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# BIFURCATION OF STEADY-STATE SOLUTIONS OF A SCALAR REACTION-DIFFUSION EQUATION IN ONE SPACE VARIABLE

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**Abstract**

We study the bifurcation of steady-state solutions of a scalar reaction-diffusion equation in one space variable by modifying a "time map" technique introduced by J. Smoller and A. Wasserman. We count the exact number of steady-state solutions which are totally ordered in an order interval. We are then able to find their Conley indices and thus determine their stabilities.

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## 1. Introduction

We study the bifurcation of steady-state solutions of a scalar reaction-diffusion equation in one space variable

$$(1.1)\qquad u_t - u_{xx} - f(u) = 0, \qquad (x, t) \in \Omega \times \mathbb{R}_+ \subset \mathbb{R} \times \mathbb{R}_+$$

together with the boundary conditions

$$(1.1')\qquad u(x, t) = 0, \qquad (x, t) \in \partial\Omega \times \mathbb{R}_+$$

and initial data $u(x, 0)$. For proper choices of $f$, equation (1.1) models some chemical and biological diffusion phenomena [2, 4, 10].

[†] Deceased. Nicholas D. Kazarinoff died in November 1991. He was a fine mathematician and a respected friend.

In this paper, we shall concern ourselves with bounded spatial regions $\Omega = \{|x| < L\}$; this requires that $u$ satisfy bounded boundary conditions at $\pm L$. Then the steady-state equation associated with (1.1) and (1.1′) is the two-point boundary value problem

$$(1.2) \qquad u'' + f(u) = 0, \qquad -L < x < L, \qquad u(-L) = u(L) = 0.$$

The real-valued function $f\colon [0, \infty) \to \mathbb{R}$ is initially assumed to be $C^2$ and to have exactly three nonnegative simple roots $0 \le s_0 < s_1 < s_2$ with $f(0) \ge 0$. Furthermore, we also assume that the area of a "hill" exceeds that of the preceding "valley".

We obtain the local bifurcation diagram of positive solutions $u$ of (1.2) satisfying

$$(1.3) \qquad s_0 < \|u\|_\infty < s_2,$$

for such a function $f$; that is, we count the exact number of nonnegative solutions in the order interval $[0, s_2) = \{u | 0 \le u < s_2\}$. Notice that phase-plane analysis shows $f(\|u\|_\infty) > 0$ if $u$ is a positive solution, and if $f(0) = 0$, then $u \equiv 0$ is always a steady-state solution (that is, for all $L$). We are interested in nonconstant positive solutions other than the trivial solution $u \equiv 0$, if it exists.

We study (1.2) through an approach due to J. Smoller and A. Wasserman [11] who studied (1.2) by the technique of "time map" $T(\alpha)$ to count the exact number of solutions of (1.2) for $f$ a cubic polynomial. Thus our method of proof is not new. We show that, for large $L$, if $f$ satisfies (2.1)–(2.3), then (1.2) has exactly three totally ordered positive solutions in the interval $(0, s_2)$ if $f(0) > 0$ and (1.2) has exactly three totally ordered positive solutions in $(0, s_2)$ other than the trivial solution $u \equiv 0$ if $f(0) = 0$ and $f'(0) < 0$.

*Note.* Our method of proof allows us to relax the $C^2$-hypothesis on $f$; we only need $f$ to be $C^1$.

The research in this paper is motivated by papers [3, 5] in which a multiplicity result of at least three totally ordered positive solutions of the Dirichlet problem

$$\Delta u + \lambda f(u) = 0 \text{ in } \Omega \ (\Omega \text{ is a smooth bounded domain in } \mathbb{R}^k \ (k \ge 1)),$$
$$(1.4) \qquad u = 0 \text{ on } \partial\Omega,$$

in the ordered interval $(0, s_2)$ was obtained separately by variational and topological index argument for function $f \in C^1$ such that $0 < s_0 < s_1 < s_2$ and satisfying

(f1) $f(y) > 0$ on $(0, s_0)$, or (f1′) $f(0) = 0$ and $f'_+(0) > 0$,
(f2) $f(s_0) = f(s_1) = f(s_2) = 0$,

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(f3) $\int_{s_0}^{s_2} f(s)\,ds > 0$,

if $\lambda$ is large enough. More precisely, it was shown in [3, 5] that (1.2) has at least one positive solution satisfying $0 < \|u\|_\infty < s_0$ and at least two positive solutions satisfying (1.3) if $\lambda$ is large enough. Moreover, these three positive solutions obtained are totally ordered (see also [7]).

