ON THE EIGENVECTOR BELONGING TO THE MAXIMAL ROOT OF A NON-NEGATIVE MATRIX

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1. By a theorem of Perron, a non-negative irreducible $(n \times n)$ matrix $A = (a_{\mu\nu})$ has a *positive* fundamental root σ, the "maximal root of A", such that the moduli of all other eigenvalues of A do not exceed σ. If we put

$$R^{(\mu)} = \sum_{\nu=1}^{n} a_{\mu\nu},\ R = \max_{\mu} R^{(\mu)};\ r = \min_{\mu} R^{(\mu)}, \quad \text{.................(1)}$$

σ lies between R and r. Since σ is not changed if A is transformed by a positive diagonal matrix $D(p_1, \ldots, p_n)$, σ lies also between the expressions

$$R_p = \max_{\mu} \sum_{\nu=1}^{n} a_{\mu\nu} p_\nu / p_\mu;\ r_p = \min_{\mu} \sum_{\nu=1}^{n} a_{\mu\nu} p_\nu / p_\mu. \quad \text{.................(2)}$$

By a theorem of Frobenius, to σ as an eigenvalue of A belongs a *positive* eigenvector $\xi = (x_1, \ldots, x_n)$, satisfying

$$\sigma x_\mu = \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu (\mu = 1, \ldots, n). \quad \text{..........................(3)}$$

For certain purposes a good estimate of the quotient

$$\gamma = \frac{\max_\nu x_\nu}{\min_\nu x_\nu} \quad \text{......................................(4)}$$

is required. In what follows we shall improve the estimates which we gave for γ in 1952 for *positive* matrices (3) † and also an upper estimate of H. Schneider (4) for non-negative matrices.‡ Our results are contained in (10), (11), (13), (14), (22), (23) and (29)-(33).

2. After making a permissible cogredient permutation of indices and a convenient norming of ξ, we can assume that

$$1 = x_1 \geqq x_2 \ldots \geqq x_n > 0. \quad \text{...............................(5)}$$

Choosing another norming we obtain an eigenvector $\eta = (y_1, \ldots, y_n)$ for which

$$y_1 + \ldots + y_n = 1;\ y_1 \geqq y_2 \geqq \ldots \geqq y_n > 0. \quad \text{.......................(6)}$$

† Refining a method used previously by Ledermann (2) in order to obtain an estimate of σ. This estimate was improved by A. Brauer (1), who did not, however, discuss γ.

‡ This estimate, however, was the first solution to the problem of estimating γ for non-negative matrices.

We have obviously

$$\gamma = \frac{1}{x_n} = \frac{y_1}{y_n}. \qquad \text{.......(7)}$$

We denote by κ_1, κ_2, m, M respectively the smallest $a_{\mu\mu}$, the smallest $a_{\mu\nu}(\mu \neq \nu)$, the smallest $a_{\mu\nu}(\mu, \nu = 1, \ldots, n)$ and the greatest $a_{\mu\nu}(\mu, \nu = 1, \ldots, n)$, i.e.

$$\kappa_1 = \min_{\mu} a_{\mu\mu},\ \kappa_2 = \min a_{\mu\nu}(\mu \neq \nu),\ m = \min_{\mu,\nu} a_{\mu\nu} = \min(\kappa_1, \kappa_2),\ M = \max_{\mu,\nu} a_{\mu\nu}. \qquad \text{.......(8)}$$

In the case of a non-negative matrix we denote by κ the smallest *positive* $a_{\mu\nu}(\mu \neq \nu)$ and by $\kappa^{(\mu)}$ the smallest *positive* $a_{\mu\nu}(\nu \neq \mu)$ in the μth row. $\kappa^{(\mu)}$ exists as A is irreducible.

3. Assuming that A is *positive* we have from

$$\sigma y_\mu = \sum_{\nu=1}^{n} a_{\mu\nu} y_\nu \quad (\mu = 1, \ldots, n) \qquad \text{.......(9)}$$

and from (6), that $M \geqq \sigma y_\mu \geqq m$, whence, since $\gamma = \dfrac{\sigma y_1}{\sigma y_n}$,

$$\gamma \leqq \frac{M}{m}, \qquad \text{.......(10)}$$

a particularly simple estimate for *positive* matrices.

