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ON THE EXTREME POINTS OF QUOTIENTS OF L^∞ BY DOUGLAS ALGEBRAS

BY

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ABSTRACT. Let B be a Douglas algebra which admits best approximation. It will be shown that the following are equivalent: (1) The unit ball of (L^∞/B) has no extreme points; (2) For any Blaschke product b with $\bar{b} \notin B$, there exists $h \in B$ such that $\|\bar{b} - h\| = 1$ and $h|_E \not\equiv 0$, where E is the essential set of B.

It will also be proven that if $B \supseteq H^\infty + C$ and its essential set E contains a closed G_δ set, then the unit ball of (L^∞/B) has no extreme points. Many known results concerning this subject will follow from these results.

1. **Introduction.** Let L^∞ denote the algebra of bounded measurable functions on the unit circle T, and let H^∞ denote the subalgebra of L^∞ consisting of all bounded analytic functions in the open unit disk D. We identify L^∞ with $C(X)$, where X is the maximal ideal space of L^∞ and $C(X)$ is the space of continuous functions on X. It is well known that $H^\infty + C$ is the smallest closed subalgebra of L^∞ which contains H^∞. A closed subalgebra B of L^∞ which contains H^∞ is called a Douglas algebra. The reader is referred to [5] and [12] for the theory of Douglas algebras and [4] for uniform algebras.

The object of this paper is to study the existence of extreme points of the unit ball of L^∞/B, where B is a Douglas algebra. The paper contains new results and new direct proofs for almost all known results.

If B is a Douglas algebra, we say that B is of type-R if the unit ball of L^∞/B has no extreme points. The problem of characterizing type-R algebras gained much attention recently. It was shown in [10] that H^∞ is not of type-R, while $H^\infty + C$ is of type-R as was shown in [1]. If E is a weak peak set for H^∞, we define $H_E^\infty = \{f \in L^\infty : f|_E \in H^\infty|_E\}$. This algebra is a Douglas algebra which is of type-R if E is a peak set for H^∞ [11], and is not of type-R if E is the support of some representing measure for H^∞ [8]. For $F \subset T$, let $L_F^\infty = \{f \in L^\infty : f$ is continuous at each $x \in F\}$, then $H^\infty + L_F^\infty$ is a Douglas algebra [3], which is of type-R if F is closed or open [15], [11].

A Douglas algebra B is said to admit a best approximation if for any $f \in L^\infty$, there exists $h \in B$ such that $\mathrm{dist}(f, B) = \inf_{g \in B} \|f + g\| = \|f - h\|$, where $\| \ \|$ denotes

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the L^∞ norm. The essential set E for B, is the smallest closed subset of $M(L^\infty)$ such that every function f in L^∞ which vanishes on E belongs to B.

All the above mentioned algebras admit best approximation, so it seemed that for one to study the type-R algebras one must assume that the algebra admits best approximation. Recently, K. Izuchi [9] was able to free most type-R algebras from this restriction, and showed for example that $H^\infty + L_F^\infty$ is of type-R for any measurable subset F of T. He showed in the same paper that if E is the essential set of a Douglas algebra B, then any one of the following conditions implies that B is of type-R.

(1) $B \neq H^\infty$ and $\hat{m}(E) > 0$, where $\hat{m}$ is the lifting Lebesgue measure on T.

(2) $B \neq H^\infty$ and E contains a closed G_δ subset of X, where X is the maximal ideal space of L^∞.

The above mentioned papers failed to give a necessary and sufficient condition for a Douglas algebra to be of type-R. In this paper we prove the following results:

THEOREM 1. *Let B be a Douglas algebra which admits best approximation. Then the following are equivalent*:

(a) *The algebra B is of type-R*,

(b) *For any Blaschke product b with $\bar{b} \notin B$, there exists $h \in B$ such that $\|\bar{b} - h\| = 1$ and $h(E) \not\equiv 0$, where E is the essential set of B.*

REMARK. The implication from (b) to (a) does not need the assumption that B admits best approximation. It would be interesting to know if one can get rid of the assumption that B admits best approximation in Theorem 1.

THEOREM 2. *If $B \supseteq H^\infty + C$ and its essential set E contains a peak set for L^∞, then B is of type-R.*

Our proof of this theorem is entirely different from the one given by K. Izuchi in [9]. It is a direct proof and depends on a recent result by P. Gorkin [6].

