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THE STRONG PERFECT GRAPH CONJECTURE FOR PLANAR GRAPHS

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1. Introduction. A graph G is called γ-perfect if $\lambda(H) = \gamma(H)$ for every vertex-generated subgraph H of G. Here, $\lambda(H)$ is the clique number of H (the size of the largest clique of H) and $\gamma(H)$ is the chromatic number of H (the minimum number of independent sets of vertices that cover all vertices of H). A graph G is called α-perfect if $\alpha(H) = \theta(H)$ for every vertex-generated subgraph H of G, where $\alpha(H)$ is the stability number of H (the size of the largest independent set of H) and $\theta(H)$ is the partition number of H (the minimum number of cliques that cover all vertices of H). For an arbitrary graph, we observe that $\lambda(H) \leqq \gamma(H)$ and $\alpha(H) \leqq \theta(H)$. A graph is called perfect if it is both γ-perfect and α-perfect. Berge [**1**; **2**; **3**] has examined perfect graphs extensively. Many familiar classes of graphs, e.g., bipartite graphs, are perfect. Moreover, α-perfect graphs and γ-perfect graphs have great importance in several fields. Shannon [**7**] has shown that graphs G such that $\alpha(G) = \theta(G)$ represent perfect channels in communication theory. Naturally, γ-perfection is important in colouring problems, both in pure colouring problems and in applied ones in block design or operations research (for a recent application of the theory of perfect graphs to a problem in refuse collection, see [**8**]). Moreover, the equations $\lambda(G) = \gamma(G)$ and $\alpha(G) = \theta(G)$ are of substantial interest in their own right, since they are complementary equations involving the complementary concepts of clique and independent set. That is, a clique in G is an independent set in G^c, the complement of G, and so $\lambda(G) = \alpha(G^c)$ and $\gamma(G) = \theta(G^c)$. Berge [**3**] has conjectured the following theorem.

STRONG PERFECT GRAPH CONJECTURE. *The following conditions on a graph G are equivalent:*

(a) *G is perfect,*
(b) *G is γ-perfect,*
(c) *G is α-perfect, and*
(d) *G and its complement G^c contain no odd-length primitive (chordless) circuits besides triangles (for short, no* OPC'*s).*

By complementarity, it suffices to show (d) $\Rightarrow$ (b) or (d) $\Rightarrow$ (c) (the converse is trivial). Omitting condition (d), we have the Weak Perfect Graph Conjecture. Here, it suffices to show that (b) $\Rightarrow$ (c) or (c) $\Rightarrow$ (b). Fulkerson [**4**] has generalized the ideas of α-perfection and γ-perfection to anti-blocking

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pairs of polyhedra and has proved what he calls the Pluperfect Graph Theorem. He posed the Weak Perfect Graph Conjecture in terms of pluperfect graphs. Recently, Lovász [**5**] extended Fulkerson's work to obtain a proof of the Weak Perfect Graph Conjecture.

In this paper we shall prove that the Strong Perfect Graph Conjecture is valid for planar graphs. Observe that the complement of a planar graph G cannot contain an OPC of length $\geqq 7$, since the complement of such a circuit is not planar. Further, the complement of an OPC of length 5 is another OPC of length 5. Thus we shall be proving the following theorem.

THEOREM 1. *The following conditions on a planar graph G are equivalent:*
(b) *G is γ-perfect,*
(c) *G is α-perfect, and*
(d′) *G contains no* OPC.

If G contains an OPC, then the subgraph consisting of just the OPC is easily seen to be neither γ-perfect nor α-perfect. By Lovász's result, it suffices to show that (d′) implies (b) or (c). We shall prove both implications since either method of proof might suggest a solution to the general problem. Actually we shall prove that for any planar graph G, (d′) implies $\lambda(G) = \gamma(G)$ and $\alpha(G) = \theta(G)$ (if G does not contain an OPC, then no subgraph H will either, and thus $\lambda(H) = \gamma(H)$ and $\alpha(H) = \theta(H)$). In section 3 we prove (d′) implies $\lambda(G) = \gamma(G)$ using basically a case-by-case argument. In section 4 we prove (d′) implies $\alpha(G) = \theta(G)$ through a set of lemmas, some of which apply to non-planar graphs.

