n\|^p_{2,p} \leqslant \sum_{k=0}^{n} \|u^j_{\varphi(k)}\|^p_{2,p} + \sum_{k=1}^{n} \varepsilon_k. \tag{4.8}$$

Taking $j = l$ in the above, we get, for $n \geqslant n_0$ (l and n_0 are defined in the beginning of the proof), $(n - n_0)\delta^p \leqslant \|v^l_n\|^p_{2,p}$, and therefore

$$(n - n_0)^{1/p}\delta \leqslant \|v^l_n\|_{2,p} \leqslant \|v_n\|_{2,p} \quad \text{for } n \geqslant n_0. \tag{4.9}$$

Let $M_j \geqslant 1$ be a bound for $\|u^j_n\|_{2,p}$, so that $M = \sum_{j=1}^{m} M_j$ is a bound of $\|u_n\|_{2,p}$. The second inequality of (4.8) yields $\|v^j_n\|_{2,p} \leqslant M_j(n + 1 + \delta^p)^{1/p}$. Therefore,

$$\|v_n\|_{2,p} \leqslant M(n + 1 + \delta^p)^{1/p}. \tag{4.10}$$

From Lemma 4.6 (ii),

$$\|v^j_{n+1}\|_{1,\infty} \leqslant \max\{\|v^j_n\|_{1,\infty}, \|u^j_{\varphi(n+1)}\|_{1,\infty}\} + \varepsilon_{n+1}.$$

Hence, by induction,

$$\|v^j_n\|_{1,\infty} \leqslant \max\{\|u^j_{\varphi(k)}\|_{1,\infty} : 0 \leqslant k \leqslant n\} + \sum_{k=1}^{n} \varepsilon_k,$$

and thus

$$\|v_n\|_{1,\infty} \leqslant m(C + \delta^p), \tag{4.11}$$

where C is a bound for $\|u_n\|_{1,\infty}$.

Next,

$$\|Lv_n\|_{0,p} \leqslant \sum_{k=0}^{n} \|Lu_{\varphi(k)}\|_{0,p} \leqslant \sum_{k=0}^{n} \varepsilon_k.$$

Therefore,

$$\|Lv_n\|_{0,p} \leqslant \delta^p. \tag{4.12}$$

Set $w_n = n^{-1/p}v_n$. By (4.9), $\|w_n\|_{2,p} \geqslant \delta(1 - (n_0/n))^{1/p}$, so that (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$. By (4.10), (w_n) is bounded in $D_p(\Omega)$. By (4.11), $\|w_n\|_{1,\infty} \leqslant \text{const.} \times n^{-1/p}$, whence $w_n \to 0$ in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$. Lastly, by (4.12), $\|Lw_n\|_{0,p} \leqslant \delta^p n^{-1/p}$, which implies that $Lw_n \to 0$ in $Y_p(\Omega)$. That $w_n \rightharpoonup 0$ in $D_p(\Omega)$ follows from its boundedness in $D_p(\Omega)$ and its convergence to 0 in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$ (see Note F2 in the appendix). □

Theorem 4.8. *Let Ω have a C^2 boundary and let L be an elliptic operator as in the preceding lemma. Then the following statements are equivalent.*

(i) $L \in \Phi_+(D_p(\Omega), Y_p(\Omega))$.

(ii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ converging to zero in $C^1_d(\bar{\Omega}; \mathbb{R}^m)$ and such that $Lu_n \to 0$ in $Y_p(\Omega)$ contains a subsequence converging to zero in $D_p(\Omega)$*.*

* Which, in turn, implies that $u_n \to 0$ in $D_p(\Omega)$ (see Note F3 of the appendix).

Proof. (i) $\Rightarrow$ (ii). Recall that a bounded sequence in $D_p(\Omega)$ converging to 0 in $C^1_d(\bar{\Omega};\mathbb{R}^m)$ is weakly convergent to zero in $D_p(\Omega)$ (see Note F2 in the appendix), so the result follows from Note B in the appendix.

(ii) $\Rightarrow$ (i). It suffices to show (by the same note) that if (u_n) is a sequence in $D_p(\Omega)$ such that $u_n \rightharpoonup 0$ in $D_p(\Omega)$ and $Lu_n \to 0$ in $Y_p(\Omega)$, then $u_n \to 0$ in $D_p(\Omega)$. If this is false, then there is a subsequence $(u_{\phi(n)})$ bounded away from zero in $D_p(\Omega)$ (which implies that it contains no subsequence converging to zero). Hence $(u_{\phi(n)})$ satisfies the conditions of Lemma 4.7, and, accordingly, there is a sequence (w_n) having the same properties as $(u_{\phi(n)})$ and, furthermore, converging to zero in $C^1_d(\bar{\Omega};\mathbb{R}^m)$. Then, by assumption, (w_n) contains a subsequence converging to zero in $D_p(\Omega)$. But this is impossible, since (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$. □

Corollary 4.9. *Let Ω have a C^2 boundary. Suppose that every bounded sequence $(u_n) \subset D_p(\Omega)$ converging to zero in $C^1_d(\bar{\Omega};\mathbb{R}^m)$ and such that $F(u_n) \to F(0)$ in $Y_p(\Omega)$ contains a subsequence converging to zero in $D_p(\Omega)$*. (This is so if F is proper on the closed bounded subsets of $D_p(\Omega)$.) Then there exists $\mu \in \boldsymbol{Z} \cup \{-\infty\}$ such that $\mathrm{D}F(u) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$ for all $u \in D_p(\Omega)$.*

Proof. By Theorem 3.9, it suffices to show that $\mathrm{D}F(0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. According to Theorem 4.8 with $L = DF(0)$, it is sufficient to show that if (u_n) is a bounded sequence from $D_p(\Omega)$ converging to zero in $C^1_d(\bar{\Omega};\mathbb{R}^m)$ and $(\mathrm{D}F(0)u_n)$ converges to 0 in $Y_p(\Omega)$, then (u_n) contains a subsequence converging to 0 in $D_p(\Omega)$. By Lemma 4.1, we have $F(u_n) - F(0) - \mathrm{D}F(0)u_n \to 0$ in $Y_p(\Omega)$. But $\mathrm{D}F(0)u_n \to 0$, and therefore $F(u_n) \to F(0)$. Hence, by assumption, (u_n) contains a subsequence converging to 0 in $D_p(\Omega)$. □

5. Operators with asymptotically periodic coefficients

In this section, we consider the case where F has a limit operator with periodic coefficients in a sense precised below. Here, Ω is unbounded and so $K = \complement\Omega$ is bounded according to our assumptions.