The largest C^*-subalgebra of $H^\infty + C$ will be denoted by QC. Thus $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$, where bar denotes complex conjugation. For $\psi \in M(L^\infty)$ we let $E_\psi = \{\phi \in M(L^\infty) : \phi(q) = \psi(q)$ for all $q \in QC\}$. We call E_ψ the QC level set corresponding to ψ. For $t \in M(QC)$ [the maximal ideal space of QC] we also write $E_t = \{\phi \in M(L^\infty) : \phi(q) = t(q)$ for all $q \in QC\}$. In this case we call E_t the QC level set corresponding to t.

Theorems 1 and 2 give most of the known results about the type-R algebras, and produce new algebras of this kind.

Notations. Throughout this paper, X will denote the maximal ideal space of L^∞. If $\alpha \in T$, then X_α is the fiber of X over α. If B is a Douglas algebra then $M(B)$ denotes the maximal ideal space of B.

2. **Type-R Algebras.** In this section we establish the following result.

THEOREM 1. *Let B be a Douglas algebra which admits best approximation. Then the following are equivalent*:

(a) B is of type-R;

(b) for every Blaschke product b, with $\bar{b} \notin B$, there exists $h \in B$ such that $\|\bar{b} - h\| = 1$ and $h|_E \not\equiv 0$, where E is the essential set of B.

The proof requires the following result.

LEMMA 1 [8, Theorem 1]. *Suppose B is a Douglas algebra which admits best approximation. For $f \in L^\infty$, $\|f + B\| = 1$, the following are equivalent*:

(a) *$f + B$ is an extreme point of the unit ball of (L^∞/B)*;

(b) *$f|_E$ has a unique best approximation h in $B|_E$, and $|f|_E + h| = 1$, where E is the essential set of B.*

Proof of Theorem 1. (a) $\Rightarrow$ (b): Suppose that there exists a Blaschke product b, $\bar{b} \notin B$ such that if $h \in B$ with $\|\bar{b} - h\| = 1$, then $h(E) \equiv 0$. We claim that $\bar{b}|_E$ admits a unique best approximation in $B|_E$. Assuming the claim for a moment, we get by Lemma 1 that $\bar{b} + B$ is an extreme point of the unit ball of L^∞/B. This contradiction shows that (a) implies (b). To prove the claim, suppose that there exists $h_0 \in B$ such that $\text{dist}(\bar{b}_{|_E}, B_{|_E}) = \|\bar{b}_{|_E} - h_{0\,|_E}\| = 1$ and $h_{0\,|_E} \not\equiv 0$. Note that $\text{dist}(\bar{b}, B) = 1$ implies that $\text{dist}(\bar{b}_{|_E}, B_{|_E}) = 1$ [8, Corollary 1]. Let $f \in L^\infty$, $\|f\| = 1$ and $f = \bar{b} - h_0$ on E. Let $h = \bar{b} - f$, then $h_{|_E} = h_{0\,|_E}$. Since E is the essential set of B, we get $h \in B$. Now $\|\bar{b} - h\| = \|f\| = 1$. Thus $\text{dist}(\bar{b}, B) = \|\bar{b} - h\| = 1$. By assumption $h(E) = 0$, hence $h_0(E) = 0$. This contradiction proves our claim and consequently ends the proof of (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Let $g \in L^\infty$ such that $\|g + B\| = 1$. Then there exists a sequence $\{g_n\}$ in B such that $\|g_n + g\| \to 1$. By [13], there exists a Blaschke product b such that $bG_n \in H^\infty + C$ for all n, where $G_n = g + g_n$. It is clear that $\bar{b} \notin B$. By (b), there exists $h \in B$ such that $\|\bar{b} - h\| = 1$ and $h_{|_E} \not\equiv 0$. By [16, Proposition 2], E is the closure of $\cup\{\text{support of } m : m \in M(B) \backslash M(L^\infty)\}$. Thus there exists $m \in M(B) \backslash M(L^\infty)$ such that $h_{|\text{supp}\, m} \not\equiv 0$ where supp m denotes the support of m. Now $\|\bar{b} - \frac{1}{2}h\| = \|\frac{1}{2}\bar{b} + \frac{1}{2}(\bar{b} - h)\| \le \frac{1}{2} + \frac{1}{2} = 1$. Let $x_0 \in \text{supp}\, m$ such that $h(x_0) \neq 0$. Since $\bar{b}(x_0) - \frac{1}{2}h(x_0) = \frac{1}{2}\bar{b}(x_0) + \frac{1}{2}(\bar{b}(x_0) - h(x_0))$, we see that $\bar{b}(x_0) - \frac{1}{2}h(x_0)$ is the average of two unequal points $\bar{b}(x_0)$ and $\bar{b}(x_0) - h(x_0)$ in the unit disk and so $|\bar{b}(x_0) - \frac{1}{2}h(x_0)| < 1$. Define f on X by $f(x) = 1 - |\bar{b}(x) - \frac{1}{2}h(x)|$, then $f \in L^\infty$, $f \ge 0$ and $f(x_0) > 0$. There exists a clopen subset W of X such that $x_0 \in W$, $W \cap \text{supp}\, m \neq \phi$, $(X \backslash W) \cap \text{supp}\, m \neq \phi$ and $a = \min\{f(x) : x \in W\} > 0$. Clearly $f \ge a \cdot \chi_W$, where χ_W is the characteristic function of W. Note that $\chi_W \notin B$, for if $\chi_W \in B$ then