2. Preliminaries. We represent a graph as a pair (V, A) where V is a finite set of vertices and A is a symmetric, irreflexive relation, called the adjacency relation (in a figure, we represent xAy with an edge between x and y). For disjoint (non-empty) subsets $V_1, V_2 \subseteq V$, we say V_1 is adjacent to V_2 if aAb for some $a \in V_1$ and $b \in V_2$. For any subset $V_1 \subset V$, $G - V_1$ is the subgraph G generated by restricting A to $V - V_1$. When V_1 is of small size, say $V_1 = \{x, y\}$, we write simply $G - x - y$. For a vertex $x \in V$, we define $N(x) = \{y \in V : yAx\}$ and $N^*(x) = N(x) \cup x$. Let the degree of x, $\deg(x)$, be $|N(x)|$. A path P is a sequence of distinct vertices $(x_1, x_2, \ldots, x_n)$ such that x_iAx_{i+1}, for $i = 1, 2, \ldots, n - 1$. A circuit is a path in which x_nAx. A path is primitive if $\sim x_iAx_j$ for $i + 1 < j$, except x_iAx_n in a primitive circuit. For any given path P from x to y, a primitive path P' from x to y can be obtained from P be deleting certain vertices. We use the abbreviation OPC to denote an odd-length primitive circuit other than a triangle. If x and y are vertices on the path P, let (x, P, y) be the path from x to y following P. Thus we will define paths of the form $(x, y, P_1, z, r, P_2, s)$.

A clique is a set of mutually adjacent vertices and an independent set is a set of mutually non-adjacent vertices. Let IS(G) denote the family of maximal independent sets of G. If the vertices of a graph are assigned colours so that

xAy implies x and y have different colours, then all the vertices of a given colour form an independent set. We use numbers as names of the colours. So we speak of a 3-vertex, i.e., a vertex with colour 3. An i-j path ($i \neq j$) in a coloured graph is a path whose vertices are alternatingly colour i and colour j. An i-j component is a component of the subgraph generated by vertices with colours i and j. An i-j interchange at the i-vertex x is a recolouring involving the vertices in C, the i-j component containing x; namely, the vertices in C exchange colours (thus x becomes a j-vertex). If I_1 and I_2 are disjoint independent sets of G, we speak of $(I_1 \& I_2)$-paths, etc.

If G is a planar graph, we shall assume it has a given planar drawing (i.e., without edges crossing). Thus it makes sense to speak of $z, y \in N(x)$ being consecutive vertices (about x) in $N(x)$. We assume the following facts about a planar graph G (see [**6**] or any basic text):

(i) $\lambda(G) \leqq 4$, and

(ii) G contains a vertex x with $\deg(x) \leqq 5$.

3. γ-perfection. We shall prove by induction that $\lambda(G) = \gamma(G)$ for any planar graph G with no OPC's. This equality is trivial if the graph has one or two vertices. The proof of the induction step involves a case-by-case argument using the following two lemmas.

LEMMA 1. *Suppose G is a graph with no* OPC's *and $G - x$ is properly coloured. If there is an i-j path in $G - x$ between a (with colour i) and b (with colour j) such that a, b are the only vertices of the path in $N(x)$, then aAb.*

Proof. If P is such a primitive i-j path from a to b, then (a, P, b, x) is an OPC unless aAb.

LEMMA 2. *Let G be a planar graph in which three vertices of G form a triangle (3-clique) T such that G has at least one vertex inside T and one vertex outside T. If $G - x$ can be k-coloured for every vertex x, then G can be k-coloured.*

Proof. By picking an x inside T, we can get a k-colouring of T and the vertices outside it. By picking an x outside T, we get a k-colouring of T and the vertices inside it. Matching the colours on T in the two colourings, we can then compose the two colourings to get a colouring of G.

Assume we have a planar graph G with no OPC's in which x is a vertex of lowest degree ($\deg(x) \leqq 5$) and $G - x$ has been coloured with colours numbered $1, 2, \ldots, k$, where $k = \lambda(G)$. By Lemma 2, we can assume that no non-consecutive vertices of $N(x)$ are adjacent. If $\lambda(G) = 1$, the result is trivial. If $\lambda(G) = 2$, then G is bipartite since it contains no triangles (3-cliques) and no OPC's. Thus G can be 2-coloured. If $\lambda(G) = 3$ but $\deg(x) \leqq 2$, then obviously x can be coloured.

$\lambda(G) = 3$; $\deg(x) = 3$. In this case, let $N(x) = a, b, c$ be coloured 1, 2, 3, respectively. Vertices a and b must be joined by a 1-2 path or else we do a 1-2 interchange at a and x is coloured 1. Then by Lemma 1, aAb. Similarly we obtain aAc and bAc. Then $N^*(x)$ is a 4-clique, but $\lambda(G) = 3$.

