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On families of finite sets no two of which intersect in a singleton

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Let X be a finite set of cardinality n, and let F be a family of k-subsets of X. In this paper we prove the following conjecture of P. Erdös and V.T. Sós.

If $n > n_0(k)$, $k \geq 4$, $|F| > \binom{n-2}{k-2}$ then we can find two members F and G in F such that $|F \cap G| = 1$.

1. Introduction and some lemmas

Let X be a finite set of cardinality n and let F be a family of k-subsets of X. Let us define

$$F_x = \{F-x \mid x \in F \in F\} .$$

We say that a family of sets is intersecting if any two members of it have non-empty intersection. Let L be a set of non-negative integers. We say that F is an (n, L, k)-system if, for any two different members F, G of F, $|G \cap F| \in L$.

The Erdös-Ko-Rado Theorem (Erdös, Ko, and Rado [4]) states that if F is an $(n, \{t, t+1, \ldots, k-1\}, k)$-system and $n > n_0(k)$; then $|F| \leq \binom{n-t}{k-t}$ with equality holding if and only if for some t-element subset Y of X, $F = \{F \subset X \mid |F| = k, Y \subseteq F\}$, where t is a positive integer.

Erdös and Sós made the following conjecture (see Erdös [2]):

If F is an $(n, \{0, 2, 3, \ldots, k-1\}, k)$-system, $k \geq 4$, $n \geq n_0(k)$,

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then $|F| \leq \binom{n-2}{k-2}$.

The aim of this paper is to prove this conjecture. For the case $k = 4$ it was proved by Katona [5].

Obviously the condition is equivalent to that for every $x \in X$, F_x is an intersecting family.

If G is an intersecting family of $(k-1)$-subsets of X then let us define

$$G^* = \Big\{E \subset X \mid E \neq \emptyset,\ \exists G_1, G_2, \ldots, G_{k^{|E|}} \text{ such that } G_i \cap G_j = E,\ 1 \leq i < j \leq k^{|E|}\Big\} ,$$

$$B(G) = \{B \in G^* \mid \nexists E \in G^* \text{ such that } E \subsetneq B\} .$$

From the definition it is evident that $G \subseteq G^*$ and consequently for every $G \in G$ there exists $B \in B(G)$ such that $B \subseteq G$. Therefore we call B the Δ-base of G .

A family $C = \{C_1, \ldots, C_s\}$ is called a Δ-system of cardinality s if for some set $K \subsetneq C_1$ we have $C_i \cap C_j = K$ for $1 \leq i < j \leq s$. K is called the kernel of the Δ-system. Erdős and Rado [3] proved that there exists a function $f(k, s)$ such that any family consisting of $f(k, s)$ different k-sets contains a Δ-system of cardinality s .

LEMMA 1. *Let* F *be an* $(n, \{0, 2, 3, \ldots, k-1\}, k)$*-system,* $k \geq 4$, $x \in X$, $1 \leq i \leq k-1$. *Then we cannot find sets* $B_1, \ldots, B_{k^i} \in B(F_x)$, *forming a* Δ*-system of cardinality* k^i *and satisfying further* $|B_j| = i + 1$ *for* $1 \leq j \leq k^i$.

Proof. Let us suppose that for $B_1, \ldots, B_{k^i}$ the lemma fails; let K be the kernel of the corresponding Δ-system.

By the definition of the Δ-base there exist sets $E_j^r \in F_x$ for $1 \leq r \leq k^i$, $1 \leq j \leq k^{i+1}$, such that for $1 \leq j < j^* \leq k^{i+1}$,

$E^r_j \cap E^r_{j*} = B_r$. As the sets $E^1_j - B_1$ are pairwise disjoint and

$$\left| \bigcup_{r=2}^{k^i} B_r \right| < k \cdot k^i = k^{i+1} ,$$

we can find a j , $1 \le j \le k^{i+1}$ such that $E^1_j - B_1$ is disjoint to the union of the B^r's . Let us define $D_1 = E^1_j$. Let us suppose that D_r is defined already for $r = 1, \ldots, s-1$. Let us set

$$C_s = \left(\bigcup_{r=1}^{s-1} D_r \right) \cup \left(\bigcup_{r=s+1}^{k^{i+1}} B_r \right) .$$

Then $|C_s| < k^{i+1}$.

