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ASYMPTOTIC VALUES ALONG JULIA RAYS

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Let f be a function meromorphic in the finite complex plane $\mathbf{C}$. If for some number θ, $0 \leqq \theta < 2\pi$, the family, $f_r(z) = f(re^{i\theta}z)$, is not normal at $z = 1$, then the ray $\arg z = \theta$ is called a *Julia ray*. Such a ray has the property that in every sector containing it, f assumes every value infinitely often with at most two exceptions. Many authors have taken this property as the definition of a Julia ray.

A function f is said to have an *asymptotic value w along an unbounded set F* if $f(z)$ tends to w as z tends to infinity along F; F is called an *asymptotic set* for f.

Consider now the following three problems:

(i) Can an entire function have one exceptional finite value for one Julia ray and another exceptional finite value for another Julia ray?
(ii) Can an entire function have one finite asymptotic value along one Julia ray and another finite asymptotic value along another Julia ray?
(iii) Can a meromorphic function have one exceptional and asymptotic value along one Julia ray and another exceptional and asymptotic value along another Julia ray?

Problem (i) was posed by C. Rényi (see [**4**, p. 10]) and solved by K. F. Barth and W. J. Schneider [**4**]. In this note we solve the related problems (ii) and (iii).

Let $A(F)$ denote as usual the set of functions continuous on F and holomorphic on the interior F^0. A closed set $F \subset \mathbf{C}$ is called a *set of asymptotic approximation* (see [**5**]) if for each $g \in A(F)$ there is an entire function f such that

$$\chi(f(z), g(z)) \to 0, \quad \text{as } z \to \infty \text{ on } F,$$

where χ denotes the chordal metric on the closed plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$.

THEOREM. (Arakelian [**1**]). *A closed set $F \subset \mathbf{C}$ is a set of asymptotic approximation if and only if $\mathbf{C}^*\backslash F$ is connected and locally connected.*

We shall call an unbounded closed set F a *strong Julia set* for an entire function f provided for every sequence $\{\zeta_n\}$ tending to ∞ on F, the family $f_n(z) =$

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$f(\zeta_n z)$ is not normal at $z = 1$. Any ray whose intersection with F is unbounded must be a Julia ray. The following shows that without further restrictions, this notion is uninteresting.

COROLLARY. *Every unbounded closed set is a strong Julia set for some entire function.*

We could easily prove this from Arakelian's theorem, however this is unnecessary. It was shown in [7] that **C** is itself a strong Julia set, and the property is clearly hereditary.

When we take our closed set to be **C** in the corollary, we have an entire function for which every unbounded curve is a Julia curve. Faced with such pathology (or equidistribution if we're optimistic) we are remined of W. Gross' example [**8**] of a meromorphic function f which maps every unbounded curve onto a dense subset of the plane. Such curves are called *Weierstrass curves* or *curves of total indetermination* for f.

These two phenomena, though similar, are fundamentally different. If F is a Julia set for f, then every value (with perhaps two exceptions) is *assumed* infinitely often *near* F; whereas if F is a Weierstrass set, every value is merely *approached*, but *on* F itself. It is because of this difference that a Julia ray can also be an asymptotic ray as in problems (ii) and (iii).

Antithetically to Weierstrass behaviour, an unbounded closed set F is called a *set of non-uniqueness* if there is a transcendental entire function with zero (or finite) asymptotic value along F.

THEOREM 1. *A closed set is a set of non-uniqueness if and only if it is contained in a proper set of asymptotic approximation.*

For details we refer to [**5**] where a similar result is proved for the disc rather than for **C**. Since the difficulties are purely topological (by Arakelian's theorem), the arguments carry over.

We consider now those Julia sets which are sets of non-uniqueness and hence are very far from being Weierstrass sets.

