ON A COVERING SURFACE OVER AN ABSTRACT RIEMANN SURFACE

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1. Let $\underline{\mathfrak{R}}$ be an abstract Riemann surface in the sense of Weyl-Radó, and $\mathfrak{R}$ an open covering surface over $\underline{\mathfrak{R}}$. If a curve $C=\{P(t)\,;\,0\leqq t<1\}$ on $\mathfrak{R}$ tends to the ideal boundary of $\mathfrak{R}$ but its projection terminates at an inner point of $\underline{\mathfrak{R}}$ as $t\to 1$, we shall say that C determines an *accessible boundary point* (which will be abbreviated by A.B.P.) of $\mathfrak{R}$ relatively to $\underline{\mathfrak{R}}$. The set of all the A.B.P.s[1] of $\mathfrak{R}$ relative to $\underline{\mathfrak{R}}$ will be called *accessible boundary* (relative to $\underline{\mathfrak{R}}$) and denoted by $\mathfrak{A}(\mathfrak{R})$ or by $\mathfrak{A}(\mathfrak{R},\underline{\mathfrak{R}})$. Throughout in this paper $\mathfrak{A}(\mathfrak{R})$ will be supposed to be non-empty.

After K. I. Virtanen [12] we shall use the notation (B_0) to denote the class of Riemann surfaces, on which no one-valued and non-constant bounded harmonic function exists.

In the first place in this note we shall define *harmonic measure* $\omega(P)$ *of* $\mathfrak{A}(\mathfrak{R})$ and show that if $\omega(P)>0$ then $\mathfrak{R}\notin(B_0)$.

We suppose next that the projection of $\mathfrak{R}$ is compact in $\underline{\mathfrak{R}}$ and that the universal covering surface $\mathfrak{R}^\infty$ of $\mathfrak{R}$ is of hyperbolic type. Then $\mathfrak{R}^\infty$ is mapped conformally onto a unit circular domain $U\colon |z|<1$, and we obtain a function $f(z)$ which maps U into $\underline{\mathfrak{R}}$, corresponding to the mappings $U\to\mathfrak{R}^\infty\to\mathfrak{R}\to\underline{\mathfrak{R}}$. If $f(z)$ tends to a value $f(e^{i\theta})$ as $z\to e^{i\theta}$ along every Stolz's path[2][3] a.e. (= almost everywhere) on $\Gamma\colon |z|=1$, $\mathfrak{R}$ will be called *of F-type* (relatively to $\underline{\mathfrak{R}}$) (cf. [7], Chap. III, § 2).

In § 5 of this note we shall show that $\omega(P)\equiv 1$ for $\mathfrak{R}$ of F-type and give a condition so that $\mathfrak{R}$ is of F-type, generalizing a result in [7].

Finally we shall remark some relations between concepts defined in this note.

2. We consider the class $\mathfrak{B}(\mathfrak{R})$ of all the non-negative continuous super-

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1) Any equivalency of A.B.P.s is not considered here.

2) By a Stolz's path we mean a path which terminates at a point on Γ and lies between two chords through the point.

3) When $f(z)$ has this property, we shall say that $f(z)$ has an angular limit at $e^{i\theta}$ and call $f(e^{i\theta})$ the angular limit at $e^{i\theta}$.

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harmonic functions $\{v(P)\}$ on $\mathfrak{R}$ such that $v(P) \leqq 1$ and $\lim v(P) = 1$ when P tends to $\mathfrak{A}(\mathfrak{R})$ along every curve determining an A.B.P. of $\mathfrak{R}$ relative to $\underline{\mathfrak{R}}$. This class is non-empty, since the constant 1 belongs to it. The lower cover (= infimum at every point) of $\mathfrak{B}(\mathfrak{R})$ is harmonic on $\mathfrak{R}$ by Perron-Brelot's principle (cf. [7], Chap. I, § 1), and will be denoted by $\mu(P, \mathfrak{A}(\mathfrak{R}))$.

First we suppose that the universal covering surface $\underline{\mathfrak{R}}'^{\infty}$ of the projection $\underline{\mathfrak{R}}'$ of $\mathfrak{R}$ into $\underline{\mathfrak{R}}$ is *of hyperbolic type*; that is, if $\underline{\mathfrak{R}}'$ is of genus zero it is conformally equivalent to a plane domain with at least three boundary points, if $\underline{\mathfrak{R}}'$ is of genus one it is open, and if the genus is greater than one $\underline{\mathfrak{R}}'$ is required to fulfill no further condition. *We define harmonic measure (function)* $\omega(P)$ *of* $\mathfrak{A}(\mathfrak{R})$ *by means of* $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}, \underline{\mathfrak{R}}))$, which may be regarded as a one-valued function on $\mathfrak{R}$.