NOTE 1. Conditions (f2) and (f3) correspond to our assumptions (2.1) and part of (2.3). However, (f1) and (f1′) are different from our assumption in the case that $f(0) = 0$ where we assume $f'_+(0) < 0$ and where Dancer [5] assumed $f'_+(0) \geq 0$ and where de Figueiredo [3] assumed $f'_+(0) > 0$.

NOTE 2. If we make change of variable $y = x/L$, then (1.2) becomes

$$(1.5) \qquad u_{yy} + L^2 f(u) = 0, \qquad |y| < 1, \qquad u(\pm 1) = 0,$$

so that if $\lambda = L^2$, we obtain a problem of the type (1.4). We prefer, however, to consider the equation (1.2) because, as we shall see, its solutions can be given a nice geometrical interpretation [10, p. 185].

REMARK. A cubic polynomial $f$ cannot satisfy the conditions (2.1)–(2.3) of Theorem 1 stated in Section 2. Nevertheless, we should remark that for $f = -(x - s_0)(x - s_1)(x - s_2)$ satisfying (f4) $0 < s_0 < s_1 < s_2$ and (f5) $\int_{s_0}^{s_2} f(s)\,ds > 0$, the problem of the complete bifurcation diagram of solutions of (1.2) is still open. Only partial results are known; see the first author's paper [13] for details.

As Smoller and Wasserman did in [11], we rewrite (1.2) as a first order system

$$(1.6) \qquad u' = v, \qquad v' = -f(u), \qquad |x| < L,$$

and we consider the phase plane for (1.6) locally illustrated in Figure 1 for $s_0 > 0$.

It is clear that positive solutions of (1.2) satisfying (1.3) correspond to those orbits of (1.6) which "begin" on the interval $(A_0, A_1)$ ($A_0 > 0$, $A_1 > 0$ with $A_0^2 = 2F(s_0)$ and $A_1^2 = 2F(s_2)$, where $F(s) \equiv \int_0^s f(u)\,du$) on the $v$-axis, and "end" on the $v$-axis, and take "time" (parameter length) $2L$ to make the journey [11]. Then, as in [11], we use the time map

$$(1.7) \qquad T(\alpha) = 2^{-1/2} \int_0^\alpha (F(\alpha) - F(u))^{-1/2}\,du, \qquad \gamma < \alpha < s_2,$$

where $\gamma \in (s_1, s_2)$ with $\int_{s_0}^{\gamma} f(s)\,ds = 0$. Notice that solutions of (1.6) correspond to curves for which $T(\alpha) = L$. This led us to investigate the shape of graph of $T$; see [10, pp. 186–187]. We write as (1.8) and (1.9) below two formulas from [11]:

$$(1.8) \qquad T'(\alpha) = 2^{-3/2} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{(\Delta F)^{3/2}} \frac{du}{\alpha},$$

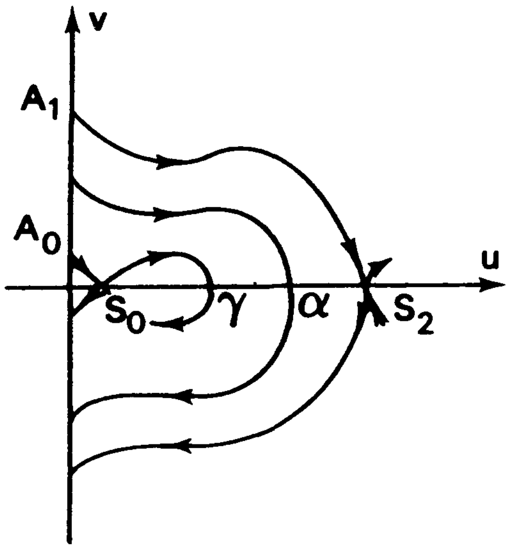


FIGURE 1

where $\Delta F = F(\alpha) - F(u)$ and $\theta(x) = 2F(x) - xf(x)$;

