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POSITIVE SOLUTIONS OF NONLOCAL SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. The paper presents the existence result for positive solutions of the differential equation $(g(x))'' = f(t, x, (g(x))')$ satisfying the nonlocal boundary conditions $x(0) = x(T)$, $\min\{x(t) : t \in J\} = 0$. Here the positive function f satisfies local Carathéodory conditions on $[0, T] \times (0, \infty) \times (\mathbb{R}\setminus\{0\})$ and f may be singular at the value 0 of both its phase variables. Existence results are proved by Leray-Schauder degree theory and Vitali's convergence theorem.

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1. Introduction. Let T be a positive number, $J = [0, T]$ and $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$. We shall discuss the singular differential equation

$$(g(x(t)))'' = f(t, x(t), (g(x(t)))'), \tag{1.1}$$

where $g \in C^0([0, \infty))$ and the positive function f satisfies local Carathéodory conditions on $J \times (0, \infty) \times \mathbb{R}_0$ ($f \in Car(J \times (0, \infty) \times \mathbb{R}_0)$) and f may be singular at the value 0 of both its phase variables.

Furthermore we shall deal with the nonlocal boundary conditions

$$x(0) = x(T), \quad \min\{x(t) : t \in J\} = 0. \tag{1.2}$$

We say that $x \in C^0(J)$ is a *solution of the boundary value problem* (BVP for short) (1.1), (1.2) if $g(x) \in AC^1(J)$ (functions having absolutely continuous derivative on J), x satisfies the boundary conditions (1.2) and (1.1) holds a.e. on J.

In this paper we are interested in finding conditions on the functions g and f in (1.1) that guarantee the existence of positive solutions to BVP (1.1), (1.2). The existence

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result is proved by regularization and sequential techniques. Any positive solution x and $(g(x))'$ for BVP (1.1), (1.2) 'go through' singularities of f somewhere inside of J.

We show that our existence result for BVP (1.1), (1.2) can be applied to obtain solutions of BVP (1.3), (1.2), where

$$(g(x(t))x'(t))' = f(t, x(t), g(x(t))x'(t)). \tag{1.3}$$

By a *solution of BVP* (1.3), (1.2) we understand a function $x \in C^1(J)$ satisfying (1.2), $g(x)x' \in AC(J)$ and (1.3) is true almost everywhere on J.

We note that only a few papers in the literature are devoted to the study of BVPs for differential equations of the form (1.1) (see [**1**], [**7**] and references therein). In [**1**] the authors consider via the method of lower and upper functions the Dirichlet problem with the differential equation $(P(x))'' + f_1(t, x) = 0$ where $P(z) = \int_0^z r(x)\,dx$, r is a continuous function and f_1 satisfies local Carathéodory conditions. Existence results for solutions of $(P(x))'' = q(t)f_2(t, x, x')$ with continuous f_2 satisfying the Dirichlet boundary conditions are given in [**7**]. Differential equations of the form $(g(x)x')' = f_3(t, x, x')$ and two-point boundary conditions were considered (in the regular case also) in [**7**]. The Dirichlet problem for differential equations of the form $(r(x)x')' = \mu q(t)f_4(t, x)$ where f_4 is singular at the value 0 of its phase variable x was studied in [**8**]–[**10**]. In [**2**] the authors give conditions for the existence of positive solutions of a more general equation $(g(x)(x')^\alpha)' = \mu q(t)f_5(t, x)(x')^\beta$ with $\alpha \in (0, \infty)$ and $\beta \in \{0, 1\}$ satisfying the Dirichlet boundary conditions. Existence results for a functional differential equation with a nonlinear functional left hand side and nonlocal boundary conditions are presented in [**4**]. In all the papers above, BVPs are considered only for local boundary conditions and, in the case that differential equations are singular at their phase variables solutions 'start' and/or 'finish', at singular points (with the exception of [**4**] and [**9**]).

In this paper the following assumptions will be used.

(H_1) $g \in C^0([0, \infty))$ is increasing, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = \infty$.

(H_2) $g \in C^0([0, \infty))$ is positive and $\lim_{u\to\infty} G(u) = \infty$, where

$$G(u) = \int_0^u g(s)\,ds, \quad u \in [0, \infty). \tag{1.4}$$

(H_3) $f \in Car(J \times (0, \infty) \times \mathbb{R}_0)$ and there exists a positive constant $a \le 1/2$ such that

$$a \le f(t, x, y) \quad \text{for a.e. } t \in J \text{ and each } (x, y) \in (0, \infty) \times \mathbb{R}_0.$$

(H_4) For a.e. $t \in J$ and each $(x, y) \in (0, \infty) \times \mathbb{R}_0$,

$$f(t, x, y) \le (h_1(x) + h_2(x))(\omega_1(|y|) + \omega_2(|y|)),$$

where $h_1, \omega_1 \in C^0([0, \infty))$ are non-negative and non-decreasing, $h_2, \omega_2 \in C^0((0, \infty))$ are positive and non-increasing.