The universal covering surface $\mathfrak{R}^{\infty}$ of $\mathfrak{R}$ is also of hyperbolic type and mapped conformally onto $U: |z| < 1$. It can be shown that the images in U of a curve determining an A.B.P. of $\mathfrak{R}$ terminate at points on $\Gamma: |z| = 1$, which are equivalent with respect to a Fuchsian group, and that, $f(z)$ denoting mapping function of U into $\underline{\mathfrak{R}}$, $f(z)$ has an angular limit at any point $e^{i\theta}$ on Γ, where an image of a determining curve of an A.B.P. terminates.[4] We shall call the set of all the points on Γ, which correspond to A.B.P.s of $\mathfrak{R}$, the image on Γ of $\mathfrak{A}(\mathfrak{R})$.

We will now give

THEOREM 1. *Let $\mathfrak{R}$ be an open covering surface over an abstract Riemann surface $\underline{\mathfrak{R}}$, and suppose that the universal covering surface of the projection $\underline{\mathfrak{R}}'$ of $\mathfrak{R}$ into $\underline{\mathfrak{R}}$ is of hyperbolic type. Then the image E on Γ of $\mathfrak{A}(\mathfrak{R})$ is linearly measurable and the value of the harmonic measure $\mu(z, E)$ in U of E is equal to the value of $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))$ at any corresponding points.*

Proof. In case $\underline{\mathfrak{R}}^{\infty}$ is of hyperbolic type, map it conformally onto $U_w: |w| < 1$. E coincides with the place on Γ, where any branch of the function corresponding to the mappings $U \to \underline{\mathfrak{R}} \to \underline{\mathfrak{R}}^{\infty} \to U_w$ has limits lying in U_w. Namely, E is the complement of the set E' on Γ, where the branch has radial limits on $|w| = 1$ or has no limit. Since E' is linearly measurable (cf. [7], Chap. IV, § 3),[5] E is so too.

In case $\underline{\mathfrak{R}}^{\infty}$ is of parabolic or elliptic type, map it conformally onto $|w| < \infty$ or $|w| \leqq \infty$. Since $\underline{\mathfrak{R}}'^{\infty}$ is of hyperbolic type, any branch of the function mapping U into the w-plane does not take at least three values w_1, w_2 and w_3. Map further the universal covering surface of the complement of w_1, w_2, w_3 onto U_ω : $|\omega| < 1$, and let $\omega = F(z)$ be any branch of the function corresponding to the composed mappings. To w_1, w_2, w_3 there correspond an enumerably infinite number

4) These results were stated in [7], Chap. III, § 1 under the assumption that the projection $\underline{\mathfrak{R}}'$ is compact in $\underline{\mathfrak{R}}$.

5) The method in proving the measurabliity of E' is available also to show the measurability of E directly.

of points $\{\omega_i\}$ on $|\omega|=1$. E is classified into the following two parts: E_1 where $F(z)$ has radial limits lying in U_ω, and E_2, which is a subset of the set E_2' where the radial limits of $F(z)$ are equal to some of $\{\omega_i\}$. E_2' is linearly measurable and its measure is zero by Riesz's theorem [9], and the measurability of E_1 follows for the same reason as in the first case. Thus $E=E_1+E_2$ is measurable.

The harmonic measure $\mu(z, E)$ of E is equal to the lower cover of the class $\mathfrak{B}(U)$ consisting of all the non-negative continuous super-harmonic functions $\{v(z)\}$ in U, each of which is $\leqq 1$ and tends to 1 as z approaches every point of E. If $v(z)$ is considered on $\mathfrak{R}^\infty$, it belongs to $\mathfrak{B}(\mathfrak{R}^\infty)$ and hence

$$\mu(P(z), \mathfrak{A}(\mathfrak{R}^\infty)) \leqq \mu(z, E).$$

Conversely let $v_1(P)$ be any function of $\mathfrak{B}(\mathfrak{R}^\infty)$ and consider it in U. Then its radial limit equals 1 at every point of E. Letting $\rho\to 1$ in inequalities

$$v_1(P(z)) \geqq \frac{1}{2\pi}\int_0^{2\pi} v_1(P(\rho e^{i\varphi}))\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)}d\varphi$$

$$\geqq \frac{1}{2\pi}\int_{e^{i\varphi}\in E} v_1(P(\rho e^{i\varphi}))\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)}d\varphi \quad (z=re^{i\theta},\ \ \rho>r),$$

we have by Lebesgue's theorem

$$v_1(P(z)) \geqq \frac{1}{2\pi}\int_{e^{i\varphi}\in E}\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi = \mu(z, E).$$

Consequently we obtain the reverse inequality

$$\mu(P(z), \mathfrak{A}(\mathfrak{R}^\infty)) \geqq \mu(z, E).$$

Thus there holds the equality and the theorem is proved.

3. As preparation for the definition of $\omega(P)$ in the case when $\underline{\mathfrak{R}}'^\infty$ is not of hyperbolic type, we shall prove the following lemma, which will be used also in §5.