Hence among the pairwise disjoint sets $E^s_1 - B_s, \ldots, E^s_{k^{i+1}} - B_s$ we can find one, say $E^s_j - B_s$, which is disjoint from C_s ; let us define $D_s = E^s_j$. Let us continue this procedure until $s = k^i$. From the definition of the D_s's it follows that they belong to F_x , and that they form a Δ-system of cardinality $k^i \ge k^{|K|}$ with kernel K , yielding $K \in F^*_x$. But this is a contradiction as $K \subsetneq B \in B(F_x)$. //

In view of Lemma 1, $B(F_x)$ contains at most $f(i+1, k^i)$ $(i+1)$-element sets for $i = 1, 2, \ldots, k-1$. Now the next lemma is obvious.

LEMMA 2. *Let* F *be an* $(n, \{0, 2, 3, \ldots, k-1\}, k)$*-system consisting of subsets of* X , $x \in X$, $k \ge 4$. *Suppose that* F^*_x *does not contain any* 1*-element set and let* $B_1, \ldots, B_v$ *be the* 2*-element sets in it. Then*

$$|F_x - \{E \in F_x \mid \exists j,\ 1 \le j \le v,\ B_j \subset E\}| < k . f(k-1, k^{k-2}) . \binom{n-4}{k-4} .$$

We need one more lemma.

LEMMA 3. *Let* F *be an* $(n, \{0, 2, 3, \ldots, k-1\}, k)$*-system*

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consisting of subsets of X *. Let* y, x *be two not necessarily different elements of* X *. If* $B \in \mathcal{B}(F_x)$ *,* $C \in \mathcal{B}(F_y)$ *, then*
$|(B \cup x) \cap (C \cup y)| \neq 1$.

Proof. If $B \in F_x$, $C \in F_y$, then the statement follows from the definition of $(n, \{0, 2, 3, \ldots, k-1\}, k)$-systems. So we may assume that, for example, $B \notin F_x$. By the definition of $\mathcal{B}(F_x)$ there exist $F_1, \ldots, F_k \in F$ forming a Δ-system with kernel $B \cup x$. As the sets $F_i - (B \cup x)$ are pairwise disjoint and in the case $(B \cup x) \cap (C \cup y) = 1$, $|(C \cup y)-(B \cup x)| \leq k - 1$, we can find an index j , $1 \leq j \leq k$, such that $|F_j \cap (C \cup y)| = 1$. If $(C \cup y) \in F$ then this is a contradiction to the definition of $(n, \{0, 2, 3, \ldots, k-1\}, k)$-systems. If $C \notin F_y$ then by the definition of $\mathcal{B}(F_y)$ there exist $G_1, G_2, \ldots, G_k \in F$ which form a Δ-system with kernel $C \cup y$. As the sets $G_i - (C \cup y)$ are pairwise disjoint and $|F_j-(C \cup y)| = k - 1$, we can find an index i , $1 \leq i \leq k$, such that $(F_j-(C \cup y)) \cap G_i = \emptyset$; that is, $|F_j \cap G_i| = 1$, a contradiction which proves the lemma.

2. The proof of the result

Let us first prove a slightly weaker result which, however, implies the conjecture of Erdös-Sós.

THEOREM 1. *Let* F *be an* $(n, \{0, 2, 3, \ldots, k-1\}, k)$*-system consisting of subsets of* X *,* $n > n_0(k)$ *. Then one of the following cases occurs:*

(i) $|F| < \binom{n-2}{k-2}$;

(ii) there exist $x \neq y \in X$ *such that*

$$F = \{F \subset X \mid |F| = k,\ x \in F,\ y \in F\} ;$$

(iii) there exists $x \in X$ *such that* $|F_x| < \binom{n-3}{k-3}$;

(iv) there exist $x \neq y \in X$ *such that*

$$|\{F \in \mathcal{F} \mid \{x, y\} \cap F \neq \emptyset\}| < \binom{n-3}{k-3} + \binom{n-4}{k-3} .$$