(H_5) $\displaystyle\int_0^1 h_2(g^{-1}(s^2))\omega_2(s)\,ds < \infty$ and

$$\lim_{u\to\infty} \int_0^u \frac{1}{K^{-1}(H_1(s))}\,ds > \frac{T}{2},$$

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where

$$K(u) = \int_0^u \frac{s}{\omega_1(s+1)+\omega_2(s)}\,ds, \quad u \in [0,\infty), \tag{1.5}$$

$$H_1(u) = \int_0^u [h_1(g^{-1}(s)+1)+h_2(g^{-1}(s))]\,ds, \quad u \in [0,\infty). \tag{1.6}$$

(H_6) $\int_0^1 h_2(G^{-1}(s^2))\omega_2(s)\,ds < \infty$ and

$$\lim_{u\to\infty} \int_0^u \frac{1}{K^{-1}(H_2(s))}\,ds > \frac{T}{2},$$

where

$$H_2(u) = \int_0^u [h_1(G^{-1}(s)+1)+h_2(G^{-1}(s))]\,ds, \quad u \in [0,\infty). \tag{1.7}$$

REMARK 1.1. Let assumptions (H_4) and (H_5) be satisfied. We show that the integral $\int_0^u (1/K^{-1}(H_1(s)))\,ds$ is convergent for all $u > 0$. Since $h_1(g^{-1}(u)+1)+h_2(g^{-1}(u)) \geq h_2(g^{-1}(1))$ and $\omega_1(u+1)+\omega_2(u) \geq \omega_2(1)$ for $u \in [0,1]$, we have $H_1(u) \geq h_2(g^{-1}(1))u$ and $K(u) \leq u^2/(2\omega_2(1))$ for these u. Hence $K^{-1}(H_1(u)) \geq \sqrt{2h_2(g^{-1}(1))\omega_2(1)u}$ for $u \in [0,\tau]$ with a $\tau > 0$ and since $K^{-1}(H_1)$ is positive and continuous on $(0,\infty)$, we see that $\int_0^u (1/K^{-1}(H_1(s)))\,ds < \infty$ for all $u > 0$. Analogously we can verify that $\int_0^u (1/K^{-1}(H_2(s)))\,ds < \infty$ for $u > 0$ if assumptions (H_4) and (H_6) are satisfied.

The paper is organized as follows. In Section 2 we prove that the solvability of BVP (1.1), (1.2) is equivalent to that of BVP (2.1), (1.2) (Lemma 2.1). Section 3 deals with a sequence of auxiliary regular BVPs to BVP (2.1), (1.2) where the nonlinearities f_n in the differential equations are regular functions on $J \times \mathbb{R}^2$. We give *a priori* bounds for their solutions x_n (Lemma 3.3) and prove their existence (Lemma 3.4) using Leray-Schauder degree theory (see, for example, [**5**]). In addition, we show that the sequence $\{f_n(t, g^{-1}(x_n(t)), x_n'(t))\}$ is uniformly absolutely continuous on J (Lemma 3.5). In Section 4 we present our main results: the existence of a positive solution to BVP (1.1), (1.2) (Theorem 4.1) and to BVP (1.3), (1.2) (Corollary 4.2). In limiting processes we use the Vitali's convergence theorem (see, for example, [**3**], [**6**]) since it is impossible to find a Lebesgue integrable majorant function for the sequence $\{f_n(t, g^{-1}(x_n(t)), x_n'(t))\}$ which is necessary for applying the Lebesgue dominated convergence theorem. We include also two examples (Examples 4.3 and 4.4) to illustrate our theory.

2. Lemma.

Let assumptions (H_1) and (H_3) be satisfied. Together with the differential equation (1.1) we consider the differential equation

$$x''(t) = f(t, g^{-1}(x(t)), x'(t)). \tag{2.1}$$

We say that x is a solution of equation (2.1) if $x \in AC^1(J)$ and x satisfies (2.1) a.e. on J.

In the next lemma we give relations between solutions of BVP (1.1), (1.2) and BVP (2.1), (1.2).

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LEMMA 2.1. *Let assumptions* (H_1) *and* (H_3) *be satisfied. If* $x(t)$ *is a solution of BVP* (1.1), (1.2)*, then the function* $u(t) = g(x(t))$, $t \in J$*, is a solution of BVP* (2.1), (1.2) *and also conversely, if* $x(t)$ *is a solution of BVP* (2.1), (1.2)*, then the function* $u(t) = g^{-1}(x(t))$, $t \in J$*, is a solution of BVP* (1.1), (1.2).

Proof. Let x be a solution of BVP (1.1), (1.2). Then $x \in C^0(J)$, $g(x) \in AC^1(J)$ and x satisfies (1.2). Set $u(t) = g(x(t))$ for $t \in J$. Then $u(0) = u(T)$, $\min\{u(t) : t \in J\} = 0$, $u \in AC^1(J)$ and $u''(t) = (g(x(t)))'' = f(t, x(t), (g(x(t))') = f(t, g^{-1}(u(t)), u'(t))$ a.e. on J. Hence u is a solution of BVP (2.1), (1.2).

Let x be a solution of BVP (2.1), (1.2). Then x satisfies (1.2) and $x \in AC^1(J)$. Let $u(t) = g^{-1}(x(t))$, $t \in J$. Then (1.2) holds with u instead of x, $u \in C^0(J)$, $g(u) = x \in AC^1(J)$ and $(g(u(t)))'' = x''(t) = f(t, g^{-1}(x(t)), x'(t)) = f(t, u(t), (g(u(t))')$ a.e. on J. Thus u is a solution of BVP (1.1), (1.2). □

REMARK 2.2. From Lemma 2.1 we see that solving BVP (1.1), (1.2) is equivalent to solving BVP (2.1), (1.2).