LEMMA. *Let the universal covering surface $\mathfrak{R}^\infty$ of $\mathfrak{R}$ be of hyperbolic type and map it conformally onto U. Suppose that the mapping function $f(z)$ of U into $\underline{\mathfrak{R}}$ has an angular limit at every point $e^{i\theta}$ belonging to a measurable set $E\subset\Gamma$. Take a finite number of points $\{\underline{P}_i\}$ $(i=1, 2, \ldots, n)$ on $\underline{\mathfrak{R}}$ and remove from $\mathfrak{R}$ all the points lying over them so that the projection of the remaining surface $\tilde{\mathfrak{R}}$ has a universal covering surface of hyperbolic type.*

Then there holds at any corresponding points

$$\mu(z, E) \leqq \mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^\infty)).$$

Proof. Map $\tilde{\mathfrak{R}}^\infty$ onto $U_\zeta: |\zeta|<1$ and denote the image on $\Gamma_\zeta: |\zeta|=1$ of $\mathfrak{A}(\tilde{\mathfrak{R}}^\infty)$ by E_ζ. Then by Theorem 1 $\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^\infty))=\mu(\zeta, E_\zeta)$. Hence we shall show $\mu(z, E)\leqq\mu(\zeta, E_\zeta)$ under the assumption that the linear measure $m(E)>0$.

Let E' be any measurable subset of positive measure of E. Any image in U_ζ of a Stolz's path terminating at a point of E' terminates at a point of E_ζ. We shall call the set of all such end-points on E_ζ the angular image on E_ζ of E'. In the following we shall show that the angular image on E_ζ of E' has a positive linear inner measure.

Consider a non-constant one-valued meromorphic function on $\mathfrak{R}$ and combine it with $f(z)$. The function $F(z)$ thus defined in U is also non-constant one-valued and meromorphic. Let $E'' \subset E'$ be the set where the limits of $f(z)$ are equal to some of $\{\underline{P}_i\}$. Then $F(z)$ has also a finite number of values as its angular limits at points of E''. E'' is measurable and Lusin-Priwaloff's theorem [2][6] shows that the linear measure of E'' is zero. Hence $m(E'-E'') = m(E')>0$. Denote the angular domain: $|\arg(1-e^{-i\theta}z)|<\frac{\pi}{4}$ at $e^{i\theta}$ by $A(\theta)$. By Egoroff's theorem we can find a closed subset F of positive linear measure of $E'-E''$ such that $f(z)$ tends to the angular limit $f(e^{i\theta})$ uniformly as $z \to e^{i\theta} \in F$ from the inside of $A(\theta)$. In the usual way we get a domain $D \subset U$, which contains an end-part of every $A(\theta)$ for $e^{i\theta} \in F$ and is bounded by a rectifiable curve C consisting of F and segments lying on the boundaries of $\{A(\theta); e^{i\theta} \in F\}$. The number of points $\{z_k\}$ corresponding to $\{\underline{P}_i\}$ and lying on $D+C$ is finite, because $f(z) \to f(e^{i\theta})$ uniformly in D and $\{f(e^{i\theta}); e^{i\theta} \in F\}$ is a closed set not containing the points $\{\underline{P}_i\}$. By removing $\{z_k\}$ from $D+C$ by rectifiable cross-cuts we obtain a simply-connected subdomain D_1 with F on its boundary. Map D_1 onto $U_x: |x|<1$. Then F is transformed to a closed set F_x of positive linear measure on $\Gamma_x: |x|=1$ in virtue of Riesz's theorem ([9], [8]). The mapping of D_1 onto a subdomain D_ζ of U_ζ is one-to-one continuous, with their boundaries included. In the mapping $U_x \to D_\zeta$ the linear measure of the image F_ζ on Γ_ζ of F_x is greater than $m(F_x)>0$ on account of the extension of Löwner's lemma (cf. [7], Chap. IV, §3), where $\zeta=0$ is supposed to correspond to $x=0$ without loss of generality. Accordingly $m(F_\zeta)>0$. Since F_ζ is contained in the angular image of F on E_ζ, the angular image on E_ζ of $E' \supset F$ has a positive linear inner measure.

