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FIXED POINTS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

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1. Introduction. In his fundamental paper [**3**], Hurwitz showed that the order of a group of biholomorphic transformations of a compact Riemann surface S into itself is bounded above by $84(g - 1)$ when S has genus $g \geqq 2$. This bound on the group of automorphisms (as we shall call the biholomorphic self-transformations) is attained for Klein's quartic curve of genus 3 [**4**] and, from this, Macbeath [**7**] deduced that the Hurwitz bound is attained for infinitely many values of g.

After genus 3, the next smallest genus for which the bound is attained is the case $g = 7$. The equations of such a curve of genus 7 were determined by Macbeath [**8**] who also gave the equations of the transformations. The equations of these transformations were found by using the Lefschetz fixed point formula. If the number of fixed points of each element of a group of automorphisms is known, then the Lefschetz fixed point formula may be applied to deduce the character of the representation given by the group acting on the first homology group of the surface. In this paper we shall determine the number of fixed points of each element of a cyclic group of automorphisms of a compact Riemann surface whose genus is at least two.

2. Fuchsian groups. We shall approach the problem using the concept of Fuchsian groups. Details of the theory are to be found in [**5**; **1**]. A *Fuchsian group* is a discrete subgroup of the hyperbolic group $\mathrm{LF}(2, R)$ of linear fractional transformations

$$w = \frac{az + b}{cz + d} \qquad (a, b, c, d \text{ real}, ad - bc = 1),$$

each such transformation mapping the complex upper half-plane D into itself. If Γ is a Fuchsian group and $z \in D$, then the images of z under Γ form a Γ-*orbit* and the orbits, with the identification topology, form the *orbit space*, denoted by D/Γ. Since we shall only be concerned with the situation where D/Γ is compact, we shall use the term *Fuchsian group* to mean a discrete subgroup of $\mathrm{LF}(2, R)$ which has a compact orbit space. The orbit space is given an analytic structure such that the projection mapping $p: D \to D/\Gamma$ is holomorphic.

If K is a normal subgroup of a Fuchsian group Γ, then the factor group $G = \Gamma/K$ acts as a group of automorphisms of the Riemann surface D/K

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for, if $x \in \Gamma$ and $z \in D$, then $xK \in \Gamma/K$, $Kz \in D/K$ and we have $(xK)(Kz) = Kxz$. This is easily seen to be independent of the choice of x in its K-coset and the choice of z in its K-orbit.

Conversely, if S is a compact Riemann surface of genus $g \geqq 2$, then S can be identified with D/K, where K is a Fuchsian group acting without fixed points in D. Moreover, if S admits a group of automorphisms G, there is a Fuchsian group Γ, with K as a normal subgroup, such that $G = \Gamma/K$ and the action of G on S coincides with that described above.

When a Fuchsian group Γ has a compact orbit space, then it is known to have the following structure:

$$\text{(1)} \qquad \begin{aligned} \text{generators:}\quad & x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma, \\ \text{relations:}\quad & x_1^{m_1} = x_2^{m_2} = \ldots . = x_r^{m_r} = 1, \\ & x_1 x_2 \ldots x_r \prod_{i=1}^{\gamma} a_i b_i a_i^{-1} b_i^{-1} = 1. \end{aligned}$$

The integers $m_1, m_2, \ldots, m_r$ are called the *periods* of Γ, and γ is the genus of the orbit space D/Γ. An element of Γ has a fixed point in D if and only if it has finite order and it is then conjugate to some power of precisely one of the x_is. A Fuchsian group which has no fixed points in D, and hence no periods, is called a Fuchsian *surface group*.