$\lambda(G) = 3$; $\deg(x) = 4$. In this case, let $N(x)$ consist of a, b, c, d in counterclockwise order about x. Remember that we assume no non-consecutive vertices of $N(x)$ are adjacent. If $N(x)$ uses only two colours, x can be easily coloured. There are two subcases to consider when $N(x)$ contains three colours. In the first subcase, shown in Figure 1, we observe that there is no 2-3 path between a and c, for otherwise Lemma 1 implies aAc. Thus we can do a 2-3 interchange at a and then colour x with 2. In the second case, shown in Figure 2, we assume that aAd or else Lemma 1 implies a 2-3 interchange

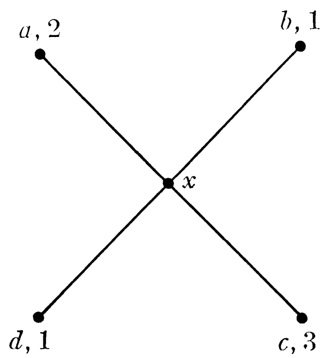

FIGURE 1
$\lambda(G) = 3$; $\deg(x) = 4$ subcase one

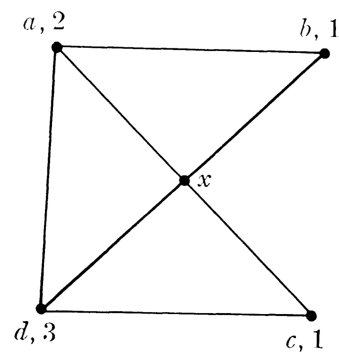

FIGURE 2
$\lambda(G) = 3$; $\deg(x) = 4$ subcase two

at d does not affect a (and then we colour x with 3). Also we assume aAb or else as before we do a 1-2 interchange at a. Similarly we can assume cAd. We now prove two lemmas which prohibit this situation as well as some similar ones in the cases where $\deg(x) = 5$.

LEMMA 3. *Let G be a planar graph with no* OPC's, *and let $G - x$ be properly coloured. Let a, b, c, d, e, x have the adjacencies and colours shown in Figure* 3 *(possibly $a = e$). Let P_2 be a* 1-2 *primitive path from b to c and P_3 a* 1-3 *primitive path from b to c. Possibly P_2 or P_3 (or both) contain one other vertex of $N(x)$, the* 1*-vertex f, where $\sim fA(N(x) - f)$. Then there exists a primitive even-length path R from b to c such that a is adjacent to only b on R, d to only c, and e, if $e \neq a$, is not adjacent to any vertex of R.*

Proof. Observe that a is adjacent to only b on P_2 (and hence a is not on P_2) or else we get a 1-2 path from a to c (or f) and Lemma 1 would imply aAc (or aAf). Similarly e is not adjacent to P_2 and d is adjacent to only c on P_3.

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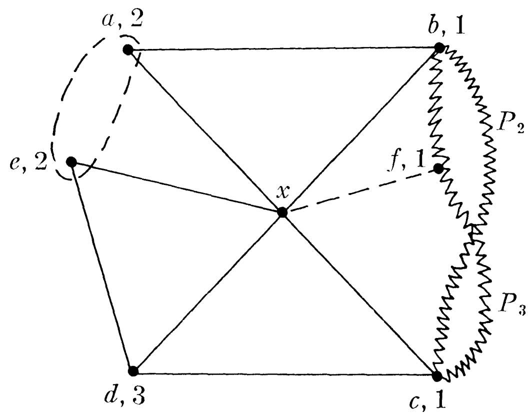

FIGURE 3
Perhaps $a = e$; see Lemma 3

Consider the path R' from b to c obtained from P_2 and P_3 by starting from b along the inside (closer-to-x) path of the pair P_2 and P_3 and, whenever the paths intersect, continuing towards c on whichever path is the inside one. Let R be a primitive path formed from R'. Observe that the only non-terminal vertices of R to which a, d, or e might be adjacent are the 1-vertices where P_2 and P_3 intersect, but since such vertices are on both paths none of a, d, or e can be adjacent to them. If R were odd-length, then (b, R, c, x) would be an OPC (or if f is on R, then either (b, R, f, x) or (f, R, c, x) must be an OPC).