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > \frac{2^{-3/2}}{\alpha^2}\int_0^\alpha (\Delta F)^{-3/2}(\phi(\alpha) - \phi(u))\,du, \tag{1.9}$$

where $\phi(x) = x\theta'(x) - \theta(x)$.

We analyze the "time map" $T$ by studying the convexity of the curve $y = f(x)$. We recall that the domain of $T$ is the open interval $(\gamma, s_2)$. Furthermore, from phase-plane analysis, we know that if $\alpha$ is near $\gamma$ or $s_2$, $T(\alpha)$ must be very large. Since $T$ is a smooth function, we see that $T(\alpha)$ must have at least one critical point, a minimum on $(\gamma, s_2)$, say at $\alpha_0$. Obviously, $T(\alpha_0) > 0$ [10].

## 2. Main Result

THEOREM 1. *Suppose $f \in C^2$, and there are numbers $0 \le s_0 < s_1 < s_2$ such that the following conditions are satisfied:*

$$f(s_0) = f(s_1) = f(s_2) = 0; \tag{2.1}$$

$$f''(x) > 0 \textit{ for } x \in (0, s_1), \qquad f''(x) < 0 \textit{ for } x \in (s_1, s_2); \tag{2.2}$$

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$$\int_{s_0}^{s_2} f(s)\,ds > 0, \textit{ and there exists } \gamma \textit{ in } (s_1, s_2) \textit{ defined by } \int_{s_0}^{\gamma} f(s)\,ds = 0$$

(2.3)

*and such that* $2F(\gamma) - \gamma f(\gamma) < 0.$

*Let* $T$ *be defined by* (1.7). *Then* $T$ *has exactly one critical point, a minimum in* $(\gamma, s_2)$.

EXAMPLE OF FUNCTIONS $f$. Choose

$$f(x) = \begin{cases} -x(x-1)(x-2), & 0 \le x \le 1, \\ -\frac{1}{4}(x-1)^3 + x - 1, & 1 < x \le 3, \end{cases}$$

where $s_0 = 0$, $s_1 = 1$, $s_2 = 3$, and $\gamma = 1 + \sqrt{2}$, or for $0 < \varepsilon \ll 1$ sufficiently small, choose

$$f(x) = \begin{cases} -(x-\varepsilon)(x-(1+\varepsilon))(x-(2+\varepsilon)), & 0 \le x \le 1+\varepsilon, \\ -\frac{1}{4}(x-(1+\varepsilon))^3 + x - (1+\varepsilon), & 1+\varepsilon < x \le 3+\varepsilon, \end{cases}$$

where $s_0 = \varepsilon$, $s_1 = 1+\varepsilon$, $s_2 = 3+\varepsilon$, and $\gamma = 1 + \sqrt{2} + \varepsilon$.

REMARK. No analysis of $f''$ was used in [11]; but it is of importance in our analysis; see also [14].

NOTE 1. Since $2F(\gamma) - \gamma f(\gamma) = 2F(s_0) - \gamma f(\gamma)$, if $s_0 = 0$ in (2.3), then the condition $2F(\gamma) - \gamma f(\gamma) = -\gamma f(\gamma) < 0$ is automatically satisfied.

NOTE 2. Conditions (2.1) and (2.2) imply (2.4) and (2.5) stated below.

$$\begin{aligned} &f(x) > 0 \quad \text{for } x \in [0, s_0) \quad \text{if } s_0 > 0, \\ &f(x) < 0 \quad \text{for } x \in (s_0, s_1), \\ &f(x) > 0 \quad \text{for } x \in (s_1, s_2). \end{aligned} \tag{2.4}$$

$$f'(s_0) < 0, \qquad f'(s_1) > 0, \quad \text{and} \quad f'(s_2) < 0. \tag{2.5}$$

PROOF OF THEOREM 1.