THEOREM 2. *Let F be an unbounded set of asymptotic approximation. Then for each $g \in A(F)$, there is an entire f such that $\chi(f(z), g(z)) \to 0$, as $z \to \infty$ on F, and such that ∂F is a strong Julia set for f.*

In particular, the answer to Problem (ii) is affirmative. In fact, we have the following:

COROLLARY. *Suppose S is a set of first category on $[0, 2\pi]$ and v is a continuous complex-valued function on $[0, 2\pi]$ (with $v(0) = v(2\pi)$). Then there is an entire function f such that for each $\theta \in S$, $\arg z = \theta$ is an asymptotic Julia ray for f with asymptotic value $v(\theta)$.*

Proof of Corollary. We may assume: $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is closed and nowhere dense. We set

$$F_n = \{re^{i\theta} : r \geqq n, \theta \in S_n\},$$

and

$$F = \bigcup_{n=1}^{\infty} F_n.$$

Also set $g(re^{i\theta}) = v(\theta)$, $re^{i\theta} \in F$. The corollary now follows directly from the theorem.

The corollary is in some sense best possible, for if S is not of first category, and f has asymptotic value zero on each ray $\arg z = \theta$, $\theta \in S$, then we may write

$$S = \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n = \{\theta \in S: |f(re^{i\theta})| \leqq 1, r \geqq n\}.$$

Since S is of second category, S_n is dense in an interval $[\theta_1, \theta_2]$ for some n. Thus f is bounded in the sector $\theta_1 \leqq \arg z \leqq \theta_2$, and hence has no Julia rays in this sector.

Proof of Theorem 2. Choose sets $P \subset F$ and $Q \subset (\mathbf{C} \backslash F)$ discrete but so "thick" that for each sequence $\{\zeta_n\}$ tending to ∞ on ∂F,

(1) $d(\zeta_n^{-1}P, 1) \rightarrow 0$ and $d(\zeta_n^{-1}Q, 1) \rightarrow 0$,

where d denotes Euclidean distance. Set $\tilde{F} = F \cup Q$. For each $q \in Q$, let $p(q)$ be a point of P at minimal distance from q. Now extend g to a function $\tilde{g}$ on $\tilde{F}$ by setting

(2) $\tilde{g}(q) = g(p(q)) + 1, \quad q \in Q.$

Now $\tilde{F}$ is a set of asymptotic approximation. Hence there is an entire f such that

(3) $\chi(f(z), \tilde{g}(z)) \rightarrow 0, \quad z \rightarrow \infty$ on $\tilde{F}$.

Let $\{\zeta_n\}$ be any sequence tending to ∞ on ∂F. Then from (1), (2) and (3), the family $f_n(z) = f(\zeta_n z)$ cannot be equicontinuous at $z = 1$ and hence cannot be normal there. This completes the proof.

Example. We shall now construct a function which resolves Problem (iii) in the affirmative. Indeed, we construct a meromorphic f which has ∞ as an exceptional asymptotic value on the Julia ray $z > 0$ and zero as exceptional asymptotic value along the Julia ray $z < 0$.

First choose an entire function h whose zeros are precisely the sequence

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$n + i$, $n = 1, 2, \ldots$, and set $\Psi(z) = h(z)/h(-z)$. By Arakelian's theorem there is an entire function φ with

$$(4) \quad \left| \varphi(x) - \log \frac{x}{|\Psi(x)|} \right| < 1, \quad x \geqq 1 \quad \text{and}$$

$$(5) \quad \left| \varphi(x) - \log \frac{1}{|\Psi(x)x|} \right| < 1, \quad x \leqq -1.$$

Actually, we didn't need the full strength of Arakelian's theorem; Carleman's theorem [**6**] would have been sufficient.

Now set $f = e^{\varphi}\Psi$. From (4) and (5) it follows that f tends to ∞ along $z > 0$ and to zero along $z < 0$. Since f has zeros close to the positive real axis and has poles close to the negative real axis, the argument invoked in the proof of Theorem 2 shows that both are Julia rays. Since f has the same zeros and poles as Ψ, it is also obvious that ∞ is an exceptional value for the Julia ray $\arg z = 0$ and zero is exceptional for $\arg z = \pi$. Thus f has all the properties required in Problem (iii).