LEMMA 4. *Let G be a planar graph with no* OPC's *and let $G - x$ be* 3-*coloured. Suppose a, b, c, d in $N(x)$ have colours* 2, 1, 1, 3, *respectively, and aAb, cAd, dAa are the only adjacencies among a, b, c, d (see Figure* 2). *Possibly $N(x)$ contains one more vertex e which is positioned between b and c and is not adjacent to a or d. Then G can be* 3-*coloured if one of the following conditions hold:*

(i) $N(x) = a, b, c, d$;

(ii) $N(x) = a, b, c, d, e$ *and e is a* 1-*vertex; or*

(iii) $N(x) = a, b, c, d, e$ *where e is a* 2-*vertex* (3-*vertex*) *and there is a* 1-2 *path* (1-3 *path*) *between b and c not containing e.*

Proof. Assume one of the three conditions apply. Suppose there exist a primitive 1-2 path P_2 from b to c and a primitive 1-3 path P_3 from b to c. If (iii) holds, then P_2 (or P_3) is chosen so as not to include e. By Lemma 3, there is an even-length primitive path R from b to c such that (a, b, R, c, d) is an OPC. So not both P_2 and P_3 can exist. By symmetry, we can assume P_2 does not exist, and so any primitive 1-2 path from a to c cannot contain b. It now follows from Lemma 1 and $\sim aAc$ that a and c are in different 1-2 components. Then perform a 1-2 interchange at c (c gets colour 2 and a is still 2). Next do

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a 1-3 interchange at b (b gets colour 3 but from Lemma 1 and $\sim bAd$ it follows that d is unaffected). If (i) holds, then x gets colour 1. If e exists, then we observe $\sim aAe$ or $\sim bAe$, for otherwise (a, b, e, c, d) is an OPC. By symmetry, we assume $\sim bAe$. If the above interchanges result in e now having colour 2 or 3, then x gets colour 1. If e now is colour 1, then perform a 1-3 interchange at e (e gets colour 3 but as before b and d are still 3). Now x is coloured 1.

$\lambda(G) = 3$; $\deg(x) = 5$. In this case there are five subcases to consider. The first subcase is shown in Figure 4. Observe that there is no 2-3 path from c to e

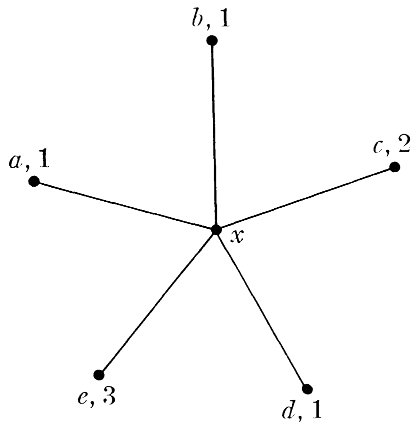

FIGURE 4
$\lambda(G) = 3$; $\deg(x) = 5$ subcase one

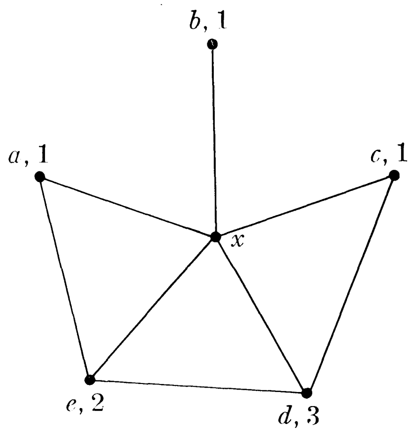

FIGURE 5
$\lambda(G) = 3$; $\deg(x) = 5$ subcase two

or else by Lemma 1, cAe. Now we do a 2-3 interchange at e (c is unaffected) and x gets colour 3. The second subcase is shown in Figure 5. Observe dAe or else we do a 2-3 interchange at d (which leaves e unaffected) and x gets colour 3. Similarly we have eAa and dAc. Now we are finished by Lemma 4 (case (ii)). The third subcase is shown in Figure 6. Lemma 1 implies there is no 1-3 path from e to b or c. Then we do a 1-3 interchange at e and let x gets colour 3. The fourth subcase is shown in Figure 7. For the usual reason, we assume aAb and aAe. Suppose $\sim cAd$. Now by Lemma 1, there is no 1-2 path from b or c to d or e. Then a 1-2 interchange at b can affect only b and c. If after such an interchange at b, c as well as b becomes colour 2, then x gets colour 1. If just b becomes colour 2, we now have the situation of the first subcase. Next suppose cAd. If there is no 1-2 path between b and c, we do a 1-2 interchange at b and get the first subcase again. So we assume there is a primitive 1-2 path P_2 between b and c. If there is no 1-3 path between b and c, we do a 1-3 interchange at c and now we have the situation of the third sub-

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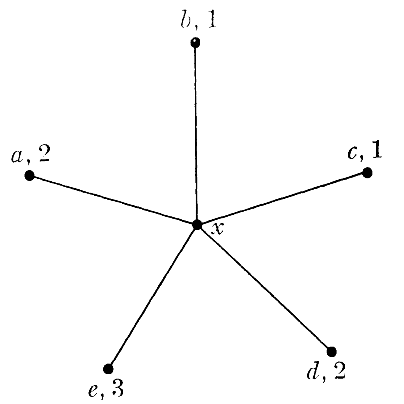