3. Auxiliary regular BVPs. For each $n \in \mathbb{N}$, define $f_n \in Car(J \times \mathbb{R}^2)$ by

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{for } t \in J,\ x \ge \frac{1}{n},\ |y| \ge \frac{1}{n}, \\ f\left(t, \frac{1}{n}, y\right) & \text{for } t \in J,\ x < \frac{1}{n},\ |y| \ge \frac{1}{n}, \\ \frac{n}{2}\left[f_n\left(t,x,\frac{1}{n}\right)\left(y+\frac{1}{n}\right) - f_n\left(t,x,-\frac{1}{n}\right)\left(y-\frac{1}{n}\right)\right] & \\ \qquad\qquad \text{for } t \in J,\ x \in \mathbb{R},\ y \in \left(-\frac{1}{n}, \frac{1}{n}\right). & \end{cases}$$

Then (H_3) and (H_4) yield (for $n \in \mathbb{N}$)

$$a \le f_n(t,x,y) \quad \text{for a.e. } t \in J \text{ and each } (x,y) \in \mathbb{R}^2 \tag{3.1}$$

and

$$f_n(t,x,y) \le (h_1(x+1) + h_2(x))(\omega_1(|y|+1) + \omega_2(|y|)) \tag{3.2}$$

for a.e. $t \in J$ and each $(x, y) \in (0, \infty) \times \mathbb{R}_0$.

Also define $\hat{g} \in C^0(\mathbb{R})$ by

$$\hat{g}(u) = \begin{cases} g(u) & \text{for } u \in [0,\infty), \\ -g(-u) + 2g(0) & \text{for } u \in (-\infty, 0). \end{cases}$$

If g satisfies assumption (H_1), then $\hat{g}$ is increasing on $\mathbb{R}$, which is the domain of the inverse function $\hat{g}^{-1}$ to $\hat{g}$.

Consider the family of regular differential equations

$$x''(t) = \lambda f_n(t, \hat{g}^{-1}(x(t)), x'(t)) + (1-\lambda)a \tag*{$(\mathrm{E})_n^\lambda$}$$

depending on the parameters $\lambda \in [0, 1]$ and $n \in \mathbb{N}$, where a appears in (H_3).

LEMMA 3.1. *Let assumptions* (H_1) *and* (H_3) *be satisfied and let* x *be a solution of BVP* $(\mathrm{E})_n^\lambda$, (1.2)*. Then there exists a unique* $\xi \in (0, T)$ *such that*

(*a*) $x(\xi) = 0$ *and* $x(t) > 0$ *for* $t \in [0, \xi) \cup (\xi, T]$,

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(*b*) x' *is increasing on* J, $x'(\xi) = 0$ *and* $|x'(t)| \geq a|\xi - t|$ *for* $t \in J$,
(*c*) $x(t) \geq \frac{a}{2}(t-\xi)^2$ *for* $t \in J$.

Proof. By (3.1),

$$x''(t) \geq a \quad \text{for a.e. } t \in J. \tag{3.3}$$

From (3.3) it follows that x' is increasing on J and then $x(0) = x(T)$ implies that x' vanishes at a unique point $\xi \in (0, T)$ and x is decreasing on $[0, \xi]$ and increasing on $[\xi, T]$. Hence the condition $\min\{x(t) : t \in J\} = 0$ yields $x(\xi) = 0$ and $x > 0$ on $[0, \xi) \cup (\xi, T]$. The validity of the inequalities in (b) and (c) follows immediately by integration of (3.3) and using $x(\xi) = x'(\xi) = 0$. □

REMARK 3.2. Lemma 3.1 shows that any solution x of BVP $(\mathrm{E})_n^\lambda$, (1.2) with $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ satisfies the inequality $x(t) > 0$ for $t \in [0, \xi) \cup (\xi, T]$ where $\xi \in (0, T)$ is the unique zero of x. Hence $\hat{g}^{-1}(x(t)) = g^{-1}(x(t))$ for $t \in J$.

LEMMA 3.3. *Let assumptions* (H_1) *and* $(H_3) - (H_5)$ *be satisfied. Let* x *be a solution of BVP* $(\mathrm{E})_n^\lambda$, (1.2). *Then there exists a positive constant* P *independent of* $\lambda \in [0, 1]$ *and* $n \in \mathbb{N}$ *such that*

$$\|x\| = \sup_{t \in J} |x(t)| < P, \quad \|x'\| < P. \tag{3.4}$$

Proof. By Lemma 3.1, there exists a unique $\xi \in (0, T)$ such that $x(\xi) = x'(\xi) = 0$, $x(t) > 0$ on $[0, \xi) \cup (\xi, T]$ and x' is increasing on J. Hence

$$\|x\| = x(0)\,(= x(T)), \quad \|x'\| = \max\{|x'(0)|, x'(T)\}. \tag{3.5}$$

In addition (see (3.2) and Remark 3.2)

$$x''(t) \leq [h_1(g^{-1}(x(t)) + 1) + h_2(g^{-1}(x(t)))][\omega_1(|x'(t)| + 1) + \omega_2(|x'(t)|)] \tag{3.6}$$

for a.e. $t \in J$. Integrating the inequality (for a.e. $t \in [0, \xi)$)

$$\frac{x''(t)x'(t)}{\omega_1(-x'(t) + 1) + \omega_2(-x'(t))} \geq [h_1(g^{-1}(x(t)) + 1) + h_2(g^{-1}(x(t)))]x'(t)$$

from $t \in [0, \xi)$ to ξ, we get

$$\int_0^{-x'(t)} \frac{s}{\omega_1(s+1) + \omega_2(s)}\, ds \leq \int_0^{x(t)} [h_1(g^{-1}(s) + 1) + h_2(g^{-1}(s))]\, ds.$$