Once established this fact, the rest of the proof of our lemma is carried as follows. The function $\mu(\zeta, E_\zeta)$ can be regarded as a one-valued bounded harmonic function in U. By Fatou's theorem it has angular limits a.e. on Γ. Denote the subset of E, where this function has angular limits less than 1, by E_1, and its angular image on E_ζ by $E_\zeta^{(1)}$. At every point of $E_\zeta^{(1)}$ there terminates a curve along which $\mu(\zeta, E_\zeta)$ tends to a value <1, and so $\mu(\zeta, E_\zeta)$ can not have the angular limit 1 at any point of $E_\zeta^{(1)}$. Hence the inner measure $\underline{m}(E_\zeta^{(1)})=0$, because if $\underline{m}(E_\zeta^{(1)})>0$ then $\mu(\zeta, E_\zeta)$ would have the angular limit 1 at a certain point of $E_\zeta^{(1)} \subset E_\zeta$. As we have seen that $\underline{m}(E_\zeta^{(1)})>0$ follows from $m(E_1)>0$,

[6] For its generalization, cf. [10] and [7], Chap. III, §2.

there must hold $m(E_1)=0$. Thus $\mu(\zeta, E_\zeta)$, which is considered as a function in U, has the radial limit 1 a.e. on E. Consequently we have $\mu(z, E)\leqq\mu(\zeta, E_\zeta)$.

Using this lemma the following theorem is proved:

THEOREM 2. *Suppose that $\underline{\mathfrak{R}}'^{\infty}$ is of hyperbolic type. Take a finite number of points $\{\underline{P}_i\}$ $(i=1, 2, \ldots, n)$ on $\underline{\mathfrak{R}}$, remove from $\mathfrak{R}$ all the points lying over them and denote the remaining surface by $\tilde{\mathfrak{R}}$. Then there holds*

$$\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))=\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty})).$$

Proof. Map $\mathfrak{R}^{\infty}$ and $\tilde{\mathfrak{R}}^{\infty}$ onto U and U_ζ, and let E and E_ζ be the images on Γ and Γ_ζ of $\mathfrak{A}(\mathfrak{R}^{\infty})$ and $\mathfrak{A}(\tilde{\mathfrak{R}}^{\infty})$ respectively. Since $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))=\mu(z, E)$ and $\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}))=\mu(\zeta, E_\zeta)$, we want to prove $\mu(z, E)=\mu(\zeta, E_\zeta)$ at corresponding points. One inequality $\mu(z, E)\leqq\mu(\zeta, E_\zeta)$ follows from the above lemma.

On the other hand, every radius terminating at a point on E_ζ is transformed to a curve in U which terminates at a point of E or at one of the inner points $\{z_n\}$ corresponding to $\{\underline{P}_i\}$. It is easily shown that E coincides with the set of all such end-points on Γ. Since the number of $\{z_n\}$ is at most enumerably infinite, the part $E'_\zeta\subset E_\zeta$ which corresponds to $\{z_n\}$ has linear measure zero. If $\zeta=0$ corresponds to $z=0$, $m(E_\zeta)=m(E_\zeta-E'_\zeta)\leqq m(E)$ on account of the extension of Löwner's lemma. Hence there follows the reverse inequality $\mu(z, E)\geqq\mu(\zeta, E_\zeta)$, and the required equality is obtained.

Let us now define the harmonic measure $\omega(P)$ of $\mathfrak{A}(\mathfrak{R})$ when $\underline{\mathfrak{R}}'^{\infty}$ is *not of hyperbolic type*. Take one or two or three points on $\underline{\mathfrak{R}}$ and remove from $\mathfrak{R}$ all the points lying over them so that the projection of the remaining surface $\tilde{\mathfrak{R}}$ has a universal covering surface of hyperbolic type. *We define harmonic measure $\omega(P)$ of $\mathfrak{A}(\mathfrak{R})$ by $\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}, \underline{\mathfrak{R}}))$.* Since every removed point of $\mathfrak{R}$ is isolated, $\omega(P)$ becomes harmonic everywhere on $\mathfrak{R}$. To avoid any possible ambiguity, we must, and shall, show that $\omega(P)$ is determined independently of the position of the points selected on $\underline{\mathfrak{R}}$.

Take a finite number of points on $\underline{\mathfrak{R}}$ in another way, remove all the points lying over them from $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$, and denote the remaining surfaces by $\hat{\mathfrak{R}}$ and $\hat{\tilde{\mathfrak{R}}}$ respectively. The universal covering surface of the projection into $\underline{\mathfrak{R}}$ of $\hat{\mathfrak{R}}$ is supposed to be of hyperbolic type here. On account of Theorem 2 we have

$$\omega(P)=\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}))=\mu(P, \mathfrak{A}(\hat{\tilde{\mathfrak{R}}}^{\infty}))=\mu(P, \mathfrak{A}(\hat{\mathfrak{R}}^{\infty})).$$

Thus the harmonic measure $\omega(P)$ of $\mathfrak{A}(\mathfrak{R})$ has been defined in all cases.

4. Prior to show a relation between $\omega(P)$ and the class (B_0), we shall state some related results obtained recently.

