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An elementary proof of part of a classical conjecture

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An elementary proof is given for the L_p conjecture, $p > 2$, which states that for a locally compact group G, $L_p(G)$ $(p > 2)$ is closed under convolution if and only if G is compact.

In this paper we present a new proof of the L_p-conjecture for $p > 2$ which is elementary and much shorter than that of Rajagopalan [5].

If G is a locally compact Hausdorff group with left Haar measure m, we define convolution as follows: If f and g are measurable functions for which the integral $\int_G f(y)g(y^{-1}x)dm(y)$ exists $ae[m]$, we say $f*g$ exists and we define $f*g(x) = \int_G f(y)g(y^{-1}x)dm(y)$ if this integral exists and $f*g(x) = 0$ otherwise.

As usual, $L_p(G)$ $(p > 1)$ denotes the linear space of all measurable functions f for which $\int_G |f|^p dm$ is finite. It is well known that $L_1(G)$ is closed under convolution in the sense that for each $f, g \in L_1(G)$ the convolution $f*g$ exists and is in $L_1(G)$. The so called "L_p-conjecture" is that $L_p(G)$ is closed under convolution iff G is compact.

To this date, the L_p-conjecture has been proven $(p > 2)$ for

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arbitrary groups and $(p > 1)$ for solvable groups. Żelazko [7] published results in 1961, proving the L_p-conjecture $(p > 1)$ for abelian groups. In 1964, Rajagopalan and Żelazko [6] proved that if $L_p(G)$ is closed under convolution for some $p > 1$ then the group G is unimodular, and in the same paper, they proved the conjecture $(p > 1)$ for solvable groups. Using his proof [4] of the L_p-conjecture $(p > 2)$ for discrete groups, and a powerful reduction theorem, Rajagopalan [5] proved the L_p-conjecture $(p > 2)$. In 1966 Leptin [3] introduced a two-valued function $I(G)$, and proved the L_p-conjecture among groups G with $I(G) < \infty$.

It is well known (see for example Hewitt and Ross [1]) that compactness of G implies that $L_p(G)$ is closed under convolution, for all $p \geq 1$. The following proof of the converse, for $p > 2$, proceeds from first principles.

THEOREM. *Let G be a locally compact Hausdorff group with left Haar measure m. Suppose $L_p(G)$ is closed under convolution for some $p > 2$. Then G is compact.*

Proof. Suppose G is not compact, so that by well known arguments we may construct a sequence $\{a_n\} \subseteq G$ and a compact symmetric neighbourhood U of the identity such that $\left\{Ua_n : n = 1, 2, \ldots\right\} \cup \left\{Ua_n^{-1} : n = 1, 2, \ldots\right\}$ is a family of pairwise disjoint subsets of G. We write Δ for the modular function on G:

$$\Delta(x) = m(Ax)/m(A) ,$$

where A is an arbitrary measurable set of positive finite measure. It is well known that Δ is independent of A, and that $\Delta\left(x^{-1}\right) = \Delta(x)^{-1}$. Because of the last equality we may assume without loss of generality that $\Delta\left(a_n\right) \geq 1$ for all n.

Choose a compact symmetric neighbourhood V of the identity such that $V^2 \subseteq U$, and define functions f, g on G by:

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$$f(x) = n^{-\frac{1}{2}}\Delta\left(a_n\right)^{-\frac{1}{p}} \quad \text{for} \quad x \in Ua_n \ , \quad n = 1, 2, \ldots ;$$

$$g(x) = n^{-\frac{1}{2}} \quad \text{for} \quad x \in a_n^{-1}V \ , \quad n = 1, 2, \ldots ;$$

and f, g vanish elsewhere.

Clearly f, g are measurable, and

$$\int_G |f|^p dm = \sum_{n=1}^{\infty} n^{-\frac{p}{2}}\Delta\left(a_n\right)^{-1}m\left(Ua_n\right) = m(U) \sum_{n=1}^{\infty} n^{-\frac{p}{2}} < \infty \ ,$$

$$\int_G |g|^p dm = \sum_{n=1}^{\infty} n^{-\frac{p}{2}}m(V) < \infty \ .$$

Thus $f, g \in L_p(G)$.

We now show that $f*g(t) = +\infty$ for $t \in V$. For all $t \in G$,

$$f*g(t) = \sum_{k,n} k^{-\frac{1}{2}}n^{-\frac{1}{2}}\Delta\left(a_k\right)^{-\frac{1}{p}} m\left(Ua_k \cap tVa_n\right)$$

$$\geq \sum_n n^{-1}\Delta\left(a_n\right)^{-\frac{1}{p}} m\left(Ua_n \cap tVa_n\right) \ .$$

But if $t \in V$ then $tVa_n \subseteq Ua_n$, so

$$m\left(Ua_n \cap tVa_n\right) = \Delta\left(a_n\right)m(V) \ .$$

Since $\Delta\left(a_n\right) \geq 1$ we deduce that

$$f*g(t) \geq \sum_n n^{-1}m(V) = \infty \quad \text{for} \quad t \in V \ .$$

This contradicts the assumption that $f*g \in L_p(G)$, proving the theorem.

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