Hence $K(-x'(t)) \leq H_1(x(t))$, where K and H_1 are defined by (1.5) and (1.6), respectively. Then

$$-x'(t) \leq K^{-1}(H_1(x(t))) \quad \text{for } t \in [0, \xi], \tag{3.7}$$

and integrating

$$-\frac{x'(t)}{K^{-1}(H_1(x(t)))} \leq 1 \quad (\text{where } 0 \leq t < \xi),$$

over $[0, \xi]$, we have

$$\int_0^{x(0)} \frac{1}{K^{-1}(H_1(s))}\, ds \le \xi. \tag{3.8}$$

Arguing as above on the inequality (for a.e. $t \in [\xi, T]$)

$$\frac{x''(t)x'(t)}{\omega_1(x'(t)+1)+\omega_2(x'(t))} \le [h_1(g^{-1}(x(t))+1)+h_2(g^{-1}(x(t)))]x'(t)$$

now on the interval $[\xi, T]$, we get

$$x'(t) \le K^{-1}(H_1(x(t))) \quad \text{for } t \in [\xi, T] \tag{3.9}$$

and

$$\int_0^{x(T)} \frac{1}{K^{-1}(H_1(s))}\, ds \le T - \xi. \tag{3.10}$$

Then (3.5), (3.8) and (3.10) imply

$$\int_0^{\|x\|} \frac{1}{K^{-1}(H_1(s))}\, ds \le \frac{T}{2}. \tag{3.11}$$

By (H_5), there is a positive constant V such that

$$\int_0^{u} \frac{1}{K^{-1}(H_1(s))}\, ds > \frac{T}{2},$$

for all $u \ge V$. Hence (3.11) yields $\|x\| < V$. Letting $t = 0$ in (3.7), $t = T$ in (3.9) and using the last inequality, we get $-x'(0) < K^{-1}(H_1(V))$ and $x'(T) < K^{-1}(H_1(V))$. Then (see (3.5)) $\|x'\| < K^{-1}(H_1(V))$ and so (3.4) is true with $P = \max\{V, K^{-1}(H_1(V))\}$. □

LEMMA 3.4. *Let assumptions* (H_1) *and* $(H_3) - (H_5)$ *be satisfied. Then BVP* $(\mathrm{E})_n^1$, (1.2) *has a solution* x *for each* $n \in \mathbb{N}$ *and* (3.4) *is true with a positive constant* P *given by Lemma* 3.3.

Proof. Fix $n \in \mathbb{N}$. Let

$$\Omega = \left\{(x, A) : (x, A) \in C^1(J) \times \mathbb{R},\ \|x\| < \max\left\{P, \frac{aT^2}{4}\right\},\right.$$
$$\left.\|x'\| < \max\left\{P, \frac{aT}{2}\right\},\ |A| < \max\left\{P, \frac{aT^2}{8}\right\}\right\}$$

and the operator $\mathcal{S} : \overline{\Omega} \to C^1(J) \times \mathbb{R}$ be defined by the formula

$$\mathcal{S}(x, A) = \left(A + \int_0^T S(t,s) f_n(s, \hat{g}^{-1}(x(s)), x'(s))\, ds,\ A + \min\{x(t) : t \in J\}\right), \tag{3.12}$$

where

$$S(t, s) = \begin{cases} s\left(\frac{t}{T} - 1\right) & \text{for } 0 \le s \le t \le T, \\ t\left(\frac{s}{T} - 1\right) & \text{for } 0 \le t < s \le T. \end{cases} \tag{3.13}$$

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We see that $S \in C^0(J \times J)$ and

$$S(t,s) < 0 \quad \text{for } (t,s) \in (0,T) \times (0,T).$$

Assume that $(x_0, A_0) \in \overline{\Omega}$ is a fixed point of $\mathcal{S}$; that is $\mathcal{S}(x_0, A_0) = (x_0, A_0)$. Then

$$x_0(t) = A_0 + \int_0^T S(t,s) f_n(s, \hat{g}^{-1}(x_0(s)), x_0'(s))\, ds, \quad t \in J, \tag{3.14}$$

$$\min\{x_0(t) : t \in J\} = 0. \tag{3.15}$$

From (3.14) we deduce that $x_0(0) = x_0(T)$ $(= A_0)$, $x_0 \in AC^1(J)$ and $x_0''(t) = f_n(t, \hat{g}^{-1}(x_0(t)), x_0'(t))$ for a.e. $t \in J$. Hence x_0 is a solution of BVP $(\mathrm{E})_n^1$, (1.2). Therefore, to prove the existence of a solution of BVP $(\mathrm{E})_n^1$, (1.2) it is sufficient to verify that

$$D(\mathcal{I} - \mathcal{S}, \Omega, 0) \neq 0, \tag{3.16}$$

where "D" stands for the Leray-Schauder degree and $\mathcal{I}$ is the identity operator on $C^1(J) \times \mathbb{R}$. The validity of (3.16) will be proved by the homotopy property. We first define the operator $\mathcal{L} : \overline{\Omega} \times [0,1] \to C^1(J) \times \mathbb{R}$ by

$$\mathcal{L}(x, A, \lambda) = \left(A + \frac{a}{2} t(t-T),\ A + (1-\lambda) x\left(\frac{T}{2}\right) + \lambda \min\{x(t) : t \in J\} \right). \tag{3.17}$$

Then $\mathcal{L}$ is a continuous operator and also $\mathcal{L}(\overline{\Omega} \times [0,1])$ is relatively compact in $C^1(J) \times \mathbb{R}$. Set $\mathcal{V} = \mathcal{I} - \mathcal{L}(\cdot, \cdot, 0)$. Then $\mathcal{V}(x, A) = (x(t) - A - at(t-T)/2,\ -x(T/2))$ for $(x, A) \in \overline{\Omega}$. We claim that $\mathcal{V}(-x, -A) \neq \nu \mathcal{V}(x, A)$, for all $(x, A) \in \partial\Omega$ and $\nu \in [1, \infty)$, so that