$$\int \chi_W \, dm = \int \chi_W^2 \, dm = \left(\int \chi_W \, dm\right)^2$$

which contradicts the fact $0 < \int \chi_W \, dm < 1$. To finish the proof of Theorem 1, it is sufficient to show that $\|g \pm a \cdot \chi_W + B\| \le 1$. Let $F_n = \frac{1}{2}hbG_n$. Then clearly $F_n \in B$. Let $\varepsilon > 0$ be given, then there exists an integer N such that $\|G_N\| \le 1 + \varepsilon$.

Now

$$\begin{aligned}\|g\pm a\cdot\chi_W+B\| &= \|G_N\pm a\cdot\chi_W+B\|\le\|G_N\pm a\cdot\chi_W-F_N\|\\ &\le\sup_{x\in X}\{|G_N(x)-F_N(x)|+a\cdot\chi_W\}\\ &\le\sup_{x\in X}\{|G_N(x)-F_N(x)|+f(x)\}\\ &\le\sup_{x\in X}\{|1-\tfrac{1}{2}b(x)h(x)|\,\|G_N\|+f(x)\}\\ &\le\sup_{x\in X}\{|\bar b(x)-\tfrac{1}{2}h(x)|\,(1+\varepsilon)+f(x)\}\\ &\le\sup_{x\in X}\{|\bar b(x)-\tfrac{1}{2}h(x)|+f(x)+\varepsilon\}\\ &=1+\varepsilon\end{aligned}$$

Since ε is an arbitrary positive number, we get $\|g\pm a\cdot\chi_W+B\|\le 1$ and this ends the proof of Theorem 1.

COROLLARY 1 [9, Theorem 4.1]. *Suppose B has the following two properties*:

(a) *B admits best approximation, and*

(b) *There exists a Blaschke product b_0 such that $\bar b_0\notin B$ and $\bigcup\{\text{supp}\,\mu_x : |x(b_0)|\neq 1$ and $x\in M(B)\}$ is dense in the essential set E of B, where μ_x is the representing measure for x. Then B is not of type-R.*

Proof. There exists a Blaschke product b such that $b\bar b_0^n\in H^\infty+C$ for all n [13]. Let $f_n=b\bar b_0^n$ then $|f_n|=1$ on X. Now for $x\in M(B)$, $|b_0(x)|\neq 1$ we have $|x(b)|=|x(f_n)\cdot x(b_0^n)|\le|x(b_0)|^n\to 0$, thus $x(b)=0$ and hence $\bar b\notin B$. Let $h\in B$ be such that $\|\bar b-h\|=1$, (such an h exists by (a)). We claim that $h_{|_E}\equiv 0$, to prove the claim we write $1=x(1-bh)=\int(1-bh)\,d\mu_x\le 1$, thus $1-bh\equiv 1$ on supp μ_x, so $h_{|\text{supp}\,\mu_x}\equiv 0$ and so by (b) we get $h_{|_E}\equiv 0$, and that proves the claim. By Theorem 1 we conclude that B is not of Type-R.