When we deal with periodic functions, it is necessary to assume them defined on the whole space $\mathbb{R}^N$. Let $T = (T_1, \dots, T_N) \in \mathbb{R}^N$ with $T_i > 0$. A mapping f defined on $\mathbb{R}^N$ is said to be periodic with period T if $f(x_1, \dots, x_i + T_i, \dots, x_N) = f(x_1, \dots, x_N)\ \forall x \in \mathbb{R}^N$. We use the following notation: for $n \in \boldsymbol{Z}$ and T as above, $nT = (nT_1, \dots, nT_N)$, and for $l \in \boldsymbol{Z}^N$, $lT = (l_1T_1, \dots, l_NT_N)$.

We maintain the previous notation for $r > 0$: B_r is the ball of centre 0 and radius r, $\tilde{B}_r = \{x \in \mathbb{R}^N : |x| > r\}$, $\Omega_r = \Omega \cap B_r$ and $\tilde{\Omega}_r = \Omega \cap \tilde{B}_r$.

Assume that there are two families of matrix-valued functions,

$$a^\infty_{\alpha\beta} : \mathbb{R}^N \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^{m\times m}, \quad 1 \leqslant \alpha, \beta \leqslant N,$$

* Therefore, $u_n \to 0$ in $D_p(\Omega)$ by Note F3 of the appendix.

and

$$c_i^\infty : \mathbb{R}^N \times (\mathbb{R}^m \times \mathbb{R}^{mN}) \to \mathbb{R}^{m\times m}, \quad 0 \leqslant i \leqslant N,$$

both continuous and periodic in x with the same period T, and verifying

$$\lim_{|x|\to\infty} |a_{\alpha\beta}(x,\xi) - a_{\alpha\beta}^\infty(x,\xi)| = 0, \tag{5.1}$$

$$\lim_{|x|\to\infty} \left| \int_0^1 \nabla_{\xi_i} b(x, t\xi)\,\mathrm{d}t - c_i^\infty(x,\xi) \right| = 0, \tag{5.2}$$

the convergence being uniform on the compact subsets of $\mathbb{R}^{m(N+1)}$.

We set

$$b^\infty(x,\xi) = \sum_{i=0}^{N} c_i^\infty(x,\xi)\cdot \xi_i, \tag{5.3}$$

so that $b^\infty(x,0) = 0$.

Note that, by Remark 2.8, $a_{\alpha\beta}^\infty$ and b^∞ are equicontinuous C^0 bundle maps. We define the limit operator F^∞ by

$$F^\infty(u) = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^\infty(\cdot,u,\nabla u)\cdot \partial_{\alpha\beta}^2 u + b^\infty(\cdot,u,\nabla u). \tag{5.4}$$

Observe that, by Lemma 2.16 and Remark 2.17, F^∞ is continuous and weakly continuous from X_p to Y_p, as well as from $X_p(\Omega)$ to $Y_p(\Omega)$, and maps bounded subsets onto bounded ones. Note also that

$$\begin{aligned} F^\infty(v) - F^\infty(0) - \Bigl(-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^\infty(\cdot,0)\cdot\partial_{\alpha\beta}^2 v + \sum_{\alpha=0}^{N} c_\alpha^\infty(\cdot,0)\cdot\partial_\alpha v\Bigr) \\ = -\sum_{\alpha,\beta=1}^{N} (a_{\alpha\beta}^\infty(\cdot,v,\nabla v) - a_{\alpha\beta}^\infty(\cdot,0))\cdot\partial_{\alpha\beta}^2 v + \sum_{\alpha=0}^{N}(c_\alpha^\infty(\cdot,v,\nabla v) - c_\alpha^\infty(\cdot,0))\cdot\partial_\alpha v. \end{aligned}$$

It follows from the equicontinuity of $a_{\alpha\beta}$ and c_α at $\xi = 0$ that F^∞ is differentiable at 0 with derivative

$$\mathrm{D}F^\infty(0)v = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^\infty(\cdot,0)\cdot\partial_{\alpha\beta}^2 v + \sum_{\alpha=0}^{N} c_\alpha^\infty(\cdot,0)\cdot\partial_\alpha v. \tag{5.5}$$

Lemma 5.1. *Let $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$ and $\mathcal{B} \subset X_p(\Omega)$ be a bounded subset. Then, for every $\varepsilon > 0$, there is an $r > 0$ such that, for every $u \in \mathcal{B}$, the following hold.*

(i) $\|F(u) - F(0) - F^\infty(u)\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon$.

(ii) $\|DF(0)u - DF^\infty(0)u\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon$.