Remarks and open problems. 1) We have defined a Julia ray for f in such a way that f has a "Picard type" behaviour in every sector about the ray. A stronger notion would replace sectors by parallel strips about the ray. This and even stronger types of Picard behaviour have been studied. The questions in this paper could be treated in these contexts also, and all of the techniques would carry over.

2) Similarly we could successfully carry on this investigation in the unit disc rather than on **C**.

3) By looking at sets of non-uniqueness, we have characterized those sets along which a transcendental entire function can have a finite asymptotic value. Infinity is different because of the maximum modulus principle. There remains the following:

Problem (iv). Characterize those sets along which a transcendental function can tend to ∞.

This problem naturally breaks into four parts depending on whether we are considering functions in the plane or in the disc and whether we are considering meromorphic or holomorphic functions. When considering the unit disc, replace the word "transcendental" by "non-constant".

The plane meromorphic case is solved by the following Runge theorem for closed sets.

THEOREM (Roth [**11**]). *Let F be any closed set in* **C**. *Then for each g holomorphic on F and $\epsilon > 0$, there is an f meromorphic on* **C** *such that $|f(z) - g(z)| < \epsilon$, $z \in F$.*

COROLLARY. *Let F be a closed set in $\mathbf{C}$ with unbounded complement. Then there is a transcendental meromorphic function with asymptotic value ∞ along F.*

Proof. Let $z_n \to \infty$ outside of F and set $g(z) = z$ on F and $g(z_n) = 0$, $n = 1, 2, \ldots$. By Roth's theorem, there is a meromorphic f with:

$$|f(z) - g(z)| < 1, \quad z \in F \cup \{z_n\}_{n=1}^{\infty}$$

Thus f is the required function.

The situation in the unit disc Δ is quite different. If f is to exist, we must surely insist that $\Delta \backslash F$ has every point of the unit circle $\partial\Delta$ as limit point. But even this is not enough. For we can construct such an F having the property that for each $e^{i\theta}$ in a set of positive measure on $\partial\Delta$, F contains a Stolz angle with vertex at $e^{i\theta}$. By a well-known uniqueness theorem [**10**, p. 72], a non-constant meromorphic function cannot tend to a value as $|z| \to 1$ along F. Thus Problem (iv) is unresolved for meromorphic functions in the unit disc.

For holomorphic functions, Problem (iv) becomes interesting when $\mathbf{C}\backslash F$ is not connected, that is, when we cannot solve it trivially by approximation theorems as we did in the meromorphic case. We state a variant of Problem (iv).

Problem (v). Let $G = \mathbf{C}$ or Δ and let $\{D_n\}$ be a sequence of disjoint discs in G, accumulating at every point of the boundary of G. Set $F = G \backslash \bigcup_{n=1}^{\infty} D_n$. Characterize those sequences for which there is an f tending to ∞ on F.

For transcendental meromorphic f on $\mathbf{C}$ we have already solved the more general problem (iv).

For entire f the problem is open, but we merely point out that some sequences work and some don't. Take a cannonical product f whose zeros grow very quickly. Then if the D_n are centered on the zeros and shrink rapidly, f tends to ∞ on F. This sequence works.

Baker and Liverpool [**3**] have constructed a sequence of discs $\{D_n\}$ whose union is a Picard set. From the definition of a Picard set, any f with three Picard exceptional values in F reduces to a constant. Hence this sequence won't work in Problem (v).

From our discussion of Problem (iv), it is clear that in the unit disc Δ some sequences don't work for Problem (v).

In the unit disc Δ some sequences *do* work for Problem (v). Bagemihl, Erdös, and Seidel [**2**] have constructed a function f holomorphic in Δ which tends to ∞ outside of a sequence of disjoint discs.

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