3. Fixed points of automorphisms. Let G be a group of automorphisms of order n of a compact Riemann surface S of genus at least two. We identify S with D/K, where K is a Fuchsian surface group and take Γ to be the Fuchsian group such that $\Gamma/K = G$. We have a projection map $p: D \to S$ and a homomorphism $p^*: \Gamma \to G$ with kernel K, such that

$$p(xz) = p^*(x)p(z)$$

for $x \in \Gamma$ and $z \in D$. Thus p maps a Γ-orbit in D onto a G-orbit in S. If $z \in D$ is fixed by some element $x \in \Gamma$, then $xz = z$, so that $p^*(x)p(z) = p(z)$ and p maps fixed points to fixed points while p^* maps the stabilizer of z to the stabilizer of $p(z)$. In fact, p^* induces an isomorphism between stabilizers; for, if p' denotes the restriction of p^* to stab(z), then p' is one-to-one since $\ker p' = \ker p^* \cap \text{stab}(z) = K \cap \text{stab}(z) = \{1\}$ since K has no fixed points. To show that p' is onto, let $y = p(z)$ and suppose that $ty = y$ for $t \in G$. Choose $x \in (p^*)^{-1}(t)$ and let $z_1 = xz$. Then

$$p(z_1) = p(xz) = p^*(x)p(z) = ty = y = p(z).$$

Hence, there is a $k \in K$ such that $kz_1 = z$ and so

$$kxz = kz_1 = z.$$

Thus $kx \in \text{stab}(z)$ and $p^*(kx) = p^*(k)p^*(x) = 1.\ t = t$ since $k \in K = \ker p^*$.

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If R_1 is a fundamental region for the Fuchsian group Γ with presentation (1), then the non-Euclidean measure of R_1 is given by

$$\int_{R_1}\int \frac{dx\,dy}{y^2} = \mu(R_1) = 2\pi\left\{2\gamma - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right\}.$$

If R_2 is a fundamental region for K, then a union of n copies tR_2 is a fundamental region for Γ when the elements t form a complete system of representatives of cosets Kt of K in Γ. Since the measure is invariant under $\mathrm{LF}(2, R)$, we have

$$n = \text{order of } G = \frac{\mu(R_2)}{\mu(R_1)} = \frac{2\pi(2g-2)}{2\pi\left\{2\gamma - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right\}},$$

where g is the genus of S and γ is the genus of Γ. We thus have a form of the Riemann-Hurwitz relation:

$$(2)\qquad 2g - 2 = n\left\{2\gamma - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right\}.$$

The points of D with non-trivial stabilizers in Γ fall into r Γ-orbits $F_1', F_2', \ldots, F_r'$, such that every point belonging to F_i' has a stabilizer which is cyclic of order m_i. When we project by p, the points of S with non-trivial stabilizers fall into r G-orbits $F_1, F_2, \ldots, F_r$ where $F_i = p(F_i')$. Since the projection of stabilizers is an isomorphism, the stabilizer of y, for $y \in F_i$, is cyclic of order m_i.

Let $F = F_1 \cup F_2 \cup \ldots \cup F_r$ and take A to be the subset of $G \times F$ given by

$$A = \{(t, y) : 1 \neq t \in G,\ ty = y\}.$$

Then the number of elements in A is given by

$$\begin{aligned}\sum_{i=1}^{r} &(\text{number of } (t, y) \in A \text{ such that } y \in F_i)\\ &= \sum_{i=1}^{r}\sum_{y \in F_i}(\text{number of } t \neq 1 \text{ with } ty = y)\\ &= \sum_{i=1}^{r}\sum_{y \in F_i}(m_i - 1)\\ &= \sum_{i=1}^{r}(m_i - 1)\,(\text{number of elements in } F_i).\end{aligned}$$

Since the order of G is n, by the orbit stabilizer relation,

$$(\text{number of elements in } F_i) \times m_i = n,$$

and so the number of elements in A is

$$\sum_{i=1}^{r}(m_i - 1)\frac{n}{m_i} = n\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

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However, the number of elements in A is also given by

$$\sum_{1\neq t\in G} (\text{number of } y \text{ such that } ty = y).$$

Thus, if we denote the number of fixed points of an element $t \in G$ by $N(t)$, we have

$$\sum_{1\neq t\in G} N(t) = n\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right). \tag{3}$$

Substituting (2) in the expression (3) yields

$$\sum_{1\neq t\in G} N(t) = 2g - 2 - n(2\gamma - 2). \tag{4}$$

If we consider the case where n is prime, then G is a cyclic group and all the elements of G, distinct from the identity, have the same fixed point set. Thus we can obtain the number of fixed points of any element of G as (4) becomes

$$(n - 1)N(t) = 2g - 2 - n(2\gamma - 2) \tag{5}$$

for any $t \in G$, $t \neq 1$.