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Proof. (i) Since $\mathcal{B}$ is bounded in $X_p(\Omega)$, and therefore in $C^1_d(\bar{\Omega};\mathbb{R}^m)$, there exists a compact set $K \subset \mathbb{R}^{m(N+1)}$ such that $(u(x), \nabla u(x)) \in K$ for every $u \in \mathcal{B}$ and $x \in \Omega$. Since the limit in (5.1) is uniform in $\xi \in K$,

$$|a_{\alpha\beta}(x,\xi) - a^{\infty}_{\alpha\beta}(x,\xi)| \leqslant \varepsilon \quad \forall x \in \tilde{\Omega}_r \quad \forall \xi \in K$$

if r is large enough. Thus

$$\|\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}^{\infty}_{\alpha\beta}(u)\|_{0,\infty,\tilde{\Omega}_r} \leqslant \varepsilon \quad \forall u \in \mathcal{B}. \tag{5.6}$$

A similar argument based on (5.2) yields

$$\left|\int_0^1 \nabla_{\xi_i} b(x,t\xi)\,\mathrm{d}t - c^{\infty}_i(x,\xi)\right| \leqslant \varepsilon \quad \forall x \in \tilde{\Omega}_r \quad \forall \xi \in K$$

if r is large enough, and thus

$$|b(x,\xi) - b(x,0) - b^{\infty}(x,\xi)| \leqslant \varepsilon \sum_{i=0}^{N} |\xi_i|.$$

Therefore, for $k = 1,\dots,m$,

$$\begin{aligned}\|\boldsymbol{b}^k(u) - \boldsymbol{b}^k(0) - \boldsymbol{b}^{\infty,k}(u)\|_{0,p,\tilde{\Omega}_r} &\leqslant \varepsilon \sum_{i=0}^{N} \||\partial_i u|\|_{0,p,\tilde{\Omega}_r} \\ &\leqslant \varepsilon \sum_{i=0}^{N}\sum_{j=1}^{m} \||\partial_i u^j|\|_{0,p,\tilde{\Omega}_r} \\ &\leqslant \varepsilon m(N+1)\|u\|_{2,p,\Omega} \quad \forall u \in \mathcal{B}.\end{aligned}$$

Thus

$$\|\boldsymbol{b}(u) - \boldsymbol{b}(0) - \boldsymbol{b}^{\infty}(u)\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon m^2(N+1)\|u\|_{2,p,\Omega}. \tag{5.7}$$

With (5.6), we get

$$\|F(u) - F(0) - F^{\infty}(u)\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon m^2(N^2+N+1)\|u\|_{2,p,\Omega} \quad \forall u \in \mathcal{B}.$$

Lastly note that ε is arbitrary and $\|u\|_{2,p,\Omega}$ is bounded. Hence the desired result follows.

(ii) The proof for this case is similar. Recall that

$$\mathrm{D}F(0)u - \mathrm{D}F^{\infty}(0)u = -\sum_{\alpha,\beta=1}^{N} (a_{\alpha\beta}(\cdot,0) - a^{\infty}_{\alpha\beta}(\cdot,0))\cdot\partial^2_{\alpha\beta}u + \sum_{\alpha=0}^{N} (\nabla_{\xi_\alpha} b(\cdot,0) - c^{\infty}_{\alpha}(\cdot,0))\cdot\partial_\alpha u.$$

Thus $\|\mathrm{D}F(0)u - \mathrm{D}F^{\infty}(0)u\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon m^2(N^2+N+1)\|u\|_{2,p,\Omega}$. □

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Corollary 5.2. *Let (u_n) be a bounded sequence from $X_p(\Omega)$ such that $u_n \to 0$ in $X_p(\Omega')$ for every bounded open subset $\Omega' \subset \Omega$. Then we have the following.*

(i) $F(u_n) - F(0) - F^\infty(u_n) \to 0$ *in* $Y_p(\Omega)$.

(ii) $(\mathrm{D}F(0) - \mathrm{D}F^\infty(0))u_n \to 0$ *in* $Y_p(\Omega)$.

Proof. (i) Let $\varepsilon >$ be given. Since (u_n) is bounded in $X_p(\Omega)$, it follows from Lemma 5.1 (i) that, for $r > 0$ large enough, we have

$$\|F(u_n) - F(0) - F^\infty(u_n)\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon \quad \forall n \in \mathbb{N}.$$

Recall that $u_n \to 0$ in $X_p(\Omega_r)$ by hypothesis and that F and F^∞ are continuous from $X_p(\Omega_r)$ to $Y_p(\Omega_r)$ by Lemma 2.16 and Remark 2.17. Therefore, $F(u_n) \to F(0)$ and $F^\infty(u_n) \to 0$ in $Y_p(\Omega_r)$, which means that $\|F(u_n) - F(0) - F^\infty(u_n)\|_{0,p,\Omega_r} \leqslant \varepsilon$ for n large enough. Thus $\|F(u_n) - F(0) - F^\infty(u_n)\|_{0,p,\Omega}$ can be made arbitrary small for n large enough.

(ii) The proof for this case is similar. First, by Lemma 5.1 (ii), we have

$$\|\mathrm{D}F(0)u_n - \mathrm{D}F^\infty(0)u_n\|_{0,p,\tilde{\Omega}_r} \leqslant \varepsilon.$$

Next, $\mathrm{D}F(0) \in \mathcal{L}(X_p(\Omega_r), Y_p(\Omega_r))$ by Theorem 2.18 with $\Omega = \Omega_r$ and it is clearly seen from (5.5) that $\mathrm{D}F^\infty(0) \in \mathcal{L}(X_p(\Omega_r), Y_p(\Omega_r))$. Therefore, $(\mathrm{D}F(0) - \mathrm{D}F^\infty(0))u_n \to 0$ in $Y_p(\Omega_r)$. Thus $\|(\mathrm{D}F(0) - \mathrm{D}F^\infty(0))u_n\|_{0,p,\Omega}$ can be made arbitrary small for n large enough. □

Given $h \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \to \mathbb{R}$, we denote by $\tau_h(f) : \mathbb{R}^N \to \mathbb{R}$ the function

$$\tau_h(f)(x) = f(x + h).$$

Corollary 5.3. *Let $\mathcal{B} \subset X_p(\Omega)$ be a bounded subset and $\Omega' \subset \mathbb{R}^N$ be a bounded open subset. Then, for every $\varepsilon > 0$, we have $\|\tau_h(F(u) - F(0)) - \tau_h F^\infty(u)\|_{0,p,\Omega'} \leqslant \varepsilon$ for every $u \in \mathcal{B}$, provided $|h|$ is large enough.*