$$D(\mathcal{I} - \mathcal{L}(\cdot, \cdot, 0), \Omega, 0) \neq 0, \tag{3.18}$$

by Theorem 8.3 in [**5**]. If not, there exist $(x_*, A_*) \in \partial\Omega$ and $\nu_* \in [1, \infty)$ such that $\mathcal{V}(-x_*, -A_*) = \nu_* \mathcal{V}(x_*, A_*)$, we then have

$$-x_*(t) + A_* - \frac{a}{2} t(t-T) = \nu_* \left(x_*(t) - A_* - \frac{a}{2} t(t-T) \right), \quad t \in J, \tag{3.19}$$

$$x_*\left(\frac{T}{2}\right) = -\nu_* x_*\left(\frac{T}{2}\right). \tag{3.20}$$

From (3.20) we obtain that $x_*(T/2) = 0$ and then (3.19) with $t = T/2$ gives $A_* = \frac{\nu_* - 1}{\nu_* + 1} \frac{aT^2}{8}$. Hence $0 \leq A_* < aT^2/8$, and so (see (3.19))

$$|x_*(t)| = \left| A_* + \frac{a(\nu_* - 1)}{2(\nu_* + 1)} t(t-T) \right| < \frac{aT^2}{4}, \quad |x_*'(t)| = \left| \frac{a(\nu_* - 1)}{2(\nu_* + 1)} (2t - T) \right| < \frac{aT}{2}.$$

We have proved that $(x_*, A_*) \notin \partial\Omega$ and so (3.18) is true. Assume now that $\mathcal{L}(\hat{x}, \hat{A}, \hat{\lambda}) = (\hat{x}, \hat{A})$, for some $(\hat{x}, \hat{A}) \in \overline{\Omega}$ and $\hat{\lambda} \in [0,1]$. Then

$$\hat{x}(t) = \hat{A} + \frac{a}{2} t(t-T), \quad t \in J, \tag{3.21}$$

$$(1 - \hat{\lambda}) \hat{x}\left(\frac{T}{2}\right) + \hat{\lambda} \min\{\hat{x}(t) : t \in J\} = 0. \tag{3.22}$$

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From (3.21) we conclude that $\hat{x}$ is a solution of equation $(\mathrm{E})_n^0$, $\hat{x}(0) = \hat{x}(T)$ $(= \hat{A})$ and $\min\{\hat{x}(t) : t \in J\} = \hat{x}(T/2)$. Then (3.22) gives $\min\{\hat{x}(t) : t \in J\} = 0$, and so $\hat{x}$ is a solution of BVP $(\mathrm{E})_n^0$, (1.2). By Lemma 3.3, $\|\hat{x}\| < P$, $\|\hat{x}'\| < P$ and then $|\hat{A}| = |\hat{x}(0)| < P$. Hence $(\hat{x}, \hat{A}) \notin \partial\Omega$. Thus (3.18) and the homotopy property yield

$$D(\mathcal{I} - \mathcal{L}(\cdot, \cdot, 1), \Omega, 0) = D(\mathcal{I} - \mathcal{L}(\cdot, \cdot, 0), \Omega, 0) \neq 0. \tag{3.23}$$

Finally, define $\mathcal{K} : \overline{\Omega} \times [0, 1] \to C^1(J) \times \mathbb{R}$ by

$$\mathcal{K}(x, A, \lambda) = \left(A + \int_0^T S(t, s)(\lambda f_n(s, \hat{g}^{-1}(x(s)), x'(s)) + (1 - \lambda)a)\, ds,\right.$$
$$\left. A + \min\{x(t) : t \in J\}\right).$$

Then $\mathcal{K}(\cdot, \cdot, 0) = \mathcal{L}(\cdot, \cdot, 1)$ and $\mathcal{K}(\cdot, \cdot, 1) = \mathcal{S}$. If we verify that

(i) $\mathcal{K}$ is a compact operator and

(ii) $\mathcal{K}(x, A, \lambda) \neq (x, A)$ for $(x, A) \in \partial\Omega$ and $\lambda \in [0, 1]$,

then (3.23) guarantees the validity of (3.16). Since $f_n \in Car(J \times \mathbb{R}^2)$, standard arguments show that $\mathcal{K}$ is a compact operator. To verify (ii), assume that $\mathcal{K}(x_*, A_*, \lambda_*) = (x_*, A_*)$, for some $(x_*, A_*) \in \overline{\Omega}$ and $\lambda_* \in [0, 1]$. Then x_* is a solution of BVP $(\mathrm{E})_n^{\lambda_*}$, (1.2) and $x_*(0) = A_*$. According to Lemma 3.3, $\|x_*\| < P$, $\|x_*'\| < P$ and then $|A_*| = |x_*(0)| < P$. Therefore $(x_*, A_*) \notin \partial\Omega$ and $\mathcal{K}$ has property (ii). □

LEMMA 3.5. *Let assumptions* (H_1) *and* $(H_3) - (H_5)$ *be satisfied and let* x_n *be a solution of BVP* $(\mathrm{E})_n^1$, (1.2). *Then the sequence*

$$\{f_n(t, g^{-1}(x_n(t)), x_n'(t))\} \subset L_1(J) \tag{3.24}$$

is uniformly absolutely continuous (*UAC*) *on* J; *that is for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_{\mathcal{M}} f_n(t, g^{-1}(x_n(t)), x_n'(t))\, ds < \varepsilon \quad (n \in \mathbb{N}),$$

whenever $\mathcal{M} \subset J$ *is measurable and* $\mu(\mathcal{M}) < \delta$, *where* $\mu(\mathcal{M})$ *denotes the Lebesgue measure of* $\mathcal{M}$.