Since we know that the "time map" $T$ has at least one critical point on $(\gamma, s_2)$, it suffices to show that $T$ has at most one critical point. For this we will show

$$T''(\alpha) > 0 \quad \text{if} \quad T'(\alpha) = 0. \tag{2.6}$$

Indeed, we shall show that $T'(\alpha) < 0$ for $\gamma < s \le p_2$ (for $p_2$ defined below) and (2.6) holds on $(p_2, s_2)$ so that $T'(\alpha)$ can vanish exactly once there (its zero lies on $(p_2, s_2)$). In (1.8), we have

$$T'(\alpha) = 2^{-3/2} \int_0^{\alpha} \frac{\theta(\alpha) - \theta(u)}{(\Delta F)^{3/2}} \frac{du}{\alpha}, \tag{2.7}$$

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where $\theta(x) = 2F(x) - xf(x)$, which gives

$$\theta'(x) = f(x) - xf'(x) \tag{2.8}$$

and

$$\theta''(x) = -xf''(x)\begin{cases} < 0 & \text{if } x \in (0, s_1), \\ > 0 & \text{if } x \in (s_1, s_2). \end{cases} \tag{2.9}$$

Now by (2.1), (2.3), and (2.4),

$$\begin{aligned} &\theta(0) = 0, \\ &\theta(s_0) = 2F(s_0) = 0 \text{ if } s_0 = 0; \qquad \theta(s_0) = 2F(s_0) > 0 \text{ if } s_0 > 0, \\ &\theta(\gamma) < 0, \text{ and} \\ &\theta(s_2) = 2F(s_2) > 0. \end{aligned} \tag{2.10}$$

So (i) if $s_0 > 0$, $\theta$ has one root at zero and at least two distinct positive roots, $q_1$ and $q_2$ with $s_0 < q_1 < \gamma < q_2 < s_2$; and (ii) if $s_0 = 0$, then $\theta$ has one root at zero and at least one positive root, $q_2$, with $\gamma < q_2 < s_2$. Also by (2.5) and (2.8), we know

(2.11)

$$\begin{aligned} &\theta'(0) = f(0) > 0 \text{ if } s_0 > 0, \\ &\theta'(s_0) = -s_0 f'(s_0) = 0 \text{ if } s_0 = 0; \qquad \theta'(s_0) = -s_0 f'(s_0) > 0 \text{ if } s_0 > 0, \\ &\theta'(s_1) = -s_1 f'(s_1) < 0, \text{ and} \\ &\theta'(s_2) = -s_2 f'(s_2) > 0. \end{aligned}$$

So by (2.9), (i) if $s_0 > 0$, then $\theta'$ has *exactly two* positive roots, $p_1$ and $p_2$ with $s_0 < p_1 < s_1 < p_2 < s_2$; and (ii) if $s_0 = 0$, then $\theta'$ has one zero root and exactly one positive root, $p_2$ with $s_1 < p_2 < s_2$. By the previous argument, (i) if $s_0 > 0$, then $\theta$ has exactly two distinct positive roots, $q_1$ and $q_2$ with $s_0 < q_1 < \gamma < q_2 < s_2$; and (ii) if $s_0 = 0$, then $\theta$ has exactly one positive root, $q_2$, with $\gamma < q_2 < s_2$. Note that

$$\theta'(p_2) = 0 \quad \text{and} \quad \theta(p_2) < 0, \tag{2.12}$$

which we use to show (2.6).

One sees that the graph of $\theta$ is as in Figure 2 for $s_0 > 0$. Thus

$$\theta(\alpha) - \theta(u) < 0 \quad \text{if } \gamma < \alpha < p_2 \text{ and } u < \alpha. \tag{2.13}$$

So

$$T'(\alpha) < 0 \quad \text{if } \gamma < \alpha \le p_2. \tag{2.14}$$

Hence, to show (2.6), we need only to consider $\alpha > p_2$.