Proof. Choose $r > 0$ as in Lemma 5.1 and increase it if necessary to have $\tilde{B}_r \subset \Omega$. Then, for $|h|$ large enough, we have $\Omega' + h \subset \tilde{B}_r \subset \Omega$. By the translation invariance of the Lebesgue measure,

$$\|\tau_h(F(u) - F(0)) - \tau_h F^\infty(u)\|_{0,p,\Omega'} = \|F(u) - F(0) - F^\infty(u)\|_{0,p,\Omega'+h}.$$

Now, by Lemma 5.1,

$$\|F(u) - F(0) - F^\infty(u)\|_{0,p,\Omega'+h} \leqslant \|F(u) - F(0) - F^\infty(u)\|_{0,p,\tilde{B}_r} \leqslant \varepsilon$$

for every $u \in \mathcal{B}$. □

Lemma 5.4 (shifted subsequence lemma). *Let $T = (T_1, \dots, T_N)$ with $T_i > 0$. If (u_n) is a bounded sequence from $X_p(\Omega)$, then either*

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(i) (u_n) *vanishes uniformly at infinity in the sense of* $C^1_d(\bar{\Omega})$*; or*

(ii) *there exists a sequence* $(l_n) \subset \boldsymbol{Z}^N$ *with* $\lim_{n\to\infty} |l_n| = \infty$*, a subsequence* $(u_{\phi(n)})$ *and a non-zero element* $\bar{u} \in X_p = W^{2,p}(\mathbb{R}^N; \mathbb{R}^m)$ *such that the sequence* $\tilde{u}_n$*, defined by* $\tilde{u}_n(x) = u_{\phi(n)}(x + l_n T)$*, is weakly convergent to* $\bar{u}$ *in* $X_p(B_k)$ *for all* $k \in \mathbb{N}^*$*.*

Proof. First of all, the definition of $\tilde{u}_n$ makes sense. Indeed, the domain of such a shifted subsequence is $\Omega - l_n T$. Let B_R be a ball containing $K = \complement\Omega$, and $\hat{T} = \min T_i$. Given $k \in \mathbb{N}^*$, there exists a $n_k \in \mathbb{N}^*$ such that $|l_n| > (k + R)/\hat{T}$ for $n \geqslant n_k$. Then, for $x \in B_k$, $|x + l_n T| \geqslant |l_n|\hat{T} - |x| \geqslant |l_n|\hat{T} - k > R$ for all $n \geqslant n_k$, and so $x + l_n T \in \Omega$ for all $x \in B_k$ and $n \geqslant n_k$. Hence B_k is in the domain of definition of $\tilde{u}_n$ for $n \geqslant n_k$. Furthermore, $\tilde{u}_n \in X_p(B_k)$ with

$$\|\tilde{u}_n\|_{2,p,B_k} \leqslant \|u_{\phi(n)}\|_{2,p,\Omega} \leqslant M \tag{5.8}$$

for some constant M and for all $n \geqslant n_k$. Therefore, it makes sense to consider $\lim_{n\to\infty} \tilde{u}_n|_{B_k}$ for any $k \in \mathbb{N}^*$.

Now we go to the proof of the alternative. Let $Q_0 = (0, T_1) \times \cdots \times (0, T_N)$. Suppose that (i) does not hold, so that there exists an $\varepsilon_0 > 0$, a sequence (x_n) such that $|x_n| \geqslant n$ and a subsequence $(u_{\psi(n)})$ such that $|u_{\psi(n)}(x_n)| + |\nabla u_{\psi(n)}(x_n)| \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$. Since $\mathbb{R}^N = \bigcup_{l\in\boldsymbol{Z}^N} (\bar{Q}_0 + lT)$, there exists $z_n \in \boldsymbol{Z}^N$ for which $y_n = x_n - z_n T \in \bar{Q}_0$. Clearly, $\lim_{n\to\infty} |z_n| = \infty$. Define $v_n(x) = u_{\psi(n)}(x + z_n T)$.

According to what has been said at the beginning of the proof, for every $k \in \mathbb{N}^*$, there exists $n_k \in \mathbb{N}^*$ from which $v_n \in X_p(B_k)$, and, furthermore, $(v_n)_{n\geqslant n_k}$ is bounded in $X_p(B_k)$. In particular, $(v_n)_{n\geqslant n_1}$ is bounded in $X_p(B_1)$, and so there is a subsequence $(v_{\theta_1(n)})$ converging weakly to some $\bar{u}_1 \in X_p(B_1)$. But, for $n \geqslant n_2$, $(v_{\theta_1(n)}) \subset (v_n)$ is bounded in $X_p(B_2)$, and again there exists a subsequence $(v_{\theta_2(n)}) \subset (v_{\theta_1(n)})$ converging weakly to some $\bar{u}_2 \in X_p(B_2)$. Clearly, $\bar{u}_2|_{B_1} = \bar{u}_1$.

Continuing the process, we construct a sequence of subsequences $(v_{\theta_k(n)})$, each of which converges weakly to $\bar{u}_k \in X_p(B_k)$, and, furthermore, $\bar{u}_{k+1}$ is an extension of $\bar{u}_k$.

Now we define $(\tilde{u}_n)$ as the diagonal subsequence $(v_{\theta_n(n)})$, i.e.

$$\tilde{u}_n(x) = u_{\psi(\theta_n(n))}(x + z_{\theta_n(n)} T) = u_{\phi(n)}(x + l_n T)$$

if we set $l_n = z_{\theta_n(n)}$ and $\phi(n) = \psi(\theta_n(n))$. On the other hand, we see that there is a function $\bar{u} : \mathbb{R}^N \to \mathbb{R}^m$ naturally defined by $\bar{u}(x) = \bar{u}_k(x)$ if $x \in B_k$.