Proof. By Lemmas 3.1 and 3.3,

$$x_n(t) \geq \frac{a}{2}(\xi_n - t)^2, \quad |x_n'(t)| \geq a|\xi_n - t| \quad \text{for } t \in J \text{ and } n \in \mathbb{N}, \tag{3.25}$$

where $\xi_n \in (0, T)$, $x_n(\xi_n) = x_n'(\xi_n) = 0$ and

$$\|x_n\| < P, \quad \|x_n'\| < P \quad \text{for } n \in \mathbb{N}, \tag{3.26}$$

where P is a positive constant. Then $g^{-1}(x_n(t)) < g^{-1}(P)$ for $t \in J$, $n \in \mathbb{N}$ and (see (3.1) and (3.2))

$$\begin{aligned} a &\leq f_n(t, g^{-1}(x_n(t)), x_n'(t)) \\ &\leq [h_1(g^{-1}(P) + 1) + h_2(g^{-1}(x_n(t)))][\omega_1(P + 1) + \omega_2(|x_n'(t)|)], \end{aligned} \tag{3.27}$$

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for a.e. $t \in J$ and for $n \in \mathbb{N}$. Now, from (3.27) and the inequalities

$$h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|) \geq h_2(g^{-1}(P))\omega_2(|x_n'(t)|),$$
$$h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|) \geq h_2(g^{-1}(x_n(t)))\omega_2(P),$$

we see that the sequence (3.24) is UAC on J if $\{h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\}$ is. From the structure of the measurable set on J we deduce that the sequence $\{h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\}$ is UAC on J if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$, $\sum_{j\in\mathbb{J}}(b_j - a_j) < \delta$, we have

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\,dt < \varepsilon \quad (n \in \mathbb{N}).$$

Therefore, let $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ be an at most countable set of mutually disjoint intervals $(a_j, b_j) \subset J$ and set

$$\mathbb{J}_n^1 = \{j : j \in \mathbb{J},\ (a_j, b_j) \subset (0, \xi_n)\},\ \ \mathbb{J}_n^2 = \{j : j \in \mathbb{J},\ (a_j, b_j) \subset (\xi_n, T)\}.$$

Then for $i \in \mathbb{J}_n^1$ and $j \in \mathbb{J}_n^2$ we have (see (3.25))

$$\begin{aligned}\int_{a_i}^{b_i} h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\,ds &\leq \int_{a_i}^{b_i} h_2\left(g^{-1}\left(\frac{a}{2}(\xi_n - t)^2\right)\right)\omega_2(a(\xi_n - t))\,dt\\ &= \frac{1}{a}\int_{a(\xi_n-b_i)}^{a(\xi_n-a_i)} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_{a_j}^{b_j} h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\,ds &\leq \int_{a_j}^{b_j} h_2\left(g^{-1}\left(\frac{a}{2}(\xi_n - t)^2\right)\right)\omega_2(a(t - \xi_n))\,dt\\ &= \frac{1}{a}\int_{a(a_j-\xi_n)}^{a(b_j-\xi_n)} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds.\end{aligned}$$

If $a_{j_n} < \xi_n < b_{j_n}$ for some $j_n \in \mathbb{J}$, then

$$\begin{aligned}\int_{a_{j_n}}^{b_{j_n}} h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\,dt &\leq \int_{a_{j_n}}^{\xi_n} h_2\left(g^{-1}\left(\frac{a}{2}(\xi_n - t)^2\right)\right)\omega_2(a(\xi_n - t))\,dt\\ &\quad + \int_{\xi_n}^{b_{j_n}} h_2\left(g^{-1}\left(\frac{a}{2}(\xi_n - t)^2\right)\right)\omega_2(a(t - \xi_n))\,dt\\ &= \frac{1}{a}\left[\int_0^{a(\xi_n-a_{j_n})} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds\right.\\ &\quad \left. + \int_0^{a(b_{j_n}-\xi_n)} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds\right].\end{aligned}$$

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Set

$$\mathcal{M}_n^1 = \mathcal{E}_n^1 \cup \bigcup_{i\in\mathbb{J}_n^1} (a(\xi_n - b_i), a(\xi_n - a_i)), \quad \mathcal{M}_n^2 = \mathcal{E}_n^2 \cup \bigcup_{j\in\mathbb{J}_n^2} (a(a_j - \xi_n), a(b_j - \xi_n)),$$

where

$$\mathcal{E}_n^1 = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}_n^1 \cup \mathbb{J}_n^2, \\ \left(0, a\left(\xi_n - a_{j_n}\right)\right) & \text{if } \{j_n\} = \mathbb{J}\setminus\left(\mathbb{J}_n^1 \cup \mathbb{J}_n^2\right), \end{cases}$$

$$\mathcal{E}_n^2 = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}_n^1 \cup \mathbb{J}_n^2, \\ \left(0, a\left(b_{j_n} - \xi_n\right)\right) & \text{if } \{j_n\} = \mathbb{J}\setminus\left(\mathbb{J}_n^1 \cup \mathbb{J}_n^2\right). \end{cases}$$

Then

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\,dt$$
$$\leq \int_{\mathcal{M}_n^1} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds + \int_{\mathcal{M}_n^1} h_2\left(g^{-1}\left(\frac{s^2}{2a}\right)\right)\omega_2(s)\,ds.$$

By (H_5), $h_2(g^{-1}(s^2/(2a)))\omega_2(s) \in L_1([0, aT])$ and, since $\mu(\mathcal{M}_n^k) \leq a\sum_{j\in\mathbb{J}}(b_j - a_j)$ for $n \in \mathbb{N}$ and $k = 1, 2$, we see that $\{h_2(g^{-1}(x_n(t)))\omega_2(|x_n'(t)|)\}$ is UAC on J which finishes the proof. □

4. Existence results and examples.

THEOREM 4.1. *Let assumptions* (H_1) *and* $(H_3)-(H_5)$ *be satisfied. Then BVP* (1.1), (1.2) *has a solution.*

Proof. By Lemma 2.1 (see also Remark 2.2), the solvability of BVP (1.1), (1.2) is equivalent to that of BVP (2.1), (1.2). Theorem 4.1 will be proved if BVP (2.1), (1.2) has a solution.