As an application for Corollary 1, let B be a Douglas algebra which admits best approximation and let m be a representing measure for some $x\in M(B)$ then $E=\text{supp}\,m$ is the essential set of B_E, $[B_E=\{f\in L^\infty : f_{|_E}\in B_{|_E}\}]$. There exists a Blaschke product b such that $x(b)=0$, so we can apply Corollary 1 to conclude that B_E is not of type-R. As a special case, H^∞_E is not of type-R, where $E=\text{supp}\,\mu_x$ and $x\in M(H^\infty)$ [8]. Note that the algebra B_E admits best approximation. This follows from the work in [14].

COROLLARY 2. *Let B be of type-R algebra which admits best approximation. If $\{f_n\}$ is a sequence in L^∞, then the algebra $A=B[f_1,f_2,\ldots,f_n,\ldots]$ is of type-R. In particular $H^\infty[f_1,f_2,\ldots,f_n,\ldots]$ is of type-R if and only if $f_i\notin H^\infty$ for some i.*

Proof. By [16, Theorem 2], the essential set E of A is the essential set of B. If b is a Blaschke product such that $\bar b\notin A$ then $\bar b\notin B$. Since B is of type-R, then there exists $h \in B$ such that $\|\bar{b}-h\|=1$, $h_{|_E} \not\equiv 0$. By Theorem 1 we get A is of type-R.

COROLLARY 3. *Let A be a Douglas algebra which admits best approximation. Suppose $A=\bigcap_{i=1}^N A_i$ where A_i is a Douglas algebra for all i. If A is of type-R then A_n is of type-R for some n, $1 \le n \le N$.*

Proof. Let E be the essential set of A and E_i be the essential set of A_i, $1 \le i \le N$. We claim that $E=\bigcup_{i=1}^N E_i$. Clearly $\bigcup_{i=1}^N E_i \subset E$. Now suppose $f \in L^\infty$ such that $f(\bigcup_{i=1}^N E_i)=0$. Then $f(E_i)=0$, for all $1 \le i \le n$ hence $f \in A_i$ for all $1 \le i \le n$ and so $f \in A$. We conclude that $E \subset \cup E_i$ from the definition of the essential set. Suppose that A_i is not of type-R, for all $1 \le i \le N$. Then there exist Blaschke products b_i such that $\bar{b}_i \notin A_i$ with the property that if $h_i \in A_i$, $\|\bar{b}_i - h_i\|=1$ then $h_i|_{E_i} \equiv 0$. Let $\bar{b}=\Pi_{i=1}^N \bar{b}_i$, then $\bar{b} \notin A$. Since A is of type-R then by Theorem 1, there exists $h \in A$ such that $\|\bar{b}-h\|=1$ and $h\,|_E \not\equiv 0$. Since $E \subset \bigcup_{i=1}^N E_i$ then $h\,|_{E_n} \not\equiv 0$ for some n, $1 \le n \le N$. Thus

$$\left\|\bar{b}_n - \prod_{\substack{i=1\\ i\neq n}}^N b_i h\right\| = 1 \quad \text{and} \quad \left(\prod_{\substack{i=1\\ i\neq n}}^N b_i h\right)\bigg|_{E_n} \not\equiv 0.$$

This contradiction ends the proof of Corollary 3.

REMARK. The above Corollary is not true if A is an arbitrary intersection of Douglas algebras. For example, let $\{S_\alpha\}$ be the family of support sets for representing measures for $H^\infty + C$, then $H^\infty + C = \bigcap_\alpha H^\infty_{S_\alpha}$ $H^\infty + C$ is of type-R, while $H^\infty_{S_\alpha}$ is not of type-R for every α.

3. **A Class Of Type-R Algebras.** In this section we give a direct and short proof of Theorem 2 [9, Theorem 3], which together with Theorem 1 covers most of the known type-R algebras.

THEOREM 2. *If $B \supseteq H^\infty + C$ is a Douglas algebra and its essential set E contains a peak set K for L^∞, then B is of type-R.*

Our proof of Theorem 2 requires the following result.