Let $\mathfrak{R}$ be a covering surface over the w-plane, K be a circular domain in the plane, and $\mathfrak{D}$ be a domain of $\mathfrak{R}$, which lies over K and whose boundary in $\mathfrak{R}$

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lies over the boundary of K. Y. Nagai [5][7] and M. Tsuji [11] found independently that if $\mathfrak{D}$ does not cover a set of positive capacity in K then $\mathfrak{R}$ has a positive boundary,[8] and Y. Nagai [5] showed that, $n(w)$ denoting the number of points of $\mathfrak{R}$ lying over w, if the set $\{w; n(w) < \sup n(w)\}$ is of positive capacity, then K and $\mathfrak{D}$ can be chosen such that $\mathfrak{D}$ does not cover a set of positive capacity in K. Further map the universal covering surface of $\mathfrak{D}$ onto U and denote the mapping function of U into the w-plane by $f(z)$. A. Mori [4] proved the following theorem: $\mathfrak{R} \notin (B_0)$ if it does not arise that almost all radial limits of $f(z)$ lie on the boundary of K; and also showed that the requirement in this theorem is fulfilled if $\mathfrak{D}$ does not cover a set of positive capacity in K.

In this section we will prove

THEOREM 3. *Let $\mathfrak{R}$ be a covering surface over an abstract Riemann surface $\underline{\mathfrak{R}}$. If the harmonic measure $\omega(P)$ of the accessible boundary $\mathfrak{A}(\mathfrak{R})$ is positive, then $\mathfrak{R} \notin (B_0)$.*

Proof. Without loss of generality we may suppose that $\underline{\mathfrak{R}}'^{\infty}$ is of hyperbolic type. Let $\{\mathfrak{S}_n\}$ be a sequence of triangulations of $\underline{\mathfrak{R}}$ such that $\mathfrak{S}_{n+1}$ is a subdivision of $\mathfrak{S}_n$ and $\mathfrak{S}_n$ becomes as fine as we please when $n \to \infty$. We denote the triangles of $\mathfrak{S}_n$ by $\{\Delta_i^{(n)}\}$ $(i = 1, 2, \ldots$; finite or infinite).[9] Map $\mathfrak{R}^{\infty}$ onto U and denote the function corresponding to $U \to \mathfrak{R}^{\infty} \to \mathfrak{R} \to \underline{\mathfrak{R}}$ by $f(z)$. The set on Γ, where the radial limits of $f(z)$ lie in $\Delta_i^{(n)}$, will be denoted by $E_i^{(n)}$. Then every $E_i^{(n)}$ is linearly measurable and the image on Γ of $\mathfrak{A}(\mathfrak{R})$ is equal to $\sum_i E_i^{(n)}$ for each n. If there is such an $E_i^{(n)}$ as $0 < m(E_i^{(n)}) < 2\pi$, its harmonic measure in U is transformed into a one-valued non-constant harmonic function on $\mathfrak{R}$. Thus the required function is obtained.

On the contrary, suppose that for every n there existed $i(n)$ such that $m(E_{i(n)}^{(n)}) = 2\pi$. Then $E_{i(n)}^{(n)} \supset E_{i(n+1)}^{(n+1)}$ and $\Delta_{i(n)}^{(n)} \supset \Delta_{i(n+1)}^{(n+1)}$. If we compose a non-constant meromorphic function $\Phi(\underline{P})$ on $\underline{\mathfrak{R}}$ and $f(z)$, the angular limits of the composed function $F(z)$ would be equal to one and the same value $\Phi(\bigcap_{n=1}^{\infty} \Delta_{i(n)}^{(n)})$ at every point of $\bigcap_{n=1}^{\infty} E_{i(n)}^{(n)}$ with $m(\bigcap_{n=1}^{\infty} E_{i(n)}^{(n)}) = 2\pi$. On account of Lusin-Priwaloff's theorem $F(z)$ would be a constant and this is a contradiction, which completes the proof.

THEOREM 4. *Let $\mathfrak{R}$ be a covering surface over an abstract Riemann surface $\underline{\mathfrak{R}}$. If $\mathfrak{R}$ does not cover a set of positive capacity on $\underline{\mathfrak{R}}$,[10] then $\omega(P) > 0$.*

7) His statement is of a slightly different form.

8) As is known, a Green's function exists on $\mathfrak{R}$ if and only if $\mathfrak{R}$ has a positive boundary. Cf. [7], Chap. II, §4.

9) $\{\Delta_i^{(n)}\}$ are made half open so that they are mutually disjoint for every fixed n.

10) This means that the image in a parameter circle, corresponding to a certain neighborhood on $\underline{\mathfrak{R}}$, is of positive capacity.

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Proof. First suppose that $\mathfrak{R}^\infty$ is of hyperbolic type, and map $\mathfrak{R}^\infty$ and $\underline{\mathfrak{R}}^\infty$ onto U and $U_w: |w|<1$ respectively. Any branch of the function corresponding to $U \to \mathfrak{R}^\infty \to \underline{\mathfrak{R}}^\infty \to U_w$ will be denoted by $w = F(z)$. $F(z)$ does not take values of a set of positive capacity in U_w and the image E on Γ of $\mathfrak{A}(\mathfrak{R})$ coincides with the place where $F(z)$ has limits lying inside U_w. Hence by Frostman's theorem [2] for functions of class (U), $m(E)>0$. Thus $\omega(P) = \mu(z, E)>0$. The case when $\underline{\mathfrak{R}}^\infty$ is not of hyperbolic type is now easily treated.