Proof. Let us argue indirectly. By Lemma 2 we may assume that for for every $x \in X$, $\mathcal{B}(\mathcal{F}_x)$ contains a set of cardinality at most 2. If it contains a 1-element set, say $\{y\}$, then the intersection property implies $\mathcal{B}(\mathcal{F}_x) = \{\{y\}\}$ and conversely $\mathcal{B}(\mathcal{F}_y) = \{\{x\}\}$. Indeed, if for some $F \in \mathcal{F}$, $|F \cap \{x, y\}| = 1$ holds then let us consider a Δ-system of cardinality k and with kernel $\{x, y\}$, consisting of members of $\mathcal{F}$ – such a system exists by the definition of $\mathcal{B}(\mathcal{F}_x)$. Now as $|F-\{x, y\}| = k - 1$, there is a member of the Δ-system, say G, which is disjoint from it; that is, $|F \cap G| = 1$, a contradiction.

Now let us suppose that $\mathcal{B}(\mathcal{F}_x)$ consists of sets of cardinality at least 2, and let $B_1, \ldots, B_v$ be the 2-element sets belonging to it. By Lemma 3 the B_i's form an intersecting family of 2-sets, and by Lemma 1 this family does not contain a Δ-system of cardinality k. As $k \geq 4$, it follows $v < k$. Now Lemma 2 and $|\mathcal{F}_x| \geq \binom{n-3}{k-3}$ imply, for $n > n_0(k)$, that there exists an i, $1 \leq i \leq v$, such that

$$|\{G \in \mathcal{F}_x \mid B_i \subset G\}| > 1/k\binom{n-3}{k-3} . \tag{1}$$

By symmetry reasons we may assume that (1) holds for $i = 1$. Let us suppose first that for some $G \in \mathcal{F}_x$, $G \not\supseteq B_1$ holds.

If $G \cap B_1 = \emptyset$ then let us choose k sets $G_1, \ldots, G_k$ belonging to $\mathcal{F}_x$ and forming a Δ-system with kernel B_1. Then $|G-B_1| = k - 1$ implies that G is disjoint from at least one of the G_i's, say from G_u. As $\mathcal{F}_x$ is an intersecting family of sets, we see that $G \cap B_1 = \emptyset$ is impossible. Hence if $G \not\supseteq B_1$ then $|G \cap B_1| = 1$. We prove now that this is impossible, too.

Let us define $H = G \cup B_1 \cup \ldots \cup B_v$, and

$$\mathcal{E}_1 = \{F-B_1 \mid B_1 \subseteq F \in \mathcal{F}_x, (F-B_1) \cap H = \emptyset\} .$$

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From (1) and $n > n_0(k)$ it follows that, for example,

$$|\mathcal{E}_1| > 1/2k\binom{n-3}{k-3} .$$

Hence there exists an element z of $X - (B_1 \cup x)$ which satisfies

$$|\{E \in \mathcal{E}_1 \mid z \in E\}| > 1/2k\binom{n-4}{k-4} . \tag{2}$$

In the case $k = 4$ we just choose $z = E - (B_1 \cup x)$ for some $E \in \mathcal{E}_1$. By a result of Erdös [1] if a family $\mathcal{D}$ of s-subsets of X, $s \geq 1$, does not contain $f(s)$ pairwise disjoint members, then $|\mathcal{D}| = O\left(\binom{n}{s-1}\right)$.

We apply this theorem for $\mathcal{D} = \{E-z \mid E \in \mathcal{E}_1, z \in E\}$, $s = k - 4$, $f(s) = s + 4$, to prove that there exist k members of $\mathcal{E}_1$, say $C_1, \ldots, C_k$, such that $C_i \cap C_j = \{z\}$, $1 \leq i < j \leq k$. In the case $k = 4$ we can choose $C_1 = \ldots = C_k = \{z\}$.