FIGURE 6
λ(G) = 3; deg(x) = 5 subcase three

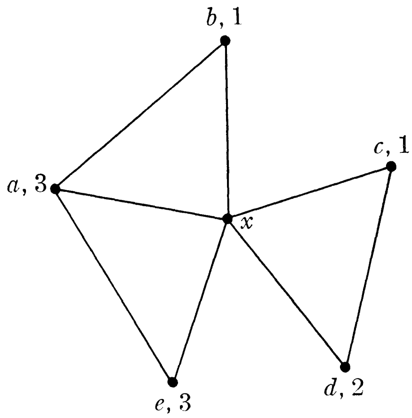

FIGURE 7
λ(G) = 3; deg(x) = 5 subcase four

case. So we assume there is a primitive 1-3 path P_3 between b and c. By symmetry, we can assume there is a primitive 2-1 path Q_1 and a primitive 2-3 path Q_3 both between d and e. By two applications of Lemma 3, we obtain even-length primitive paths R and S such that (a, b, R, c, d, S, e) is an OPC.

The fifth subcase is shown in Figure 8. For the usual reason, we assume aAb and aAe. Further assume that b, c, d, e are all in the same 1-2 component, or else a 1-2 interchange at b will yield one of the previous subcases. Consider now the form of a primitive 1-2 path P from b to e. It must include c or d, for otherwise Lemma 1 implies bAe. One possibility is $P = (b, c, d, e)$, but then

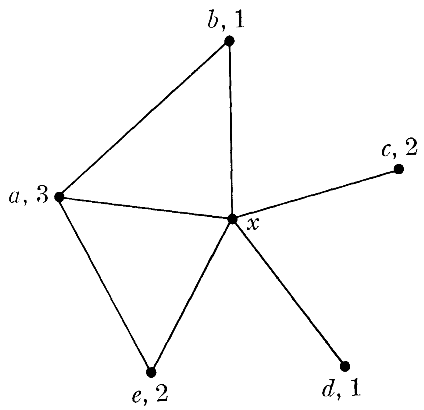

FIGURE 8
λ(G) = 3; deg(x) = 5 subcase five

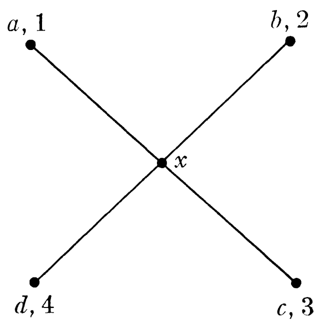

FIGURE 9
λ(G) = 4; deg(x) = 4

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(a, b, c, d, e) is an OPC. A second possibility is $P = (b, Q, d, e)$, where Q is a primitive 1-2 path from b to d, and now Lemma 4 (case iii) applies. The last possibility is $P = (b, c, Q, e)$, where Q is a primitive 1-2 path from c to e, and again Lemma 4 (case iii) applies. This finishes the proof when $\lambda(G) = 3$.

$\lambda(G) = 4$. If $\deg(x) \leqq 3$, the result is obvious. If $\deg(x) = 4$, we need only worry about the case shown in Figure 9. By Lemma 1, there can be no 1-3 path between a and c. Then do a 1-3 interchange at a and give x colour 1. When $\deg(x) = 5$, the same argument works (the two possible configurations are shown in Figures 10 and 11). This completes our inductive proof that $\lambda(G) = \theta(G)$ if G is a planar graph with no OPC's.

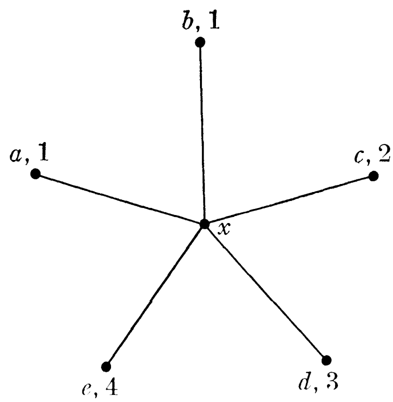

FIGURE 10

$\lambda(G) = 4$; $\deg(x) = 5$ subcase one

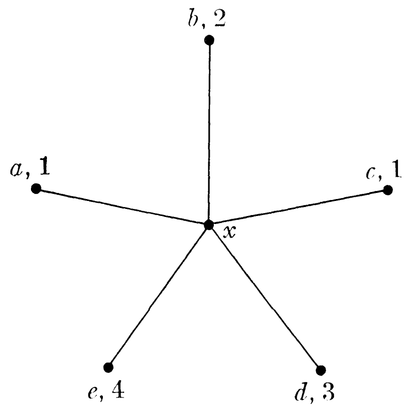

FIGURE 11

$\lambda(G) = 4$; $\deg(x) = 5$ subcase two

4. α-perfection. In this section we shall prove by induction that $\alpha(G) = \theta(G)$ for any planar graph G with no OPC's. For a graph G with one or two vertices, it is obvious that $\alpha(G) = \theta(G)$. Now let $G = (V, A)$ be a planar graph with no OPC's such that:

(*) $\alpha(G - V_1) = \theta(G - V_1)$ for any non-empty $V_1 \subseteq V$.