COROLLARY. Let $\mathfrak{D}$ and $\underline{\mathfrak{D}}$ be domains of $\mathfrak{R}$ and $\underline{\mathfrak{R}}$ respectively such that $\mathfrak{D}$ lies over $\underline{\mathfrak{D}}$ and the boundary of $\mathfrak{D}$ in $\mathfrak{R}$ does not lie over the inside of $\underline{\mathfrak{D}}$. If $\mathfrak{D}$ does not cover a set of positive capacity in $\underline{\mathfrak{D}}$ then $\omega(P)$ of $\mathfrak{A}(\mathfrak{R})$ is positive.

For, the harmonic measure of $\mathfrak{A}(\mathfrak{D}, \underline{\mathfrak{D}})$ is positive by Theorem 4. On account of the extension of Löwner's lemma $\omega(P)$ of $\mathfrak{A}(\mathfrak{R}, \underline{\mathfrak{R}})$ is greater than it and hence is positive.

5. Theorem 3 is trivial when $\omega(P)$ is not a constant, and is interesting only when $\omega(P) \equiv 1$.

THEOREM 5. *Let* $\mathfrak{R}$ *be a covering surface of* F-*type over* $\underline{\mathfrak{R}}$. *Then* $\omega(P) \equiv 1$.

Proof. If $\underline{\mathfrak{R}}'^\infty$ is of hyperbolic type, $\omega(P) = \mu(z, E) \equiv 1$ by Theorem 1, where E is the image on Γ of $\mathfrak{A}(\mathfrak{R})$.

In the case when $\underline{\mathfrak{R}}'^\infty$ is not so, define $\tilde{\mathfrak{R}}$ as in § 3 and map $\tilde{\mathfrak{R}}^\infty$ onto $U_\zeta: |\zeta| <1$. We shall denote the image on $|\zeta| = 1$ of $\mathfrak{A}(\tilde{\mathfrak{R}})$ by E_ζ, and the set on Γ, where the mapping function of U into $\underline{\mathfrak{R}}$ has angular limits, by E. Then by Lemma in § 3 there follows $\mu(z, E) \leqq \mu(\zeta, E_\zeta)$ at corresponding points. Since $m(E) = 2\pi$, we have $\omega(P) = \mu(\zeta, E_\zeta) = \mu(z, E) \equiv 1$.

We next give a condition under which $\mathfrak{R}$ becomes of F-type, by

THEOREM 6. (*Extension of Theorem* 3.3 *in* [7] .) *Let* $\mathfrak{R}$ *be a covering surface over an abstract Riemann surface* $\underline{\mathfrak{R}}$ *such that the projection of* $\mathfrak{R}$ *is compact in* $\underline{\mathfrak{R}}$, *and denote the number of points of* $\mathfrak{R}$ *lying over* $\underline{P} \in \underline{\mathfrak{R}}$ *by* $n(\underline{P})$, *computing the multiplicity at each branch point of* $\mathfrak{R}$. *If the set* $\underline{E} = \{\underline{P} \in \underline{\mathfrak{R}};\ n(\underline{P}) < N = \sup n(\underline{P})\}$ *is of positive capacity on* $\underline{\mathfrak{R}}$, *then* $\mathfrak{R}$ *is of* F-*type.*

Proof. The set $\underline{E}_k = \{\underline{P};\ n(\underline{P}) \leqq k\}$ is a closed set for each k. Since $\underline{E} = \bigcup_{0 \leqq k < N} \underline{E}_k$ and is of positive capacity, there exitst the smallest number k_0 for which $\underline{E}_{k_0}$ is of positive capacity. If $k_0 = 0$ there follows $\mathfrak{R} \notin (\mathrm{B}_0)$ from Theorems 4 and 3. The set $\underline{E}^b_{k_0} - \underline{E}^b_{k_0} \cap \underline{E}_{k_0-1}$ for $k_0>0$ is also of positive capacity, where $\underline{E}^b_{k_0}$ denotes the boundary in $\underline{\mathfrak{R}}$ of $\underline{E}^b_{k_0}$. Let $\underline{P}_0$ be an arbitrary point of its transfinite kernel. There lie $l \leqq k_0$ points of $\mathfrak{R}: P_1, P_2, \ldots, P_l$, over $\underline{P}_0$. Over a sufficiently small neighborhood $\underline{N}$ on $\underline{\mathfrak{R}}$ of $\underline{P}_0$ there exists another connected piece $\mathfrak{D}$ of $\mathfrak{R}$ than those containing $\{P_j\}$ $(1 \leqq j \leqq l)$. Since this domain

$\mathfrak{D}$ does not cover a set of positive capacity in $\underline{N}$, $\omega(P)>0$ by Corollary of Theorem 4 and hence $\mathfrak{R}\notin(B_0)$ by Theorem 3.[11] Thus $\mathfrak{R}$ has a positive boundary.