In the case $n \geqq g$ (n not necessarily prime) there are only two possibilities for γ, namely 0 and 1. For, if we assume $\gamma > 1$, then, from (2), we obtain $2g - 2 \geqq 2n$, contrary to $n \geqq g$.

For prime n, $n \geqq g$, we thus have two cases to consider:

(a) $\gamma = 1$. In this case (5) reduces to $N(t) = 2(g - 1)/(n - 1)$. If $n > g$, then we have $0 < N(t) < 2$, and so $N(t) = 1$. In this case let y_0 be the unique fixed point of t; then there is a single Γ-orbit of fixed points in D so that Γ has only a single period. Thus Γ is given by generators t', a, b and relations $t'^n = t'aba^{-1}b^{-1} = 1$. Then $p^*(t') = p^*(bab^{-1}a^{-1}) = 1$ since G is abelian. Thus $t' \in K$, which contradicts the fact that K is a surface group. Hence the only possibility is that $n = g$, in which case

$$N(t) = \frac{2(g - 1)}{n - 1} = 2.$$

(b) $\gamma = 0$. If $n = g$, then, from (5), $(n - 1)N(t) = 2n - 2 - n(-2)$ and so

$$N(t) = 2 + \frac{2n}{n - 1}.$$

Since this must be an integer, the only possibilities for n are two and three. Since these are both prime, the two cases can occur and

$$n = g = 2 \text{ yields } N(t) = 6,$$

while

$$n = g = 3 \text{ yields } N(t) = 5.$$

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When $n > g$,

$$N(t) = \frac{2g + 2n - 2}{n - 1} = 2 + \frac{2g}{n - 1}; \tag{6}$$

whence $N(t) > 2$ and, since $2g + 2n - 2 < 4n - 2$,

$$N(t) < \frac{4n - 2}{n - 1} = 4 + \frac{2}{n - 1} \leqq 5 \quad \text{since } n \geqq 3.$$

Hence $2 < N(t) < 5$ so that $N(t) = 3$ or 4.

If $N(t) = 3$, then, from (6), this is equivalent to $n = 2g + 1$ while $N(t) = 4$ is equivalent to $n = g + 1$.

Thus we have shown the following.

THEOREM 1. *There are only two possible prime orders greater than g for a group of automorphisms G of a Riemann surface S of genus g.*

(i) $n = 2g + 1$: *each element of G has three fixed points.*

(ii) $n = g + 1$: *each element of G has four fixed points.*

This agrees with the results obtained by Lewittes [**6**] for the case where S is a hyperelliptic surface.

All the cases mentioned do occur and for each case we give a Fuchsian group Γ and a homomorphism θ from Γ onto Z_n, the cyclic group of order n. Then N, the kernel of θ, will be seen to be a surface group and $S = D/N$ will be a surface with Z_n as a group of automorphisms.

Let z be a generator of Z_n and let γ be the genus of Γ.

$\gamma = 1, n = g.$ $\quad \Gamma: x_1^n = x_2^n = x_1x_2aba^{-1}b^{-1} = 1;$
$\theta(x_1) = z,\ \theta(x_2) = z^{n-1},\ \theta(a) = \theta(b) = 1.$

$\gamma = 0, n = g = 2.$ $\quad \Gamma: x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = x_1x_2x_3x_4x_5x_6 = 1;$
$\theta(x_i) = z,\ 1 \leqq i \leqq 6.$

$\gamma = 0, n = g = 3.$ $\quad \Gamma: x_1^3 = x_2^3 = x_3^3 = x_4^3 = x_5^3 = x_1x_2x_3x_4x_5 = 1;$
$\theta(x_i) = z,\ 1 \leqq i \leqq 4,\ \theta(x_5) = z^2.$

$\gamma = 0, n = 2g + 1.$ $\quad \Gamma: x_1^n = x_2^n = x_3^n = x_1x_2x_3 = 1;$
$\theta(x_1) = \theta(x_2) = z,\ \theta(x_3) = z^{n-2}.$

$\gamma = 0, n = g + 1.$ $\quad \Gamma: x_1^n = x_2^n = x_3^n = x_4^n = x_1x_2x_3x_4 = 1;$
$\theta(x_1) = \theta(x_2) = z,\ \theta(x_3) = \theta(x_4) = z^{n-1}.$

In each case no element of finite order lies in the kernel of θ and so, for $N = \ker \theta$, we have that N is a surface group. If the genus of the orbit space D/N is g', then it is straightforward to verify that $g' = g$ using relation (2) for Γ and N. Thus, in each case, $S = D/N$ is a surface of genus g with the required property.