LEMMA 2 [6, Theorem 2.13]. *Let F be a non-empty clopen subset of X_α. Then F contains a non-trivial QC level set.*

Proof of Theorem 2. Let b be a Blaschke product such that $\bar{b} \notin B$. Let $K \cap X_\alpha \neq \phi$ for some $\alpha \in T$, and pick $x_0 \in K \cap X_\alpha$. The set $G=\{x \in X : b(x)=b(x_0)\}$ is a peak set for H^∞ with peaking function $b(x_0)/(2b(x_0)-b)$. Thus the set $L=G \cap K \cap X_\alpha$ is a non-empty peak set for L^∞. Let f be a peaking function for L. By [7, p. 171], f assumes 1 on a non empty clopen subset F of X_α. Since f is 1 on L only, we get $F \subseteq L$. By Lemma 2 there exists a non-trivial QC level set E_{t_0} such that $E_{t_0} \subseteq F$. Thus $b\,|_{E_{t_0}} = b(x_0)$. By [2], there exists an open subset V of $M(QC)$ with $b\,|_{E_{t_0}}$ constant for each $t \in V$. Let q_1 be

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a QC function with $q_1(t_0)=1$ and $q_1(t)=0$ for all $t \in M(QC)\setminus V$ and $0 \le q_1 \le 1$. Let $h=\bar{b}q_1$. Then clearly $h \in QC$ and $h|_{E_{t_0}} \not\equiv 0$. Now, $1 \le \|\bar{b}-h\| \le \|\bar{b}(1-q_1)\| \le \|1-q_1\| \le 1$. Thus $\|\bar{b}-h\|=1$. By Theorem 1, we get that B is of type-R. This ends the proof of Theorem 2.

From Theorem 2, we get the following known results:

COROLLARY 4. *If $B=H^\infty+L^\infty_F$, F is a measurable subset of T then B is of type-R.*

COROLLARY 5. *If B is a Douglas algebra such that $\hat{m}(\Gamma)>0$, where Γ is the essential set of B, then B is of type-R.*

Proof. Since $\hat{m}(\Gamma)>0$ then by [4, page 18] there exists a non-empty clopen set W in $M(L^\infty)$ such that $W \subset \Gamma$. Since W is a closed G_δ-set, by Theorem 2 we get B is of type-R.

Finally, we end the paper with the following question: Does there exist a maximal antisymmetric set E for $H^\infty+C$ such that H^∞_E is of type-R?

REFERENCES

1. S. Axler, I. Berg, N. Jewell and A. Shields, *Approximation by compact operators and the space $H^\infty+C$*, Annals of Math., **109** (1979), 601–612.

2. K. F. Clancey and J. A. Gosselin, *On the local theory of Toeplitz operators*, Illinois J. Math. **22** (1978), 449–458.

3. A. M. Davie, T. W. Gamelin and J. Garnett, *Distance estimates and point-wise bounded density*, Trans. Amer. Math. Soc. **175** (1973), 37–68.

4. T. W. Gamelin, *Uniform algebras*, Prentice-Hall (1969).

5. J. Garnett, *Bounded analytic functions*, Academic Press (1981).

6. P. Gorkin, *Decomposition of the maximal ideal space of L^∞*, Ph.D. thesis, Michigan State University, 1982.

7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall (1962).

8. K. Izuchi and R. Younis, *On the quotient space of two Douglas algebras*, submitted.

9. K. Izuchi, *Extreme points of unit balls of quotients of L^∞ by Douglas algebras*, preprint.

10. P. Koosis, *Weighted quadratic means of Hilbert transforms*, Duke Math. J. **38** (1971), 609–634.

11. D. Leucking and R. Younis, *Quotients of L^∞ by Douglas algebras and Best approximation*, Transaction, Amer. Math. Soc. **V276** (1983), 699–706.

12. D. Sarson, *Function theory on the unit circle*, Virginia Poly. Inst. and State Univ. (1979).

13. C. Sundberg, *A note on algebras between L^∞ and H^∞*, Rocky Mount. J. Math. **11** (1981), 333–336.

14. R. Younis, *Best approximation in certain Douglas algebras*, Proc. Amer. Math. Soc. **80** (1980), 639–642.

15. R. Younis, *Properties of certain algebras between L^∞ and H^∞*, J. Funct. Anal. **44** (1981), 381–387.

16. R. Younis, *M-ideals of L^∞/H^∞ and support sets*, to appear in Illinois J. Math.

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