Map $\mathfrak{R}^\infty$, which is of hyperbolic type, onto U, and consider a Green's function $G(P)$ on $\mathfrak{R}$ as a function in U. The angular limit of $G(P(z))$ is equal to 1 at every point of a set G_z of linear measure 2π (cf. [6], Chap. VII). In a similar manner as in the proof of Lemma in §3, we get a domain D in U such that it contains an end-part of the angular domain: $|\arg(1-e^{-i\theta}z)|<\frac{\pi}{2}-\frac{1}{p}$ (>0) at every point $e^{i\theta}$ of a closed set $F_n\subset G_z$ with $m(F_n)>2\pi-\frac{1}{n}$ and is bounded by a rectifiable curve C and $G(P(z))\to 0$ uniformly as $z\to F_n$ from the inside of D. Since $G(P_j)>0$ $(1\leqq j\leqq l)$, the image of $\{P_j\}$ in D or on C consists of a finite number of points. We remove these points from $D+C$ by rectifiable cross-cuts such that the remaining domain D_1 is simply-connected and F_n lies on its boundary. Map D_1 onto $U_\zeta: |\zeta|<1$ and consider in U_ζ the function $f(z)$ which maps U into $\underline{\mathfrak{R}}$. Since the image on $\mathfrak{R}$ of D_1 dose not contain points near $\{P_j\}$, it does not cover a set of positive capacity on $\underline{\mathfrak{R}}$. Hence by Theorem 3.3 in [7] $f(z(\zeta))$ has angular limits a.e. on $\Gamma_\zeta: |\zeta|=1$.

Now we denote the angular domain: $|\arg(1-e^{-i\theta}z)|<\frac{\pi}{2}-\frac{2}{p}$ at $e^{i\theta}$ by $A_p(\theta)$. By the method in proving the angular proportionality at boundary points in conformal mapping (cf. [1]), we can show that an end-part of $A_p(\theta)$ at $e^{i\theta}\in F_n$ is transformed to a domain inside an angular domain at $\zeta(e^{i\theta})$ when D_1 is mapped onto U_ζ. Thus $f(z)$ has a limit from the inside of $A_p(\theta)$ at the image $e^{i\theta}$ of a point on Γ_ζ where $f(z(\zeta))$ has an angular limit. By Riesz's theorem the image on Γ of any null set on Γ_ζ is a null set. Therefore $f(z)$ has a limit from the inside of $A_p(\theta)$ at every point $e^{i\theta}$ of a set of measure $2\pi-\frac{1}{n}$. By letting $n\to\infty$ we see that $f(z)$ has limits everywhere on Γ from the inside of $A_p(\theta)$, except on a set H_p with $m(H_p)=0$. Hence $f(z)$ has an angular limit at every point of $\Gamma-\bigcup_{p=1}^{\infty}H_p$. Since $m(\bigcup_{p=1}^{\infty}H_p)=0$, $f(z)$ has an angular limit a.e. on Γ. Thus $\mathfrak{R}$ is of F-type.

6. In the following we shall see some relations between various concepts defined in this note, under the assumption that $\underline{\mathfrak{R}}'^\infty$ is *not of hyperbolic type*; if this is of hyperbolic type the relations are stated in simpler forms.

First we supose that $\mathfrak{R}$ ***has a null boundary***. The surface $\tilde{\mathfrak{R}}$ which is defined in §3 has also a null boundary by Lemma 1.3 in [7]. Since no bounded and non-constant continuous superharmonic function exists on a surface with null boundary by Lemma 1.2 in [7], the upper classes $\mathfrak{B}(\mathfrak{R})$ and $\mathfrak{B}(\tilde{\mathfrak{R}})$ contain merely the constant 1. Thus $\mu(P, \mathfrak{A}(\mathfrak{R}))=\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}))\equiv 1$. On the other hand

11) Here we see that Theorem 6 does not serve as an example of the application of the fact, which follows from Theorems 5 and 3, that $\mathfrak{R}$ of F-type does not belong to (B_0).