Since $(\tilde{u}_n)_{n\geqslant n_k}$ is a subsequence of $(v_{\theta_k(n)})$, we have that $(\tilde{u}_n)$ converges weakly to $\bar{u}$ in $X_p(B_k)$ for all $k \in \mathbb{N}^*$. Therefore, according to (5.8),

$$\|\bar{u}\|_{2,p,B_k} \leqslant \liminf_{n\to\infty} \|\tilde{u}_n\|_{2,p,B_k} \leqslant M$$

for all $k \in \mathbb{N}^*$, and so $\bar{u} \in X_p$.

It remains to show that $\bar{u} \neq 0$. Choose $k \in \mathbb{N}^*$ such that $\bar{Q}_0 \subset B_k$. By the compactness of the embedding $W^{2,p}(Q_0; \mathbb{R}^m) \hookrightarrow C^1(\bar{Q}_0; \mathbb{R}^m)$, we have that $\tilde{u}_n \to \bar{u}$ in $C^1(\bar{Q}_0; \mathbb{R}^m)$, and hence $\|\tilde{u}_n\|_{1,\infty,Q_0} \to \|\bar{u}\|_{1,\infty,Q_0}$. But

$$\begin{aligned}\|\tilde{u}_n\|_{1,\infty,Q_0} &\geqslant \tfrac{1}{2}(|\tilde{u}_n(y_{\theta_n(n)})| + |\nabla\tilde{u}_n(y_{\theta_n(n)})|)\\ &= \tfrac{1}{2}(|u_{\psi(\theta_n(n))}(x_{\theta_n(n)})| + |\nabla u_{\psi(\theta_n(n))}(x_{\theta_n(n)})|)\\ &\geqslant \tfrac{1}{2}\varepsilon_0.\end{aligned}$$

Therefore, $\|\bar{u}\|_{1,\infty,Q_0} \geqslant \frac{1}{2}\varepsilon_0$, whence $\bar{u} \neq 0$. □

Theorem 5.5. *Let Ω have a C^2 boundary. Suppose that the following hold.*

(i) *There exists $u^0 \in D_p(\Omega)$ for which $\mathrm{D}F(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$.*

(ii) $\{u \in X_p;\ F^\infty(u) = 0\} = \{0\}$.

Then F is proper on the closed bounded subsets of $D_p(\Omega)$.

Proof. By Corollary 4.5, it suffices to show that if (u_n) is a bounded sequence from $D_p(\Omega)$ and $(F(u_n))$ converges to some y in $Y_p(\Omega)$, then (u_n) vanishes uniformly at infinity in the sense of $C^1_d(\bar{\Omega})$. After replacing F by $F - F(0)$ and y by $y - F(0)$, we can assume that $F(0) = 0$. Let us show that case (ii) of Lemma 5.4 cannot occur. By contradiction, suppose there is a sequence $(l_n) \subset \boldsymbol{Z}^N$ with $\lim_{n\to\infty} |l_n| = \infty$ and a subsequence $(u_{\phi(n)})$ such that the sequence $(\tilde{u}_n)$ defined by $\tilde{u}_n(x) = u_{\phi(n)}(x + l_nT)$ has a non-zero weak limit $\bar{u}$ in $X_p(B_k)$.

It is enough to show that $F^\infty(\bar{u}) = 0$. Let $\tilde{y}_n$ be defined by $\tilde{y}_n(x) = y(x + l_nT) = \tau_{l_nT}(y)(x)$. According to the proof of Lemma 5.4, for all $k \in \mathbb{N}^*$, $\tilde{y}_n \in X_p(B_k)$ from a certain rank n_k, and it is bounded by a constant independent of k.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$, and choose $k \in \mathbb{N}^*$ such that B_k contains the support of ψ. Recall that $\tilde{u}_n \rightharpoonup \bar{u}$ in $X_p(B_k)$ and $F^\infty : X_p(B_k) \to Y_p(B_k)$ is weakly continuous, and so $F(\tilde{u}_n) \rightharpoonup F(\bar{u})$ in $Y_p(B_k)$. Thus

$$\begin{aligned}\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x &= \int_{B_k} \psi F^\infty(\bar{u})\,\mathrm{d}x\\ &= \lim_{n\to\infty} \int_{B_k} \psi F^\infty(\tilde{u}_n)\,\mathrm{d}x\\ &= \lim_{n\to\infty} \int_{B_k} \psi \tau_{l_nT} F^\infty(u_{\phi(n)})\,\mathrm{d}x.\end{aligned} \tag{5.9}$$

On the other hand, according to Corollary 5.3 (we assumed that $F(0) = 0$),

$$\tau_{l_nT}F(u_{\phi(n)}) - \tau_{l_nT}F^\infty(u_{\phi(n)}) \xrightarrow[n\to\infty]{} 0 \quad \text{in } Y_p(B_k). \tag{5.10}$$

For $n \geqslant n_k$ and $j = 1, \dots, m$, we have

$$\begin{aligned}\|(\tau_{l_nT}F(u_{\phi(n)}) - \tilde{y}_n)_j\|^p_{0,p,B_k} &= \int_{B_k} |\tau_{l_nT}F_j(u_{\phi(n)})(x) - \tau_{l_nT}y_j(x)|^p\,\mathrm{d}x\\ &= \int_{|z-l_nT|<k} |F_j(u_{\phi(n)})(z) - y_j(z)|^p\,\mathrm{d}z\\ &\leqslant \|F_j(u_{\phi(n)}) - y_j\|^p_{0,p,\Omega} \xrightarrow[n\to\infty]{} 0 \quad \text{by assumption.}\end{aligned} \tag{5.11}$$