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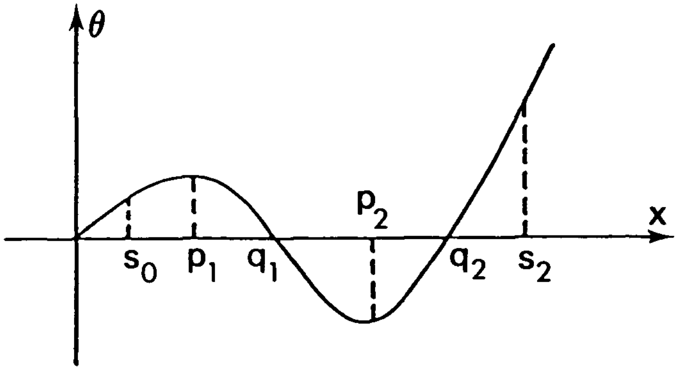


FIGURE 2

Now, from (1.9), we find that

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > \frac{2^{-3/2}}{\alpha^2}\int_0^\alpha (\Delta F)^{-(3/2)}(\phi(\alpha) - \phi(u))\,du, \tag{2.15}$$

in which

$$\phi(x) = x\theta'(x) - \theta(x). \tag{2.16}$$

So by (2.9)

$$\phi'(x) = x\theta''(x) = -x^2 f''(x)\begin{cases} < 0 & \text{if } x \in (0, s_1), \\ > 0 & \text{if } x \in (s_1, s_2). \end{cases}$$

It is easy to see that by (2.12) and (2.16),

$$\phi(0) = 0, \qquad \phi(p_2) = p_2\theta'(p_2) - \theta(p_2) = -\theta(p_2) > 0. \tag{2.18}$$

Thus, we obtain the graph of $\phi$, given in Figure 3 for $s_0 > 0$; note that $s_1 < p_2 < s_2$.

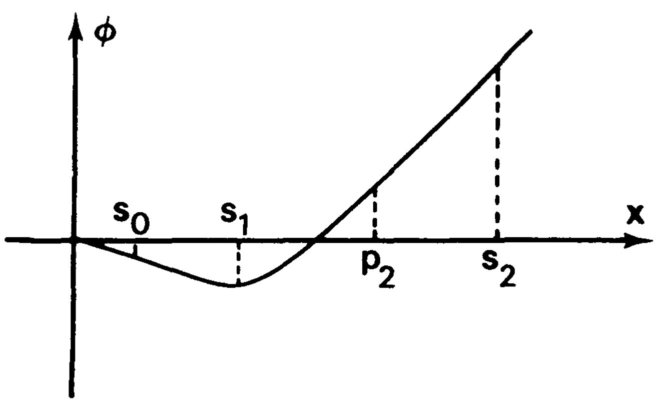


FIGURE 3

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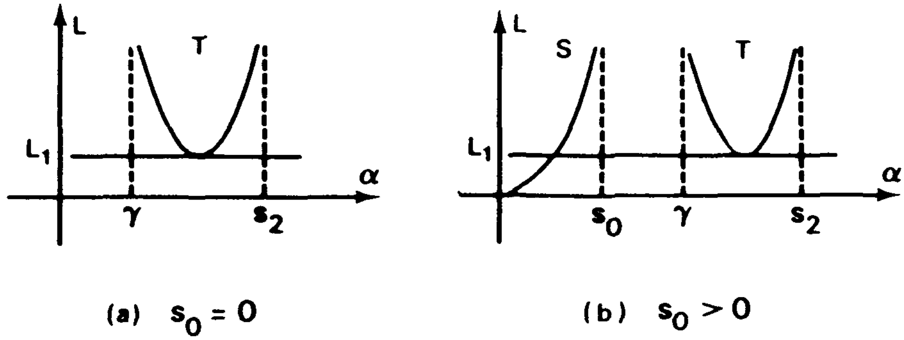


FIGURE 4

We conclude that in the integrand of (2.15),

$$\phi(\alpha) - \phi(u) > 0 \quad \text{if } p_2 < \alpha < s_2 \text{ and } u < \alpha. \tag{2.19}$$

Thus

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > 0 \quad \text{for} \quad p_2 < \alpha < s_2, \tag{2.20}$$

and if $T'(\alpha) = 0$ for some $\alpha$, $p_2 < \alpha < s_2$, then $T''(\alpha) > 0$. This and (2.14) imply (2.6). So $T'$ vanishes at most once on $(\gamma, s_2)$. Hence $T$ has exactly one critical point, a minimum on $(\gamma, s_2)$. This completes the proof of Theorem 1.