4. Cyclic groups. We now use equation (4) to investigate the case when G is a cyclic group of automorphisms of the surface S of order n. Necessary and sufficient conditions for the existence of a homomorphism θ from a Fuchsian group Γ onto a cyclic group Z_n, having a surface group as its kernel, were given by Harvey [**2**, pp. 36–37].

If t is any element of G whose powers exhaust G and, if d is any divisor of n, then t^d generates a subgroup of G of order $d' = n/d$. Denoting this subgroup by $G_{d'}$, there is a subgroup $\Gamma_{d'}$ of Γ corresponding to $G_{d'}$, of genus $\gamma_{d'}$ say. We may form the orbit space $S/G_{d'}$ in the same manner as for orbit spaces of D. Then $D/\Gamma_{d'} \simeq (D/K)/(\Gamma_{d'}/K) \simeq S/G_{d'}$ so that $\gamma_{d'}$ is the genus of $S/G_{d'}$. Applying (4) to this subgroup, the elements distinct from the identity in $G_{d'}$ have a total number of fixed points given by

$$\sum_{i=1}^{d'-1} N(t^{id}) = 2g - 2 - d'(2\gamma_{d'} - 2). \tag{7}$$

LEMMA 1. *If* $(m, n) = m'$ *is the highest common factor of* m *and* n, *then*

$$N(t^m) = N(t^{m'}).$$

Proof. Let $F(t^\alpha)$ denote the set of points fixed by t^α; then, clearly,

$$F(t^\alpha) \subseteq F(t^{\beta\alpha})$$

for any integer β.

Thus, for $m = \alpha m'$, $n = \beta m'$, we have

$$F(t^{m'}) \subseteq F(t^{\alpha m'}) = F(t^m).$$

Since α and β are co-prime, by the Euclidean algorithm, there are integers λ and μ such that

$$\lambda\alpha + \mu\beta = 1.$$

Then $t^{\lambda m} = t^{\lambda\alpha m'} = t^{m'(1-\mu\beta)} = t^{m'} \cdot t^{-\mu n} = t^{m'}$. Hence, $F(t^m) \subseteq F(t^{\lambda m}) = F(t^{m'})$ so that $F(t^m) = F(t^{m'})$ and $N(t^m) = N(t^{m'})$, as required.

We now consider the sum

$$-\sum_{1 \neq d \mid n} \mu(d)[2g - 2 - d'(2\gamma_{d'} - 2)] \tag{8}$$

for $dd' = n$, where $\mu(d)$ is the Möbius function defined by $\mu(1) = 1$, $\mu(d) = 0$ if d has a squared factor, and $\mu(\rho_1\rho_2 \ldots \rho_k) = (-1)^k$ if all the primes ρ_j are distinct. Using (7), the sum (8) is equal to

$$-\sum_{1 \neq d \mid n} \mu(d) \sum_{i=1}^{d'-1} N(t^{id}),$$

so that if $1 < m < n$ and $(m, n) = m' \neq 1$, then each fixed point of t^m is counted $-\sum_{1 \neq d \mid m'} \mu(d)$ times. Then, since $\sum_{d \mid m'} \mu(d) = 0$, each is counted only once. If $m' = 1$, then the fixed points of t^m are not counted in the sum. Thus the total number of fixed points of G is given by

$$\sum_{(m,n)=1} N(t^m) - \sum_{1 \neq d \mid n} \mu(d)[2g - 2 - d'(2\gamma_{d'} - 2)]. \tag{9}$$

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Now, from Lemma 1, each t^m such that $(m, n) = 1$ has the same number of fixed points as t. The number of such m is given by the Euler function

$$\varphi(n) = n\prod_{\rho|n}\left(1 - \frac{1}{\rho}\right) \qquad (\rho \text{ prime}).$$