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Equations (5.9)–(5.11) together give

$$\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x = \lim_{n\to\infty} \int_{B_k} \psi \tilde{y}_n \,\mathrm{d}x.$$

But, for each component $(\tilde{y}_n)_j$, $j = 1,\dots,m$, we have

$$\begin{aligned}
\int_{B_k} \psi(\tilde{y}_n)_j \,\mathrm{d}x &= \int_{B_k} \psi(x) y_j(x + l_n T)\,\mathrm{d}x \\
&= \int_{B(l_n T,k)} \psi(z - l_n T) y_j(z)\,\mathrm{d}z \\
&\leqslant \biggl(\int_{B(l_n T,k)} |y_j|^p\biggr)^{1/p} \biggl(\int_{B(l_n T,k)} |\psi(z - l_n T)|^q \,\mathrm{d}z\biggr)^{1/q} \quad \biggl(\frac{1}{p} + \frac{1}{q} = 1\biggr) \\
&\leqslant \biggl(\int_{B(l_n T,k)} |y_j|^p\biggr)^{1/p} \biggl(\int_{\mathbb{R}^N} |\psi|^q\biggr)^{1/q} \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

since $B(l_n T, k) \subset \tilde{B}_{|l_n T|-k}$, so that the result follows from Note G3 in the appendix.

Thus, finally,

$$\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x = 0 \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^N),$$

and consequently $F^\infty(\bar{u}) = 0$. □

The last theorem shows that, together, the semi-Fredholmness of F and the non-existence of non-trivial solutions of the limit problem $F^\infty(u) = 0$ are sufficient conditions for the properness of F on $D_p(\Omega)$. Are they also necessary conditions? We already know by Corollary 4.9 that the semi-Fredholmness is necessary. It turns out that, in the case of $\Omega = \mathbb{R}^N$, the second condition is also necessary.

Accordingly, assume in the sequel that conditions (2.9)–(2.12) and (3.1) are satisfied on the whole space $\mathbb{R}^N$. Consequently, the results obtained so far are true and will be applied with $\Omega = \mathbb{R}^N$.

Lemma 5.6. *Let $1 < q < \infty$ and $k \in \mathbb{N}$. Given $u \in W^{k,p}(\mathbb{R}^N;\mathbb{R}^m)$ and a sequence $(h_n) \subset \mathbb{R}^N$ such that $\lim_{n\to\infty} |h_n| = \infty$, set $\tilde{u}_n(x) = u(x + h_n)$. Then $\tilde{u}_n \to 0$ in $W^{k,p}(\Omega';\mathbb{R}^m)$ for every bounded open subset $\Omega' \subset \mathbb{R}^N$. In particular, $\tilde{u}_n \rightharpoonup 0$ in $W^{k,p}(\mathbb{R}^N;\mathbb{R}^m)$.*

Proof. For $m = 1$, this is Lemma 4.8 of [**8**]. The conclusion is then clear, since convergence, respectively weak convergence in $W^{k,p}(\mathbb{R}^N;\mathbb{R}^m)$, is equivalent to this type of convergence of each component in $W^{k,p}(\mathbb{R}^N)$. □

Theorem 5.7. *The following statements are equivalent.*

(i) *F is proper on the closed bounded subsets of X_p.*

(ii) *Every sequence $(u_n) \subset X_p$ such that $u_n \rightharpoonup 0$ in X_p and $F(u_n) \to F(0)$ in Y_p contains a subsequence converging in X_p*.*

* Which implies that $u_n \to 0$ in X_p.

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(iii) *There exists $u^0 \in X_p$ for which $DF(u^0) \in \Phi_+(X_p, Y_p)$ and the equation $F^\infty(u) = 0$ has no non-zero solution in X_p.*

Proof. (i) ⇒ (ii) is evident since a weakly convergent sequence is bounded. (iii) ⇒ (i) is Theorem 5.5 (with $\Omega = \mathbb{R}^N$). It remains to prove (ii) ⇒ (iii). That $\mathrm{D}F(0) : X_p \to Y_p$ is semi-Fredholm follows from Corollary 4.9. Consider now an element $u \in X_p$ such that $F^\infty(u) = 0$. Set $u_n(x) = u(x + nT)$ so that $F^\infty(u_n) = 0$ by the periodicity of the coefficients of F^∞. By Lemma 5.6, $u_n \to 0$ in $X_p(\Omega')$ for every open bounded subset $\Omega' \subset \mathbb{R}^N$, and hence, by Corollary 5.2 (i), $F(u_n) - F(0) \to 0$ in Y_p. Also, $u_n \rightharpoonup 0$ in X_p, and therefore, by hypotheses, (u_n) contains a convergent subsequence $(u_{\phi(n)})$. Its limit is necessarily 0 (by the uniqueness of the weak limit in X_p). But recall that $\|u_n\|_{2,p,\mathbb{R}^N} = \|u\|_{2,p,\mathbb{R}^N}$. Hence $u = 0$. □

Remark 5.8. Note that strict ellipticity on the compact subsets of $\mathbb{R}^N$ and strict ellipticity on $\mathbb{R}^N$ are equivalent for F^∞. This is due to the periodicity of $a^\infty_{\alpha\beta}$.