Let B be a member of $\mathcal{B}(\mathcal{F}_z)$ for which $|\{F \in \mathcal{F}_z \mid B \subseteq F\}|$ is maximal. Then, as we proved it already for x, it follows that

$$|\{F \in \mathcal{F}_z \mid B \subset F\}| > 1/k\binom{n-3}{k-3} . \tag{3}$$

From Lemma 3 we know that $(B \cup z) \cap (B_1 \cup x) = 1$ is impossible. We prove now that these two sets cannot be disjoint either. Otherwise from a Δ-system $F_1, \ldots, F_k$ consisting of members of $\mathcal{F}$, and having kernel $B \cup z$, we could choose a set, say F_i, satisfying $F_i \cap (B_1 \cup x) = \emptyset$. But then there is an index j, $1 \leq j \leq k$, such that $C_j \cap F_i = \{z\}$. Now setting $G_j = (C_j \cup B_1 \cup x \cup z) \in \mathcal{F}$, $|G_j \cap F_i| = 1$ is a contradiction, proving $(B \cup z) \cap (B_1 \cup x) \neq \emptyset$.

As $|B| \leq 2$, it follows now that $|B| = 2$ and $B \subset (B_1 \cup x)$.

If $B = B_1$ then from a Δ-system $F_1, \ldots, F_k$ consisting of members of $\mathcal{F}$ and having kernel $B \cup z$ we can choose a set, say F_i, which is

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disjoint from $G - B$. But then we have $|F_i \cap (G \cup x)| = 1$, a contradiction.

Hence $x \in B$. This in turn implies $((B \cup z)-x) \in \mathcal{B}(\mathcal{F}_x)$, a contradiction since $z \notin (B_1 \cup \ldots \cup B_v)$. This final contradiction proves that $G \not\supseteq B_1$ is impossible; that is, for every $G \in \mathcal{F}_x$, $G \supseteq B_1$ holds, and in particular $v = 1$. Hence $\mathcal{B}(\mathcal{F}_x) = \{B_1\}$.

Let $B_1 = \{y_1, y_2\}$. We assert that $\mathcal{B}(\mathcal{F}_{y_1}) = \{\{y_2\}\}$.

Otherwise it follows from the definition of $\mathcal{B}(\mathcal{F}_{y_1})$ that $\{x, y_2\}$ is a member of it. Then repeating the argument applied to x for y_1 we obtain that $\mathcal{B}(\mathcal{F}_{y_1}) = \{\{x, y_2\}\}$; that is, every member of $\mathcal{F}$ which contains y_1 contains x and y_2 as well. Consequently we have

$$|\{F \in \mathcal{F} \mid F \cap \{x, y_1\} \neq \emptyset\}| \leq |\{F \subset X \mid |F| = k, \{x, y_1, y_2\} \subset F\}| = \binom{n-3}{k-3} < \binom{n-3}{k-3} + \binom{n-4}{k-3},$$

contradicting the indirect assumptions.

So far we have proved that for every $x \in X$ either there exists a $y \in X$ such that $\mathcal{B}(\mathcal{F}_x) = \{\{y\}\}$, $\mathcal{B}(\mathcal{F}_y) = \{\{x\}\}$, or there exist $y, z \in X$ such that $\mathcal{B}(\mathcal{F}_x) = \{\{y, z\}\}$, $\mathcal{B}(\mathcal{F}_y) = \{\{z\}\}$, $\mathcal{B}(\mathcal{F}_z) = \{\{y\}\}$.

Now let $\{x_1, y_1\}, \ldots, \{x_w, y_w\}$ be the collection of all the different unordered pairs satisfying $x_i, y_i \in X$, $\mathcal{B}(\mathcal{F}_{y_i}) = \{\{x_i\}\}$, $\mathcal{B}(\mathcal{F}_{x_i}) = \{\{y_i\}\}$, $1 \leq i \leq w$. By Lemma 3 all the elements x_i, y_i are different; that is, they form w pairwise disjoint 2-subsets of X.

As we proved it is possible to divide the remaining elements of X into w classes $Z_1, \ldots, Z_w$ such that for $1 \leq i \leq w$, $z_i \in Z_i$, we have $\mathcal{B}(\mathcal{F}_{z_i}) = \{\{x_i, y_i\}\}$. So we proved that $\mathcal{F}$ is contained in the following family of subsets of X:

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$$F^* = \{F \subset X \mid |F| = k,\ F \cap (Z_i \cup \{x_i, y_i\}) \neq \emptyset \text{ implies } \{x_i, y_i\} \subset F,\ 1 \le i \le w\} .$$

If $w = 1$ then either (i) or (ii) holds. So we may assume $w \ge 2$. All we have to prove now is that in this case $|F^*| < \binom{n-2}{k-2}$.