This result can be improved to a certain extent and generalised to the case of non-negative matrices with $a_{\mu\nu} > 0$ $(\mu \neq \nu)$ if we use the values of κ_1 and κ_2. We prove in this case, if $\kappa_1 \leqq \kappa_2$, that

$$\gamma \leqq \frac{M + \kappa_2 - \kappa_1}{\kappa_2} \quad (\kappa_1 \leqq \kappa_2), \qquad \text{.......(11)}$$

which is a better estimate than (10) and also holds for $\kappa_1 = 0$.

A still more precise result is obtained if we introduce

$$M_1 = \max_{\mu} a_{\mu\mu}, \quad M_2 = \max_{\mu \neq \nu} a_{\mu\nu}. \qquad \text{.......(12)}$$

Then we prove that

$$\gamma \leqq \frac{\max(M_1 + \kappa_2 - \kappa_1, M_2)}{\kappa_2}. \qquad \text{.......(13)}$$

(11) is obviously contained in (13) and it suffices to prove (13).

If we add to each $a_{\mu\mu}$ the quantity $\kappa_2 - \kappa_1$, then a new matrix B is obtained, for which σ is replaced by $\sigma + \kappa_2 - \kappa_1$, but for which all y_ν in (9) remain unchanged; therefore γ is unchanged too. On the other hand, for the matrix B, κ_1 and M_1 become κ_2 and $M_1 + \kappa_2 - \kappa_1$ respectively, while κ_2 and M_2 remain unchanged. Further, m becomes κ_2, and M becomes $\max(M_1 + \kappa_2 - \kappa_1, M_2)$. (13) then follows immediately from (10) applied to B.

4. We sometimes obtain a more precise result if we use (12) directly in (9). Then we have

$$\sigma y_\mu \leqq M_1 y_\mu + M_2(1-y_\mu), \quad (\sigma + M_2 - M_1) y_\mu \leqq M_2$$

and

$$\sigma y_\mu \geqq \kappa_1 y_\mu + \kappa_2(1-y_\mu), \quad (\sigma + \kappa_2 - \kappa_1) y_\mu \geqq \kappa_2$$

and therefore

$$\gamma \leqq \frac{M_2}{\kappa_2} \frac{\sigma + \kappa_2 - \kappa_1}{\sigma + M_2 - M_1}. \qquad \text{(14)}$$

Here, the second fraction is increasing or decreasing with σ according as $(M_2 - M_1) - (\kappa_2 - \kappa_1)$ is positive or negative. In the first case σ can be replaced by any *upper bound* as, for instance, R_p. In the second case we have to replace σ by a lower bound such as r_p.

5. We consider now an irreducible *non-negative* matrix A for which κ_2 could also be zero. We have from (3) that, for an $a_{\mu\nu} \neq 0$, $\mu \neq \nu$,

$$(\sigma - a_{\mu\mu}) x_\mu \geqq a_{\mu\nu} x_\nu \geqq \kappa^{(\mu)} x_\nu \quad (a_{\mu\nu} \neq 0, \ \mu \neq \nu). \qquad \text{(15)}$$

We now assert that there exists a sequence of indices, $n = \mu_0, \mu_1, \mu_2, \ldots, \mu_{k-1}$, such that $n \neq \mu_1 \neq \mu_2 \neq \ldots \neq \mu_{k-1} \neq 1$ and

$$a_{n\mu_1} a_{\mu_1\mu_2} \ldots a_{\mu_{k-1}1} \neq 0. \qquad \text{(16)}$$

Indeed, call an index τ *connected* with n if either $\tau = n$ or there exists a "chain" $a_{n\mu_1}, a_{\mu_1\mu_2}, \ldots, a_{\mu_s\tau}$ of *positive* elements of A, and consider the set of all different indices $\tau_1 = n, \tau_2, \ldots, \tau_s$ connected with n. If 1 is not connected with n, then denote the non-empty set of indices among $1, 2, \ldots, n-1$ not connected with n, by $\lambda_1 = 1, \lambda_2, \ldots, \lambda_{n-s}$. Then we must have

$$a_{\mu\nu} = 0 \begin{pmatrix} \mu = \tau_1, \ldots, \tau_s \\ \nu = \lambda_1, \ldots, \lambda_{n-s} \end{pmatrix}$$

and A is not irreducible.