If $\alpha(G) = \alpha(G - x) + 1$ for some x, then using (*) we get

$$\theta(G) \leqq \theta(G - x) + 1 = \alpha(G - x) + 1 = \alpha(G).$$

However $\alpha(G) \leqq \theta(G)$, and so $\alpha(G) = \theta(G)$. Thus we are finished if we can show that G cannot satisfy the following condition:

(**) $\alpha(G) = \alpha(G - x) = \theta(G - x) = \theta(G) - 1$ for all x in G.

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We first prove several lemmas. Then a case-by-case analysis shows that G cannot satisfy (**).

LEMMA 5. *For any graph G satisfying (*) and (**), $\alpha(G) = \alpha(G - C)$ where C is any clique of G.*

Proof. Clearly $\alpha(G) \leqq \alpha(G - C) + 1$. So let $\alpha(G) = \alpha(G - C) + 1$. Since by (*), $\alpha(G - C) = \theta(G - C)$, then $\theta(G) \leqq \theta(G - C) + 1 = \alpha(G - C) + 1 = \alpha(G)$, contradicting (**).

LEMMA 6. *For any graph G satisfying (*) and (**) and for any distinct x, y in G, there exists an $I \in \mathrm{IS}(G - y)$ such that $x \in I$.*

Proof. Recall $\mathrm{IS}(G - y)$ is the family of maximal independent sets of $G - y$. Let S be a minimal clique covering $G - x$ and let C be the clique in S containing y. Then using (*) and (**), $\alpha(G - x - C) = \theta(G - x - C) = \theta(G - x) - 1 = \alpha(G - x) - 1 = \alpha(G) - 1$. However by Lemma 5, $\alpha(G - C) = \alpha(G)$. So $\alpha(G - x - C) = \alpha(G - C) - 1$. This equality implies x is in every $\mathrm{IS}(G - C)$. Since $\alpha(G - C) = \alpha(G - y)$, $\mathrm{IS}(G - C) \subseteq \mathrm{IS}(G - y)$.

For any independent set I, let $I_x = I \cap N(x)$.

LEMMA 7. *For any vertex x in the graph G and for any $I \in \mathrm{IS}(G - x)$, $I_x \neq \emptyset$.*

Proof. By (**), any $I \in \mathrm{IS}(G - x)$ is of size $\alpha(G)$. If $I_x = \emptyset$, then $I \cup x$ is an independent set of size $\alpha(G) + 1$.

COROLLARY 7.1. *If G is a graph satisfying (*) and (**), then $N^*(x)$ is not a clique for any x in G.*

Proof. If $N^*(x)$ is a clique, then by Lemma 5, $\mathrm{IS}(G - N^*(x)) \subseteq \mathrm{IS}(G)$. However for $I \in \mathrm{IS}(G - N^*(x))$, $I_x = \emptyset$, contradicting Lemma 7.

LEMMA 8 (Berge). *If G satisfies (*) and contains a clique C such that $G - C$ is not connected, then $\theta(G) = \alpha(G)$* (this is a weakened form of Lemma 2 in [**2**]).

COROLLARY 8.1. *If G is a planar graph satisfying (*) and (**), then for any vertex x, no non-consecutive vertices a, c in $N(x)$ can be adjacent.*

Proof. Suppose aAc. Since $G - a - c - x$ is not connected, then by Lemma 8, $\theta(G) = \alpha(G)$, but this contradicts (**).

LEMMA 9. *Let G be a graph with no* OPC's *and satisfying (*) and (**). Let x be a vertex and let $I, I' \in \mathrm{IS}(G)$ be such that $\sim aAb$ or $a = b$ for any $a \in I_x$, $b \in I_x'$. Then there exists $I'' \in \mathrm{IS}(G)$ such that $I_x'' = I_x \cap I_x'$.*

Proof. Consider the $(I \,\&\, I')$-components of G. Observe that we can get a new pair of independent sets by doing an $(I \,\&\, I')$-interchange (like a 1-2

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interchange) in one or more, but not all, $(I \& I')$-components. Observe that any such interchanges produce new maximal independent sets, for if I'' and I''' are the independent sets resulting from such interchanges and $|I''| < \alpha(G)$, then $|I'''|$ must be greater than $\alpha(G)$.