The total number of fixed points of G is also given by $2g - 2 - n(2\gamma_n - 2)$, so that equating this with (9) yields

$$\varphi(n)N(t) - \sum_{1\neq d|n} \mu(d)[2g - 2 - d'(2\gamma_{d'} - 2)] = 2g - 2 - n(2\gamma_n - 2).$$

Hence $\varphi(n)N(t) = \sum_{d|n} \mu(d)[2g - 2 - d'(2\gamma_{d'} - 2)]$ and, since

$$\sum_{dd'=n} \mu(d) = 0,$$

we have

$$N(t) = \frac{1}{\varphi(n)}\sum_{d|n} d'\mu(d)(2 - 2\gamma_{d'}). \tag{10}$$

For t^m we have $N(t^m) = N(t^{m'})$, where $m' = (m, n)$ and $t^{m'}$ generates the subgroup $G_{n/m}$; thus by applying (10) to this group we have the following result.

THEOREM 2. *Let G be a cyclic group, of order n, of automorphisms of a compact Riemann surface S of genus $g \geqq 2$ and, for $d|n$, let γ_d denote the genus of the orbit space S/G_d, where G_d is the subgroup of G of order d. For t a generator of G, the number of fixed points of t^m is given by*

$$N(t^m) = \frac{1}{\varphi(n')}\sum_{dd'=n'} d'\mu(d)(2 - 2\gamma_{d'}),$$

where $n' = n/(m, n)$.

We may alternatively compute the number of fixed points of an element of G using the periods of the Fuchsian group Γ. Let r_d be the number of periods m_i of Γ such that $m_i = d$. Since the projection of stabilizers is an isomorphism, if $r_d \neq 0$, then d must divide n. Since the ordering of the periods of Γ is immaterial, we may suppose that Γ has generators

$$x_{21}, x_{22}, \ldots, x_{2r_2}, x_{31}, \ldots, x_{n-1,r_{n-1}}, x_{n1}, \ldots x_{nr_n}; \quad a_1, b_1, \ldots, a_\gamma, b_\gamma$$

and relations

$$x_{ij}^i = 1 \qquad (j = 1, 2, \ldots, r_i; \quad i = 2, 3, \ldots, n),$$

$$\prod_{i=2}^{n}\prod_{j=1}^{r_i} x_{ij}\prod_{k=1}^{\gamma} a_k b_k a_k^{-1} b_k^{-1} = 1.$$

For convenience, i has been allowed to take all values from 2 up to n although, for example, r_{n-1} will always be zero since $(n - 1, n) = 1$.

As before, we take t to be a generator of G and for $d|n$ we denote $t^{n/d}$ by t_d so that t_d has order d.

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Suppose that $y \in S$ is a fixed point of G and that the stabilizer of y in G has order q. If y is a fixed point of t_d, for $d > 1$, then $t_d \in \operatorname{stab}(y)$ and so $d|q$. The fixed points of the elements of G can be divided into classes C_q characterized by the order q of the stabilizer. Now, for $y \in C_q$, there is an element t', of order q, such that $t'y = y$ and t' generates the subgroup G_q of G. Since $t_q \in G_q$, there is an integer β such that $t'^{\beta} = t_q$ and hence $t_q y = y$ so that t_q, being of order q, generates the stabilizer of y. Any point in the G-orbit of y has a conjugate stabilizer, and hence is also a member of C_q.

Let $\{x_{qi}\}$ denote the conjugacy class of x_{qi} in Γ. Then x_{qi} fixes a point $z \in D$ and, for $x \in \Gamma$, $xx_{qi}x^{-1}$ fixes $x(z)$ and $p(x(z)) = p^*(x)p(z)$ so that if z' is the fixed point of a conjugate of x_{qi}, then $p(z')$ belongs to the G-orbit of $p(z)$. Since the stabilizer of z has order q, so has the stabilizer of $p(z)$, and hence $p(z) \in C_q$. Conversely, if $y \in C_q$, then, for $z \in p^{-1}(y)$, z has a stabilizer of order q and $\operatorname{stab}(z)$ is a finite cyclic subgroup of Γ, which thus has a generator conjugate to x_{qi} for some i.