We can therefore assume (16) and have from (15), for $\mu = \mu_0 = n, \mu_1, \ldots, \mu_{k-1} \neq 1$, that

$$x_{\mu_{t+1}} / x_{\mu_t} \leqq \frac{\sigma - a_{\mu_t\mu_t}}{\kappa^{(\mu_t)}} \quad (t = 0, \ldots, k-1). \qquad \text{(17)}$$

If we put now

$$\phi_\mu = \frac{\sigma - a_{\mu\mu}}{\kappa^{(\mu)}}, \qquad \text{(18)}$$

we have from (17) that

$$x_{\mu_{t+1}} / x_{\mu_t} \leqq \phi_{\mu_t} (t = 0, 1, \ldots, k-1). \qquad \text{(19)}$$

6. Multiplying all inequalities in (19) and using (5), we obtain

$$\frac{1}{x_n} \leqq \prod_{t=0}^{k-1} \phi_{\mu_t}. \qquad \text{(20)}$$

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Here we have $k \leqq n-1$ and all μ_t are distinct. We therefore obtain from (7) the result:

If the expressions $\dfrac{\sigma - a_{\mu\mu}}{\kappa^{(\mu)}}$ *in decreasing order are denoted by*

$$\psi_1 \geqq \psi_2 \geqq \dots \geqq \psi_n, \qquad \text{(21)}$$

then

$$\gamma \leqq \psi_1 \prod_{\nu=2}^{n-1} \max(1, \psi_\nu). \qquad \text{(22)}$$

Introducing here κ_1 and κ we obtain the bound found by H. Schneider (4), namely,

$$\gamma \leqq \left(\frac{\sigma - \kappa_1}{\kappa}\right)^{n-1}, \qquad \text{(23)}$$

since, as is easy to see from (15) for $\mu = 1$, $\dfrac{\sigma - \kappa_1}{\kappa} \geqq 1$.

7. We show now by a fairly general example that in (22) and (23) the equality sign can certainly occur. Consider, for a sequence of positive numbers

$$\kappa^{(2)} < 1, \ \kappa^{(3)} < 1, \ \dots, \ \kappa^{(n)} < 1, \ \kappa^{(1)} = \frac{1}{\kappa^{(2)} \dots \kappa^{(n)}} \qquad \text{(24)}$$

and a $\kappa_1 > 0$, the non-negative irreducible matrix

$$\begin{pmatrix} \kappa_1 & 0 & 0 & . & . & . & . & . & \kappa^{(1)} \\ \kappa^{(2)} & \kappa_1 & 0 & . & . & . & . & . & 0 \\ 0 & \kappa^{(3)} & \kappa_1 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & & \kappa^{(n)} & \kappa_1 \end{pmatrix}, \qquad \text{(25)}$$

then equations (3) become

$$\begin{aligned} x_1(\sigma - \kappa_1) &= \kappa^{(1)} x_n \ , \\ x_2(\sigma - \kappa_1) &= \kappa^{(2)} x_1 \ , \\ . \ . \ . \ . & \ . \ . \ . \ . \\ x_n(\sigma - \kappa_1) &= \kappa^{(n)} x_{n-1}. \end{aligned} \qquad \text{(26)}$$

Since all x_ν are positive, (24) gives $(\sigma - \kappa_1)^n = \kappa^{(1)} \kappa^{(2)} \dots \kappa^{(n)} = 1$,

$$\sigma - \kappa_1 = 1, \ \sigma = \kappa_1 + 1, \ \phi_\mu = \frac{1}{\kappa^{(\mu)}} \qquad \text{(27)}$$

and therefore by (24) and (4)

$$\gamma = \frac{x_1}{x_n} = \kappa^{(1)} = \frac{1}{\kappa^{(2)} \kappa^{(3)} \dots \kappa^{(n)}} = \prod_{\nu=1}^{n-1} \psi_\nu. \qquad \text{(28)}$$

If we here take $\kappa^{(2)} = \kappa^{(3)} = \ldots = \kappa^{(n)} = \kappa$, then $\gamma = \kappa^{-(n-1)}$ and by (27) the equality sign holds in (23).