If $c \in I \cap I'$, then c must be an isolated point in the $(I \& I')$-subgraph. We claim that if an $(I \& I')$-component C (with more than one vertex) contains a vertex a in I_x, then C contains no vertices of I_x'. Suppose there exists $b \in C \cap I_x'$. Since $a \neq b$ (for otherwise C consists of only a), there exists a primitive $(I \& I')$-path P from a to b of odd length k ($k > 1$ since by assumption $\sim aAb$). Assume b is the only vertex of I_x' on P (if not, pick a new b and shorten P). Similarly, we assume a is the only vertex of I_x on P. Then (a, P, b, x) is an OPC, and the claim follows. Now in all non-isolated $(I \& I')$-components containing vertices of I_x, do an $(I \& I')$-interchange. This turns I into a new maximal independent set I'' where $I_x'' = I_x \cap I_x'$.

COROLLARY 9.1. *Let G be a graph with no* OPC's *and satisfying* $(*)$ *and* $(**)$. *Let x be any vertex of G. There cannot exist I, $I' \in$* IS$(G - x)$ *such that $\sim aAb$ and $a \neq b$ for any $a \in I_x$, $b \in I_x'$.*

Proof. If such I and I' exist, then by Lemma 9 there exists $I'' \in$ IS$(G - x)$ where $I_x'' = I_x \cap I_x' = \emptyset$. Then I'' violates Lemma 7.

COROLLARY 9.2. *Let G be a graph with no* OPC's *and satisfying* $(*)$ *and* $(**)$. *For any vertex x of G, some pair of vertices in $N(x)$ is adjacent.*

Proof. Suppose no pair of vertices in $N(x)$ is adjacent. For each $a \in N(x)$, pick $I^a \in$ IS$(G - x - a) \subseteq$ IS(G) (inclusion follows from Lemma 5). Then by multiple application of Lemma 9 to the I^a's, we obtain $I \in$ IS$(G - x)$ with $I_x = \bigcap_{a \in N(x)} I_x^a = \emptyset$. This violates Lemma 7.

LEMMA 10. *Let G be a planar graph with no* OPC's *and satisfying* $(*)$ *and* $(**)$. *Then for any vertex x with* $\deg(x) \leqq 5$ *and for any $I \in$* IS$(G - x)$, $|I_x| \geqq 2$.

Proof. By Lemma 7, $|I_x| \geqq 1$. Suppose that $I \in$ IS$(G - x)$ can be chosen so that $I_x = a$ for some $a \in N(x)$. Suppose $deg(x) = 5$ and let $N(x)$ be arranged as in Figure 12, but the adjacencies shown there need not hold (we omit the proof when $\deg(x) < 5$; it is similar but much less involved). Recall that by Corollary 8.1, no non-consecutive vertices of $N(x)$ are adjacent. Observe that by Corollary 9.1, for any $I' \in$ IS$(G - a - x)$, a is adjacent to I_x'. If $\sim aAb$, then for any $I' \in$ IS$(G - a - x)$, or $I' \in$ IS$(G - a - e - x)$ if aAe, a is not adjacent to I_x'. Hence aAb, and similarly aAe. Pick $I^e \in$ IS$(G - a - b - x)$ and $I^b \in$ IS$(G - a - e - x)$. Then for a to be adjacent to I_x^e and I_x^b, it follows that $e \in I_x^e$ and $b \in I_x^b$.

Consider the $(I^b \& I^e)$-component B containing b. Then $e \in B$, for otherwise an $(I^b \& I^e)$-interchange at b turns I^b into an $I' \in$ IS$(G - a - x)$ such that

a is not adjacent to I_x'. Thus there is a primitive $(I^b \,\&\, I^e)$-path Q from b to e in B. From Lemma 1 it follows that $Q = (b, c, d, e)$, or $Q = (b, P, d, e)$, where $d \in I_x{}^b$ and P is a primitive even-length path from b to d, or finally $Q = (b, c, P, e)$, where $c \in I_x{}^e$ and P is a primitive even-length path from c to e. We eliminate the first possibility since if it were true, (a, b, c, d, e) would be an OPC. By symmetry, we can assume $Q = (b, P, d, e)$. Now consider the $(I \,\&\, I^b)$-component C containing a and b. We claim $d \notin C$. If $d \in C$, then there is a primitive even-length $(I \,\&\, I^b)$-path P' from b to d. We now apply Lemma 3 where the vertices of I are considered 2-vertices, the vertices of I^b 1-vertices and the vertices of I^e 3-vertices. Then by Lemma 3, there exists an even-length path R from b to d such that (a, b, R, d, e) is an OPC. This proves our claim that $d \notin C$.