By Lemma 3.4, BVP $(\mathrm{E})_n^1$, (1.2) has a solution x_n for each $n \in \mathbb{N}$. Also Lemmas 3.1 and 3.3 guarantee the validity of inequalities (3.25) and (3.26), where P is a positive constant and $\xi_n \in (0, T)$, $x_n(\xi_n) = x_n'(\xi_n) = 0$. In addition (see Lemma 3.5), $\{f_n(t, g^{-1}(x_n(t)), x_n'(t))\}$ is UAC on J and therefore $\{x_n'(t)\}$ is equicontinuous on J. Going if necessary to a subsequence, we can assume, by the Arzelà-Ascoli theorem and the compactness principle, that $\{x_n\}$ is convergent in $C^1(J)$ and $\{\xi_n\}$ in $\mathbb{R}$. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \xi_n = \xi$. Then x satisfies the boundary conditions (1.2) and (see (3.25)) $x(t) \geq (a/2)(\xi - t)^2$, $|x'(t)| \geq a|\xi - t|$ for $t \in J$. Thus $x(t) > 0$ and $|x'(t)| > 0$ for $t \in J\setminus\{\xi\}$, and $f(t, g^{-1}(x(t)), x'(t))$ is defined almost everywhere. Also

$$\lim_{n\to\infty} f_n(t, g^{-1}(x_n(t)), x_n'(t)) = f(t, g^{-1}(x(t)), x'(t)) \quad \text{for a.e. } t \in J.$$

Now, by the Vitali's convergence theorem, $f(t, g^{-1}(x(t)), x'(t)) \in L_1(J)$ and

$$\lim_{n\to\infty} \int_0^t f_n(s, g^{-1}(x_n(s)), x_n'(s))\,ds = \int_0^t f(s, g^{-1}(x(s)), x'(s))\,ds \quad (t \in J).$$

Letting $n \to \infty$ in the equalities

$$x_n'(t) = x_n'(0) + \int_0^t f_n(s, g^{-1}(x_n(s)), x_n'(s))\, ds \quad (t \in J,\ n \in \mathbb{N}),$$

we get

$$x'(t) = x'(0) + \int_0^t f(s, g^{-1}(x(s)), x'(s))\, ds \quad (t \in J).$$

Hence $x \in AC^1(J)$ and x is a solution of BVP (2.1), (1.2). □

COROLLARY 4.2. *Let assumptions* $(H_2) - (H_4)$ *and* (H_6) *be satisfied. Then BVP* (1.3), (1.2) *has a solution.*

Proof. Using the function G defined in (1.4) we can write equation (1.3) in the form

$$(G(x(t))'' = f(t, x(t), (G(x(t))'), \tag{4.1}$$

which is equation (1.1) with G instead of g. Since assumption (H_6) is obtained from assumption (H_5) with G instead of g, we see that BVP (4.1), (1.2) has a solution x, by Theorem 4.1, such that $x \in C^0(J)$ and $G(x) \in AC^1(J)$. Set $y(t) = G(x(t))$ for $t \in J$. Then $y \in AC^1(J)$ and from $x(t) = G^{-1}(y(t))$ we see that $x \in C^1(J)$ by (H_2), and so $g(x)x' = (G(x))' \in AC(J)$. Consequently, x is a solution of BVP (1.3), (1.2). □

EXAMPLE 4.3. Consider the differential equation

$$(x^p)'' = c_0\left(1 + c_1 x^\alpha + \frac{c_2}{x^\beta}\right)\left(1 + c_3|(x^p)'|^\gamma + \frac{c_4}{|(x^p)'|^\delta}\right), \tag{4.2}$$

where $p \in (0,\infty)$, $c_0, c_2, c_4 \in (0,\infty)$, $c_1, c_3 \in [0,\infty)$, $\alpha, \beta, \gamma \in (0,\infty)$, $\delta \in (0,1)$ and $2\beta < p(1-\delta)$. Equation (4.2) is the special case of (1.1) with $g(u) = u^p$ satisfying (H_1), and

$$f(t,x,y) = c_0\left(1 + c_1 x^\alpha + \frac{c_2}{x^\beta}\right)\left(1 + c_3|y|^\gamma + \frac{c_4}{|y|^\delta}\right). \tag{4.3}$$

We see that (H_3) is true with $a = \min\{1/2, c_0\}$ and (H_4) with

$$h_1(u) = c_0(1 + c_1 u^\alpha),\quad h_2(u) = \frac{c_0 c_2}{u^\beta},\quad \omega_1(u) = 1 + c_3 u^\gamma,\quad \omega_2(u) = \frac{c_4}{u^\delta}.$$

We now verify (H_5). Notice that

$$\int_0^1 h_2(g^{-1}(s^2))\omega_2(s)\, ds = c_0 c_2 c_4 \int_0^1 s^{-(\delta + \frac{2\beta}{p})}\, ds = \frac{p c_0 c_2 c_4}{(1-\delta)p - 2\beta} < \infty$$

and by a calculation we can show that there exist positive constants A, B and $u_0 \in (0,\infty)$ such that for $u \geq u_0$ we have

$$H_1(u) = \int_0^u [h_1(g^{-1}(s) + 1) + h_2(g^{-1}(s))]\, ds < \begin{cases} Au^{\frac{p+\alpha}{p}} & \text{if } c_1 > 0, \\ Au & \text{if } c_1 = 0, \end{cases}$$