8. We now choose μ in (3) such that $R = R^{(\mu)}$. Then by (5)

$$x_\mu(\sigma - a_{\mu\mu}) \geqq a_{\mu 1} + (R - a_{\mu\mu} - a_{\mu 1})x_n = (R - a_{\mu\mu})x_n + (1 - x_n)a_{\mu 1},$$

where for $\mu = 1$ we write 0 for $a_{\mu 1}$. In any case it follows that

$$x_\mu(\sigma - a_{\mu\mu}) \geqq (R - a_{\mu\mu})x_n, \quad \sigma - a_{\mu\mu} \geqq (R - a_{\mu\mu})x_n, \quad x_n \leqq \frac{\sigma - a_{\mu\mu}}{R - a_{\mu\mu}}.$$

As $\sigma/R \leqq 1$, we do not decrease the right-hand bound by replacing $a_{\mu\mu}$ by κ_1 and so

$$x_n \leqq \frac{\sigma - \kappa_1}{R - \kappa_1}, \quad \gamma \geqq \frac{R - \kappa_1}{\sigma - \kappa_1}. \qquad \text{(29)}$$

Here σ can be replaced by any upper bound, for instance an R_p.

On the other hand, taking μ in (3) such that $r = R^{(\mu)}$, we have

$$x_\mu(\sigma - a_{\mu\mu}) \leqq r - a_{\mu\mu} - a_{\mu n}(1 - x_n),$$

where for $\mu = n$ we write 0 for $a_{\mu n}$. In any case it follows that

$$x_\mu(\sigma - a_{\mu\mu}) \leqq r - a_{\mu\mu}, \quad x_n(\sigma - a_{\mu\mu}) \leqq r - a_{\mu\mu}, \quad x_n \leqq \frac{r - a_{\mu\mu}}{\sigma - a_{\mu\mu}}.$$

As $r/\sigma \leqq 1$, we do not decrease the right-hand bound replacing $a_{\mu\mu}$ by κ_1 and hence

$$x_n \leqq \frac{r - \kappa_1}{\sigma - \kappa_1}, \quad \gamma \geqq \frac{\sigma - \kappa_1}{r - \kappa_1}. \qquad \text{(30)}$$

Here σ can be replaced by any lower bound, for instance r_p.

Multiplying (29) and (30) we obtain

$$x_n \leqq \sqrt{\frac{r - \kappa_1}{R - \kappa_1}}, \quad \gamma \geqq \sqrt{\frac{R - \kappa_1}{r - \kappa_1}}. \qquad \text{(31)}$$

The inequalities (29), (30), (31) *hold, by continuity, in the case of non-negative irreducible matrices.*

In particular we obtain the estimate depending only on R and r, namely,

$$x_n \leqq \sqrt{\frac{r}{R}}, \quad \gamma \geqq \sqrt{\frac{R}{r}}. \qquad \text{(32)}$$

9. We now use (3) for $\mu = n$. Then it follows by (5), if $\kappa_2 > 0$, that

$$x_n(\sigma - a_{nn}) \geqq a_{n1} + (R^{(n)} - a_{nn} - a_{n1})x_n = (R^{(n)} - a_{nn})x_n + a_{n1}(1 - x_n).$$

The expression on the right is $\geqq (r - a_{nn})x_n + \kappa_2(1 - x_n)$, and hence

$$x_n(\sigma - a_{nn}) \geqq x_n(r - a_{nn} - \kappa_2) + \kappa_2,$$

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from which we obtain

$$x_n \geqq \frac{\kappa_2}{\sigma - r + \kappa_2}, \quad \gamma \leqq \frac{\sigma - r + \kappa_2}{\kappa_2}; \quad \text{..........................(33)}$$

here σ can be replaced by any upper bound, for instance, R_p.

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