LEMMA 2. *The G-orbits of fixed points of class C_q are in one-to-one correspondence with the conjugacy classes of the x_{qi}, $i = 1, 2, \ldots, r_q$.*

Proof. Define $f(\{x_{qi}\}) = Gy_1$, where $y_1 = p(z_1)$ is the projection of the fixed point z_1 of x_{qi} in D; then, by the above remarks, f is a well-defined mapping onto the G-orbits in C_q.

Suppose that $f(\{x_{qi}\}) = f(\{x_{qj}\})$; then, if z_2 is the fixed point of x_{qj} in D, there is a $t' \in G$ such that

$$t'p(z_1) = p(z_2).$$

Take $x \in (p^*)^{-1}(t')$; then $p(x(z_1)) = t'p(z_1) = p(z_2)$, and so there is a $k \in K$ such that $kx(z_1) = z_2$. Thus

$$(kx)^{-1}x_{qj}kx(z_1) = (kx)^{-1}x_{qj}(z_2) = (kx)^{-1}(z_2) = z_1$$

and $(kx)^{-1}x_{qj}kx \in \operatorname{stab}(z_1)$.

Now $\operatorname{stab}(z_1)$ is generated by x_{qi} so that x_{qj} is conjugate to some power of x_{qi} and the only way that this can happen is for $x_{qi} = x_{qj}$. Thus f is one-to-one and yields the correspondence asserted.

By the orbit stabilizer relation, the number of points in the G-orbit of $y \in C_q$ is $n/q = q'$, say. The number of conjugacy classes in Γ of elements of order q is r_q. Thus the number of points in C_q is $q'r_q$.

THEOREM 3. *Let r_q denote the number of periods m_i of Γ such that $m_i = q$. Then if $d'|n$, $d' \neq n$, the number of fixed points of $t^{d'}$ for t a generator of G is given by*

$$N(t^{d'}) = \sum_{\delta\delta' = d'} \delta' r_{\delta d},$$

where $dd' = n$.

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Proof. Now, if $t^{d'}$ fixes a point $y \in C_q$, then $t^{d'} = t_d \in \text{stab}(y)$ and we have seen that t_q generates stab(y), and so every point of C_q is fixed by t_d. Conversely, if $d|q$, say $q = \alpha d$, then

$$t_q^{\alpha} = t^{n\alpha/q} = t^{n/d} = t_d$$

and for each q, such that $d|q$, there is a class of fixed points C_q of t_d. Hence the total number of fixed points of t_d is given by

$$N(t_d) = \sum_{q:\, d|q|n} q' r_q,$$

where $qq' = n$.

If we now let $\delta' = q'$, then since $d|q$, $\delta' = q'|d'$ and we take δ given by $\delta\delta' = d'$ so that

$$q'\delta d = \delta'\delta d = d'd = n$$

and so $q = \delta d$. Thus

$$N(t_d) = \sum_{\delta\delta' = d'} \delta' r_{\delta d}.$$

For a cyclic group of automorphisms G of a Riemann surface S we thus have two expressions for the number of fixed points of an element of G; one in terms of the genera of factor spaces of S by subgroups of G and the other in terms of the periods of the Fuchsian group Γ covering G.

Then, for $s > 1$ such that s divides n, by Theorem 2,

$$N(t_s) = \frac{1}{\varphi(s)} \sum_{dd'=s} d\mu(d')(2 - 2\gamma_d)$$

and by Theorem 3,

$$N(t_s) = \sum_{\delta\delta' = s'} \delta' r_{\delta s} \quad \text{for } ss' = n.$$

Equating these two expressions yields:

$$(11)_s \qquad \sum_{dd'=s} d\mu(d')(2 - 2\gamma_d) = \varphi(s) \sum_{\delta\delta'=s'} \delta' r_{\delta s}.$$

Write $\lambda(s) = \varphi(s) \sum_{\delta\delta'=s'} \delta' r_{\delta s}$ for $s > 1$, $s|n$, and let $\lambda(1) = 2 - 2g$; we regard the equation $2 - 2\gamma_1 = \lambda(1)$ as $(11)_1$. Then, if $d(n)$ is the number of divisors of n, we have $d(n)$ possible orbit genera γ_d.