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$$K(u) = \int_0^u \frac{s}{\omega_1(s+1)+\omega_2(s)}\,ds > \begin{cases} Bu^{2-\gamma} & \text{if } c_3 > 0, \\ Bu^2 & \text{if } c_3 = 0. \end{cases}$$

Hence there exists $u_1 \geq u_0$ such that for $u \geq u_1$ we have

$$K^{-1}(H_1(u)) < \begin{cases} \sqrt{\frac{A}{B}}u^{\frac{p+\alpha}{2p}} & \text{if } c_1 > 0,\ c_3 = 0, \\ \sqrt{\frac{A}{B}u} & \text{if } c_1 = 0,\ c_3 = 0, \\ \sqrt[2-\gamma]{\frac{A}{B}}u^{\frac{p+\alpha}{p(2-\gamma)}} & \text{if } c_1 > 0,\ c_3 > 0, \\ \sqrt[2-\gamma]{\frac{A}{B}}u^{\frac{1}{2-\gamma}} & \text{if } c_1 = 0,\ c_3 > 0. \end{cases}$$

Finally, from the last inequalities we deduce that if one of the cases

(a) $\alpha < p$ if $c_1 > 0$, $c_3 = 0$,
(b) $c_1 = c_3 = 0$,
(c) $\alpha < p(1-\gamma)$ if $c_1 > 0$, $c_3 > 0$,
(d) $\gamma \in (0, 1)$ if $c_1 = 0$ and $c_3 > 0$

occurs, we have

$$\lim_{u\to\infty} \int_0^u \frac{1}{K^{-1}(H_1(s))}\,ds = \infty.$$

Applying Theorem 4.1, BVP (4.2), (1.2) has a solution if one of the cases (a)–(d) is satisfied.

EXAMPLE 4.4. Consider the differential equation

$$\left(\frac{x'(t)}{(\max\{1, x(t)\})^p}\right)' = c_0(x(t))^\alpha + \frac{c_1}{(x(t))^\beta} + \frac{c_2}{|x'(t)|^\gamma}, \tag{4.4}$$

where $p \in (0, 1)$, $\alpha, \beta, \gamma, c_i$ are positive constants ($i = 0, 1, 2$) and

$$2\beta + \gamma < 1, \quad \alpha < 1 - p. \tag{4.5}$$

Equation (4.4) is the special case of (1.3) with $g(u) = 1/(\max\{1, u\})^p$ satisfying (H_2) since

$$G(u) = \int_0^u g(s)\,ds = \begin{cases} u & \text{for } u \in [0, 1], \\ \frac{u^{1-p}-p}{1-p} & \text{for } u \in (1, \infty), \end{cases}$$

and

$$f(t, x, y) = c_0 x^\alpha + \frac{c_1}{x^\beta} + \frac{c_2}{(\max\{1, x\})^{p\gamma}|y|^\gamma}.$$

We can see that (H_3) is satisfied with $a = \min\{1/2, c_0, c_1\}$ and (H_4) with

$$h_1(u) = cu^\alpha, \quad h_2(u) = c\left(\frac{1}{u^\beta} + \frac{1}{(\max\{1, u\})^{p\gamma}}\right), \quad \omega_1(u) = 1, \quad \omega_2(u) = \frac{1}{u^\gamma},$$

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where $c = \max\{c_0, c_1, c_2\}$. We shall show that (4.5) guarantees the validity of (H_6). Since

$$G^{-1}(u) = \begin{cases} u & \text{for } u \in [0, 1], \\ \sqrt[1-p]{(1-p)u + p} & \text{for } u \in (1, \infty), \end{cases}$$

we have

$$\int_0^1 h_2(G^{-1}(s^2))\omega_2(s)\, ds = c \int_0^1 \left(\frac{1}{s^{2\beta+\gamma}} + \frac{1}{s^\gamma} \right) ds < \infty.$$

Further for $u \geq 1$,

$$\begin{aligned} H_2(u) &= \int_0^u [h_1(G^{-1}(s) + 1) + h_2(G^{-1}(s))]\, ds = c \int_0^1 \left[(s+1)^\alpha + \frac{1}{s^\beta} + 1 \right] ds \\ &\quad + c \int_1^u \left[(\sqrt[1-p]{(1-p)s + p} + 1)^\alpha + \frac{1}{(\sqrt[1-p]{(1-p)s + p}\,)^\beta} \right. \\ &\quad \left. + \frac{1}{(\sqrt[1-p]{(1-p)s + p}\,)^{p\gamma}} \right] ds \end{aligned}$$

and, for $u \geq 0$, we have

$$K(u) = \int_0^u \frac{s}{\omega_1(s+1) + \omega_2(s)}\, ds = \int_0^u \frac{s^{1+\gamma}}{1 + s^\gamma}\, ds.$$

Thus there exist a positive constant A and $u_1 \in (1, \infty)$ such that

$$H_2(u) < Au^{1+\frac{\alpha}{1-p}}, \quad K(u) > Au^2 \quad \text{for } u \geq u_1. \tag{4.6}$$

Now from (4.6) we deduce that

$$K^{-1}(H_2(u)) < \sqrt{u^{1+\frac{\alpha}{1-p}}} \quad (u \geq u_2), \tag{4.7}$$

where $u_2\,(\geq u_1)$ is a sufficiently large number. Since $\alpha < 1 - p$ by (4.5), we see that

$$\lim_{u\to\infty} \int_0^u \frac{1}{K^{-1}(H_2(s))}\, ds = \infty.$$

We have verified that (H_6) is true. Applying Theorem 4.1, BVP (4.4), (1.2) has a solution.

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