We prove this by induction on w and for every $n > k$.

By symmetry reasons we may assume that $|Z_1| \le |Z_2|$. We count the number of members of F^* according to the cardinality of their intersection with $Z_1 \cup \{x_1, y_1\}$. Let us define $n_1 = |Z_1 \cup \{x_1, y_1\}|$. Then $|Z_1| \le |Z_2|$ implies $n_1 \le n - n_1$. Using the induction hypothesis or the estimate for the case $w = 1$, we obtain

$$|F^*| \le \binom{n_1-2}{k-2} + \sum_{i=2}^{k-2} \binom{n_1-2}{k-2-i}\binom{n-n_1-2}{i-2} + \binom{n-n_1-2}{i-2} . \tag{4}$$

As

$$\binom{n_1-2}{k-2} = \big((n_1-2)/(k-2)\big)\binom{n_1-3}{k-3} < n_1\binom{n-n_1-2}{k-3},$$

and

$$\binom{n_1-2}{k-2-i} \le \binom{n_1}{k-i} , \quad i = 2, \ldots, k-2 ,$$

it follows from (4),

$$|F^*| < n_1\binom{n-n_1-2}{k-3} + \sum_{i=2}^{k-2}\binom{n_1}{k-i}\binom{n-n_1-2}{i-2} + \binom{n-n_1-2}{k-2}$$
$$= \sum_{i=2}^{k}\binom{n_1}{k-i}\binom{n-n_1-2}{i-2} = \binom{n-2}{k-2} . \qquad //$$

THEOREM 2. *Let* F *be an* $(n, \{0, 2, 3, \ldots, k-1\}, k)$*-system,* $k \ge 4$. *Suppose that* $n > n_0(k) + 2\binom{n_0(k)}{k}$, *where* $n_0(k)$ *is the bound from Theorem* 1. *Then either there exist two different elements* x, y *such that* $F = \{F \subset X \mid |F| = k,\ \{x, y\} \subset F\}$ *or* $|F| < \binom{n-2}{k-2}$.

Proof. Let us argue indirectly and let F be a counter-example. Let $|F| = \binom{n-2}{k-2} + d$, where d is a non-negative integer. We may apply Theorem 1 to F. Hence either there exists $x \in X$ such that $|F_x| < \binom{n-3}{k-3}$ or there exist two different elements x, y in X such that

$$|\{F \in F \mid F \cap \{x, y\} \neq \emptyset\}| < \binom{n-3}{k-3} + \binom{n-4}{k-3} .$$

In the first case let us define $X_1 = X - x$ and in the second $X_1 = X - \{x, y\}$. In both cases we define $F_1 = \{F \subset X_1 \mid F \in F\}$. Then F_1 is an $(|X_1|, \{0, 2, 3, \ldots, k-1\}, k)$-system of cardinality at least

$$\binom{|X_1|-2}{k-2} + d + 1 .$$

Now we apply Theorem 1 to the family F_1, and we construct a set X_2 and a family of subsets, F_2, of X_2 such that F_2 is an $(|X_2|, \{0, 2, 3, \ldots, k-1\}, k)$-system of cardinality at least $\binom{|X_2|-2}{k-2} + d + 2$, and so on, and so on until we get a set X_r and a family of k-subsets of X_r, F_r such that $|X_r| \leq n_0(k)$.

Now the method of construction implies that

$$|F_r| \geq \binom{|X_r|-2}{k-2} + d + \binom{n_0(k)}{k} > \binom{n_0(k)}{k} ,$$

a contradiction since the number of k-subsets of X_r is $\binom{|X_r|}{k} \leq \binom{n_0(k)}{k}$. //

REMARK. One might conjecture that for an arbitrary integer s and $k > k_0(s)$, $n > n_0(k)$, any family of more than $\binom{n-s-1}{k-s-1}$ k-subsets of an n-set contains two members intersecting in a set of cardinality s. The author can prove it only for $c_k\binom{n-s-1}{k-s-1}$, where c_k is a large constant

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depending only on k .

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