On the other hand, note that $\det(\sum_{\alpha,\beta=1}^N (\eta_\alpha \eta_\beta a_{\alpha\beta}(x, 0)))$ is a homogeneous polynomial of order $2m$ in η, so it could be written as* $P(x, \eta) = \sum_{|\gamma|=2m}^N p_\gamma(x)\eta^\gamma$, and the coefficients are an algebraic combination of the components of the matrices $a_{\alpha\beta}$. Similarly,

$$\det\biggl(\sum_{\alpha,\beta=1}^N (\eta_\alpha \eta_\beta a^\infty_{\alpha\beta}(x, 0))\biggr) = P^\infty(x, \eta) = \sum_{|\gamma|=2m}^N p^\infty_\gamma(x)\eta^\gamma.$$

Thus, from (5.1), it follows that, given $\varepsilon > 0$, $|p_\gamma(x) - p^\infty_\gamma(x)| \leqslant \varepsilon$ for $|x|$ large enough. Therefore, $|P(x, \eta) - P^\infty(x, \eta)| \leqslant N^{2m}\varepsilon|\eta|^{2m}$, and thus, if $P(x, \eta) \geqslant \lambda|\eta|^{2m}$, we have $P^\infty(x, \eta) \geqslant (\lambda - N^{2m}\varepsilon)|\eta|^{2m}$. Therefore, the strict ellipticity condition (in $\mathbb{R}^N$) for $a_{\alpha\beta}(\cdot, 0)$ is equivalent to the strict ellipticity of $a^\infty_{\alpha\beta}(\cdot, 0)$.

Note that this reasoning also proves the stability of the ellipticity condition, i.e. an elliptic system remains elliptic after a small enough perturbation of its leading coefficients.

Appendix A. Some results about sequences and the function spaces used in the paper

In what follows, X, Y and Z are real Banach spaces, $\mathcal{L}(X, Y)$ is the Banach space of all linear and bounded operators from X to Y, $X' = \mathcal{L}(X, \mathbb{R})$ is the dual of X, $X \hookrightarrow Y$ means that X is imbedded in Y and $X \hookrightarrow_{\mathrm{comp}} Y$ means that the imbedding is compact.

Note A. $L \in \mathcal{L}(X, Y)$ is semi-Fredholm if $\operatorname{rge} L$ is closed and at least one among $\dim \operatorname{Ker} L$ and $\operatorname{codim} \operatorname{rge} L$ is finite. The index of L is $\mu = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{rge} L \in \boldsymbol{Z} \cup \{\pm\infty\}$. We denote by $\Phi_\mu(X, Y)$ the set of semi-Fredholm operators of index μ, and

$$\Phi_+(X, Y) = \bigcup_{\mu \in \boldsymbol{Z} \cup \{-\infty\}} \Phi_\mu(X, Y).$$

When $F \in C^1(X, Y)$ is not necessarily linear, it is semi-Fredholm if, for every $u \in X$, $\mathrm{D}F(u) \in \mathcal{L}(X, Y)$ is semi-Fredholm. In this paper, we used three properties of semi-Fredholm operators.

* $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{N}^N$ is a multi-index.

(i) The index of semi-Fredholm operators is a locally constant function. This property is due to the fact that $\Phi_\mu(X,Y)$ is open in $\mathcal{L}(X,Y)$.

(ii) If $F \in \Phi_\mu(X,Y)$ and $K \in \mathcal{L}(X,Y)$ is compact, then $F + K \in \Phi_\mu(X,Y)$ (stability under a compact perturbation).

(iii) Yood's criterion, which states that when L is linear, L is proper on the closed bounded subsets of X if and only if $L \in \Phi_+(X,Y)$.

For more details, see [**8**, p. 141] and the references therein.

Note B. Let $L : X \to Y$ be a continuous operator. L is proper on the closed bounded subsets of X if, for every compact K from Y and every closed bounded subset B of X, we have that $L^{-1}(K) \cap B$ is compact. This condition is clearly equivalent to the following.

(i) Every bounded sequence (u_n) from X such that $(L(u_n))$ converges contains a convergent subsequence.

When X is reflexive*, properness on the closed bounded subsets is equivalent to the following.

(ii) For every sequence $(u_n) \subset X$ such that $u_n \rightharpoonup u$ and $L(u_n)$ converges, we have $u_n \to u$.

Proof. (ii) $\Rightarrow$ (i). Let (u_n) be bounded and (Lu_n) convergent. Since X is reflexive, there exists a subsequence $(u_{\varphi(n)})$† converging weakly to some u, but $(Lu_{\varphi(n)})$ also converges. Therefore, $u_{\varphi(n)} \to u$.

(i) $\Rightarrow$ (ii). Let $u_n \rightharpoonup u$ and $L(u_n)$ converges. If (u_n) does not converge to u, there exist $\varepsilon_0 > 0$ and a subsequence $(u_{\varphi(n)})$ such that $\|u_{\varphi(n)} - u\| \geqslant \varepsilon_0$. But $(u_{\varphi(n)})$ has the same properties of (u_n). Therefore, it contains a convergent subsequence (to u, by uniqueness of the weak limit). But this contradicts the above inequality. □

Note C. Let $L : X \to Y$ be an operator (not necessary linear or continuous). L is compact if it transforms bounded subsets onto relatively compact ones (i.e. with compact closure). L is completely continuous if it transforms weakly convergent sequences into strongly convergent ones. We used the fact that when X is reflexive and L is linear and bounded, complete continuity and compactness for L are equivalent. More precisely, one can prove the following.

(1) If X is reflexive and L is completely continuous, then L is compact.

(2) If L is weakly continuous and compact, then it is completely continuous (hence continuous). One can argue by contradiction.

* The reflexivity condition is used in this work to ensure that every bounded sequence from X contains a weakly convergent subsequence.

† φ and ψ are strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$.

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Note D. Let $\Omega \subset \mathbb{R}^N$ be an open set, with $\partial\Omega$ bounded and Lipschitz and $N < p < \infty$. Then the functions of $W^{1,p}(\Omega)$ are bounded and Hölder continuous on $\bar{\Omega}$ (see [**1**, Theorem 5.4]). Now let $u \in W^{1,p}(\Omega)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\Omega)$ (see [**1**, Theorem 3.18]), there exists a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ converging to u in $W^{1,p}(\Omega)$, and hence in $L^\infty(\Omega)$. Accordingly, for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\|u - u_{n_0}\|_{0,\infty,\Omega} \leqslant \varepsilon$, and therefore, $\forall x \in \Omega$, $|u(x)| \leqslant |u_{n_0}(x)| + \varepsilon$. But, for $|x|$ large enough $u_{n_0}(x) = 0$, we have $|u(x)| \leqslant \varepsilon$, which means that $\lim_{x\in\Omega,\,|x|\to\infty} |u(x)| = 0$.