Define

$$\mu_{sd} = d\mu(d') \quad \text{if } d|s,\ dd' = s,$$
$$= 0 \qquad \text{otherwise.}$$

Take M to be the matrix (μ_{sd}), where the rows and columns of M are labelled with the divisors of n. Let $\boldsymbol{\gamma}$ be the column vector with elements $(2 - 2\gamma_d)$ and let $\boldsymbol{\lambda}$ be the column vector with elements $\lambda(s)$. Then the equations $(11)_s$ may be rewritten $M\boldsymbol{\gamma} = \boldsymbol{\lambda}$.

Let

$$\mu_{de}' = 1/d \quad \text{if } e|d,$$
$$= 0 \qquad \text{otherwise,}$$

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and take M' to be the matrix (μ_{de}') labelled in the same manner as M.

LEMMA 3. *M' is the inverse of M.*

Proof. The (s, e) element of MM' is

$$\sum_{d|n} \mu_{sd}\mu_{de}'.$$

Now $\mu_{de}' = 0$ unless $e|d$ and $\mu_{sd} = 0$ unless $d|s$. Hence, if e does not divide s, $\sum_{d|n} \mu_{sd}\mu_{de}' = 0$ while, for $e|s$,

$$\begin{aligned}\sum_{d|n} \mu_{sd}\mu_{de}' &= \sum_{e|d|s} d\mu\left(\frac{s}{d}\right)\cdot\frac{1}{d}\\ &= \sum_{e|d|s} \mu\left(\frac{s}{d}\right)\\ &= \sum_{d/e|s/e} \mu\left(\frac{s}{e}\Big/\frac{d}{e}\right).\end{aligned}$$

Hence, for $s/e > 1$, the (s, e) element of MM' is zero, while, for $s = e$, it is unity. Thus MM' is the unit matrix and M' is the inverse of M.

We now have $\gamma = M'\lambda$ or, re-written,

$$2 - 2\gamma_d = \sum_{s|n} \mu_{ds}'\lambda(s) = \sum_{s|d} \frac{\lambda(s)}{d}.$$

Thus

$$\begin{aligned}d(2 - 2\gamma_d) &= 2 - 2g + \sum_{1\neq s|d} \varphi(s) \sum_{\delta\delta'=s'} \delta' r_{\delta s}\\ &= 2 - 2g - \sum_{\delta\delta'=n} \delta' r_\delta + \sum_{s|d} \varphi(s) \sum_{\delta\delta'=s'} \delta' r_{\delta s}.\end{aligned}$$

If $\delta s = b$, then, taking $bb' = n$, the coefficient of r_b in the second summation is

$$\sum_{s|b, s|d} \varphi(s)\delta' = \sum_{s|(b,d)} \varphi(s)b' = b' \sum_{s|(b,d)} \varphi(s) = b'(b, d),$$

where (b, d) is the highest common factor of b and d. Hence

$$\begin{aligned}d(2 - 2\gamma_d) &= 2 - 2g - \sum_{\delta\delta'=n} \delta' r_\delta + \sum_{bb'=n} b'(b, d) r_b\\ &= 2 - 2g + \sum_{bb'=n} b' r_b[(b, d) - 1].\end{aligned}$$

In the case $d = n$, since $(b, n) = b$, this reduces to

$$n(2 - 2\gamma_n) = 2 - 2g + \sum_{bb'=n} n r_b\left(1 - \frac{1}{b}\right)$$

which is equivalent to (2).

We have thus proved the following result.

THEOREM 4. *Let G be a cyclic group of automorphisms of order n acting on a Riemann surface S of genus at least two. Suppose that the Fuchsian group*

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covering G has r_b periods b, for each b dividing n, and, for $d|n$, let G_d be the subgroup of G of order d. Then the orbit space S/G_d has genus γ_d, given by

$$\gamma_d = 1 + \frac{1}{d}(g-1) - \frac{1}{2d}\sum_{bb'=n} b' r_b[(b,d) - 1],$$

where (b, d) denotes the highest common factor of b and d.

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