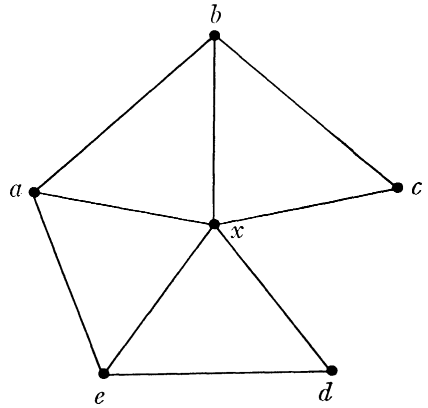

FIGURE 12
See Lemma 10

Now an $(I \,\&\, I^b)$-interchange in C turns I into a new $I' \in \mathrm{IS}(G - x)$ with $I_x' = b$. Since $I_x{}^e$ must be adjacent to I_x' (by Corollary 9.1), it follows that $c \in I_e^x$ and bAc (so $c \notin I_x{}^b$). Consider the $(I \,\&\, I^e)$-component D containing e (and a) and the $(I' \,\&\, I^e)$-component D' containing c (and b). If $c \in D$, then there exists a primitive even-length $(I \,\&\, I^e)$-path P from c to e. If $e \in D'$, then there exists a primitive even-length $(I' \,\&\, I^e)$-path P' from c to e. If both P and P' exist, then by Lemma 3 (I^e is 1-vertices, I' is 2-vertices, I is 3-vertices) there exists a path R from c to e such that (a, b, c, R, e) is an OPC. Thus either $c \notin D$ or $e \notin D'$. If $c \notin D$, then an $(I \,\&\, I^e)$-interchange in D turns I into a new $I'' \in \mathrm{IS}(G - x)$ with $I_x'' = e$, but now I_x' and I_x'' are not adjacent as required by Corollary 9.1. If $e \notin D'$, an $(I \,\&\, I^e)$-interchange in D' turns I' into a new $I''' \in \mathrm{IS}(G - x)$ with $I_x''' = c$, but now I_x and I_x''' are not adjacent. This finishes the proof when $\deg(x) = 5$.

Now we are ready for a case-by-case proof that a planar graph cannot satisfy (*) and (**) and still have no OPC's. We examine different possible

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adjacencies among $N(x)$ when $\deg(x) \leqq 5$. In each possibility we aim to produce an $I \in \mathrm{IS}(G - x)$ with $|I_x| \leqq 1$, and then we are done by Lemma 10. By Corollary 8.1, we assume that no non-consecutive vertices of $N(x)$ are adjacent. The arguments for $\deg(x) = 1, 2, 3$ and 4 are fairly direct and are left to the reader.

Suppose $\deg(x) = 5$ and $N(x)$ consists, in counterclockwise order, of a, b, c, d, e. We assume aAb by Corollary 9.2. We consider three subcases:

Case 1. The only adjacency in $N(x)$ is aAb.

Case 2. No vertex of $N(x)$ is adjacent to more than one other vertex of $N(x)$, yet $N(x)$ has another adjacency besides aAb.

Case 3. A vertex of $N(x)$ is adjacent to both possible neighbouring vertices of $N(x)$.

We give the argument for Case 3 only. In this case, not all consecutive pairs are adjacent, for otherwise (a, b, c, d, e) would be an OPC. Without loss of generality we can assume aAb, aAe and $\sim dAe$. Suppose cAd. Then for an $I \in \mathrm{IS}(G - c - d - x)$, $I_x = a$ or $= b, e$. So assume $I_x = b, e$. Now for $I' \in \mathrm{IS}(G - a - e - x)$, $I_x' = b, c$ or $= b, d$ (or else $|I_x'| \leqq 1$). Now apply Lemma 9 to I and I'. Next suppose $\sim cAd$. Pick $I \in \mathrm{IS}(G - c)$ such that $a \in I_x$ (such an I exists by Lemma 6). Then $I_x = a$ or $I_x = a, d$. Similarly, pick $I' \in \mathrm{IS}(G - d)$ such that $a \in I_x'$, and so $I_x' = a$ or $I_x' = a, c$. Now apply Lemma 9 to I and I' to obtain an $I'' \in \mathrm{IS}(G)$ with $|I_x''| = 1$. This violates Lemma 10.

This completes the proof that if G has no OPC's, then $\alpha(G) = \theta(G)$.

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