From the preceding, we deduce that $W^{2,p}(\Omega) \hookrightarrow C_d^1(\bar{\Omega})$. The imbeddings are compact if, in addition, Ω is bounded.

Note E. Let $N < p < \infty$ and $u \in L^p(\Omega)$. Define the operator T by $Tv := uv$. Then $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

Proof. $Tv \in L^p(\Omega)$ since $v \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Define ϕ_n for $n \in \mathbb{N}$ by $\phi_n(x) = 1$ if $|x| \leqslant n$ and zero elsewhere. Let $T_n v = \phi_n uv$ and $\Omega_n = \{x \in \Omega : |x| < n\}$. Since $W^{1,p}(\Omega_n) \hookrightarrow_{\text{comp}} L^\infty(\Omega_n)$, then $T_n : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Now

$$\|Tv - T_n v\|_p^p = \int_{|x|>n} |u|^p|v|^p \leqslant \|v\|_\infty^p \int_{|x|>n} |u|^p \leqslant \text{const.} \times \|v\|_{1,p}^p \int_{|x|>n} |u|^p.$$

This means that T is the uniform limit of a sequence of compact operators, and so it is itself compact (see also Note G3). □

Note F1. Let X be reflexive, $u \in X$ and (u_n) be a bounded sequence from X. Suppose that every weakly convergent subsequence of (u_n) converges weakly to u (i.e. the limit is independent of the subsequence, or (u_n) has a unique weak cluster point). Then $u_n \rightharpoonup u$.

Proof. If not, there exist $f \in X'$, $\varepsilon_0 > 0$ and a subsequence $(u_{\varphi(n)})$ such that $|\langle f, u_{\varphi(n)}\rangle - \langle f, u\rangle| \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$. But $(u_{\varphi(n)})$ is bounded, and therefore it contains a subsequence $(u_{\varphi(\psi(n))})$ converging weakly to some l. By assumption, $l = u$, so that $u_{\varphi(\psi(n))} \rightharpoonup u$. But this contradicts the above inequality. □

Note F2. Let X be reflexive, $X \hookrightarrow Y$, $u \in X$ and (u_n) be a bounded sequence in X such that $u_n \to u$ in Y. Then $u_n \rightharpoonup u$ in X.

Proof. If not, there exist $f \in X'$, $\varepsilon_0 > 0$ and a subsequence $(u_{\varphi(n)})$ such that, $\forall n \in \mathbb{N}$, $|\langle f, u_{\varphi(n)}\rangle - \langle f, u\rangle| \geqslant \varepsilon_0$. But $(u_{\varphi(n)})$ is bounded, so it contains a subsequence $(u_{\varphi(\psi(n))})$ converging weakly to some v in X (and hence in Y). By the uniqueness of the weak limit in Y, we have $v = u$. Therefore, $u_{\varphi(\psi(n))} \rightharpoonup u$ in X. But this contradicts the definition of $(u_{\varphi(n)})$. □

Application. $X = W^{2,p}(\Omega; \mathbb{R}^m)$, $Y = C_d^1(\bar{\Omega}; \mathbb{R}^m)$, with $N < p < \infty$ and $\partial\Omega$ bounded and Lipschitz.

Note F3. Let $L : X \to Z$ have the following property: if (u_n) is bounded in X, $u_n \to 0$ in Y and $L(u_n)$ converges in Z, then (u_n) contains a subsequence converging to zero in X. Then, for (u_n) as above, we have, in fact, $u_n \to 0$ in X.

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Proof. If not, there exists a subsequence, bounded away from zero in X. But this subsequence has all the properties of (u_n), so, by hypotheses, it contains a subsequence converging to zero in X, which contradicts its definition. □

Note G1. Let $A \subset \mathbb{R}^N$ be a measurable set and $1 \leqslant q \leqslant \infty$. Let $u_n \to u$ in $L^\infty(A)$ and $v_n \to v$ in $L^q(A)$. Then $u_n v_n \to uv$ in $L^q(A)$.

Proof. This is because $\|u_n v_n - uv\|_q \leqslant \|u_n - u\|_\infty \|v_n\|_q + \|u\|_\infty \|v_n - v\|_q$. □

Note G2. Let $u_n \to u$ in $L^\infty(A)$ and $v_n \rightharpoonup v$ in $L^q(A)$ for $1 \leqslant q < \infty$. Then we have that $u_n v_n \rightharpoonup uv$ in $L^q(A)$.

Proof. Let $f \in L^{q'}(A)$. Then

$$\int_A f(u_n v_n - uv) = \int_A f(u_n v_n - uv_n) + \int_A fu(v_n - v).$$

The result follows from the fact that $fu \in L^{q'}(A)$ and

$$\left| \int_A f(u_n v_n - uv_n) \right| \leqslant \|f\|_{q'} \|v_n\|_q \|u_n - u\|_\infty.$$

□

Note G3. Let $f \in L^1(A)$. Then the functional μ defined on the measurable subsets of A by $\mu(G) = \int_G |f|$ is a measure on A, and, as any measure, it satisfies $\lim_{n\to\infty} \mu(G_n) = \mu(\bigcap G_n)$ for every decreasing family of subsets (G_n) from A. This is why

$$\lim_{n\to\infty} \int_{|x|>n} |f| = \int_{\bigcap \tilde{B}_n} |f| = 0,$$

since $\bigcap \tilde{B}_n = \varnothing$.

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