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Theorem 3 shows that $\omega(P)=\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}))\equiv 0$. If $\mathfrak{R}^{\infty}$ is of parabolic type, this has a null boundary and hence $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))\equiv 1$. We shall show that $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))\equiv 0$ if $\mathfrak{R}^{\infty}$ is of hyperbolic type. Any curve determining an A.B.P. of $\mathfrak{R}$ converges to an ideal boundary component of $\mathfrak{R}$.[12] M. Tsuji [11] showed that the image E_0 on Γ of the ideal boundary of $\mathfrak{R}$ has linear measure zero in the mapping of $\mathfrak{R}^{\infty}$ onto U. Hence any image of a determining curve of an A.B.P. terminates at a point of E_0, and the lower cover of the class consisting of all the non-negative continuous superharmonic functions $\{v(z)\}$ not greater than 1 and with $\lim_{z\to E_0} v(z)=1$ is zero. For any $\varepsilon>0$ and an arbitrary point z_0, we can find in this class a function $v_0(z)$ with $v_0(z_0)<\varepsilon$. If $v_0(z)$ is regarded as a function on $\mathfrak{R}^{\infty}$, it belongs to $\mathfrak{B}(\mathfrak{R}^{\infty})$. By the arbitrarinesses of z_0 and ε, the lower cover $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))$ of $\mathfrak{B}(\mathfrak{R}^{\infty})$ is zero constantly.

Let us now pass to the case where $\mathfrak{R}$ *has a positive boundary*. Set $\mathfrak{R}-\tilde{\mathfrak{R}}=\{P_n\}$ and let $G_n(P)$ be the Green's function on $\mathfrak{R}$ with its pole at P_n. For an arbitrary point $P_0\in\tilde{\mathfrak{R}}$, the function $g(P)=\sum_n \frac{1}{n^2}\cdot\frac{G_n(P)}{G_n(P_0)}$ represents a harmonic function on $\tilde{\mathfrak{R}}$ in virtue of Harnack's theorem. For any $\varepsilon>0$ and $v(P)\in\mathfrak{B}(\mathfrak{R})$, $\min(1, v(P)+\varepsilon g(P))$ belongs to $\mathfrak{B}(\tilde{\mathfrak{R}})$ if it is considered as a function on $\tilde{\mathfrak{R}}$. ε and $v(P)$ being arbitrary, there follows $\mu(P, \mathfrak{A}(\mathfrak{R}))\geqq\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}))$. Conversely any $v(P)\in\mathfrak{B}(\tilde{\mathfrak{R}})$ belongs to $\mathfrak{B}(\mathfrak{R})$ if the value 1 is supplemented to $v(P)$ at $\mathfrak{R}-\tilde{\mathfrak{R}}$. Hence $\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}))\geqq\mu(P, \mathfrak{A}(\mathfrak{R}))$ and the equality follows. Further there holds $\mu(P, \mathfrak{A}(\mathfrak{R}))\geqq\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))$, because any $v(P)\in\mathfrak{B}(\mathfrak{R})$ considered on $\mathfrak{R}^{\infty}$ belongs to $\mathfrak{B}(\mathfrak{R}^{\infty})$. It is yet unknown whether there is or not a case when a proper inequality holds. Since, for any $v(P)\in\mathfrak{B}(\mathfrak{R}^{\infty})$ and $\varepsilon>0$, $\min(1, v(P)+\varepsilon g(P))\in\mathfrak{B}(\tilde{\mathfrak{R}}^{\infty})$, we can conclude the inequality $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))\geqq\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}))$. At present we have no example in which the inequality of this relation is proper. The relations are summarized in

$$\mu(P, \mathfrak{A}(\mathfrak{R}))=\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}))\geqq\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))\geqq\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty})).$$

Generalizing the definition in [7], Chap. IV, § 2, we will say that a covering surface $\mathfrak{R}$ with positive boundary over $\underline{\mathfrak{R}}$ is *of* D-*type* (relatively to $\underline{\mathfrak{R}}$), if any upper bounded continuous subharmonic function $u(P)$ is non-positive whenever $\overline{\lim}\, u(P)\leqq 0$ as $P\to\mathfrak{A}(\mathfrak{R})$ along every determining curve of an A.B.P. Since, for any $v(P)\in\mathfrak{B}(\mathfrak{R})$, $1-v(P)$ may be taken as above $u(P)$ and conversely, for any such a $u(P)<M$ (>0), $\min(1, 1-u(P)/M)\in\mathfrak{B}(\mathfrak{R})$, we find that $\mathfrak{R}$ is of D-type if and only if $\mu(P, \mathfrak{A}(\mathfrak{R}))\equiv 1$. Taking Theorem 4.1 in [7] into account, for $\mathfrak{R}$ with positive boundary we can write

$$\begin{array}{ccc} \mu(P, \mathfrak{A}(\mathfrak{R}))\equiv 1 & \leftrightarrows & \text{D-}type \\ \downarrow & & \updownarrow \\ \omega(P)\equiv 1 & \leftarrow & \text{F-}type, \end{array}$$

[12] For the definition of an ideal boundary component, cf. [7], Chap. III, § 5.

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where $\widehat{\underset{\smile}{\downarrow}}$ means that this is known to us only in a special case. Theorem 4.2 in [7] is included in this scheme. Here are left some questions open still.

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