REMARK 1. If one reviews the proof, one sees that the requirements of the smoothness and convexity conditions on function $f$ in (2.2) can be weakened; we can replace $f \in C^2$ by $f \in C^1$, and weaken (2.2) by requiring that $\theta$ and $\phi$ satisfy

(2.21) $\theta'(x) = f(x) - xf'(x)$ is strictly decreasing in $(0, s_1)$ and strictly increasing in $(s_1, s_2)$, and

(2.22) $\phi(x) = -2F(x) + 2xf(x) - x^2 f'(x)$ is strictly decreasing in $(0, s_1)$ and strictly increasing in $(s_1, s_2)$.

REMARK 2. Condition (2.2) can be weakened as $f'' > 0$ in $(0, d)$, $f'' < 0$ in $(d, s_2)$ for $d \in (c_1, c_2)$, where $c_1$ is the critical point of $f$ in $(s_0, s_1)$ and $c_2$ is the critical point of $f$ in $(s_1, s_2)$.

REMARK 3. If $s_0 > 0$, then the solution with $\|u\|_\infty \in (0, s_0)$ cannot undergo bifurcation. To see this we consider $T(\alpha)$ defined by (1.9) on $(\gamma, s_2)$ and on $(0, s_0)$, we define the "time map" $S(\alpha)$ by

$$S(\alpha) = 2^{-1/2} \int_0^\alpha (F(\alpha) - F(u))^{-1/2}\, du, \qquad 0 < \alpha < s_0. \tag{2.23}$$

By (2.1) and (2.2), $f(0) > 0$, $f(s_0) = 0$ and $f$ is strictly decreasing in

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$(0, s_0)$. It is easy to see that $S(0^+) = 0$, $S(s_0^-) = +\infty$ and $S$ is strictly increasing in $(0, s_0)$; see also [7]. Combining the solution branches of $S$ and $T$, we see that the bifurcation diagram of (1.2) takes the form in Figure 4(a), (b). Therefore, for $L > L_1$, there are exactly three positive solutions if $s_0 > 0$ and exactly two positive solutions if $s_0 = 0$ in the order interval $(0, s_2)$ for (1.2) if the function $f$ satisfies (2.1)–(2.3).

## 3. One Generalization

Suppose $f$ has $2m+1$ $(m \geq 2)$ nonnegative simple roots $0 \leq s_0 < s_1 < s_2 < \cdots < s_{2m-1} < s_{2m}$ and also assume that the area of a "hill" exceeds that of the preceding "valley". Similarly to (1.7), for $n = 1, 2, \ldots, m$, we define $A_n > 0$ by $A_n^2 = 2F(s_{2n})$, and the "time map"

$$T_n(\alpha) = 2^{-1/2} \int_0^\alpha (F(\alpha) - F(u))^{-1/2}\, du, \qquad \gamma_n < \alpha < s_{2n}, \tag{3.1}$$

where $\gamma_n \in (s_{2n-1}, s_{2n})$ with $\int_{s_{2n-2}}^{\gamma_n} f(s)\, ds = 0$. As before, solutions of (1.6) correspond to curves for which $T_n(\alpha) = L$. Our argument in Section 2 can be easily modified to show that each "time map" $T_n$ has exactly one critical point for each $n = 1, 2, \ldots, m$. We now state without proof the following generalized theorem (recall that $\theta(x) = 2F(x) - xf(x)$ and $\phi(x) = x\theta'(x) - \theta(x) = -2F(x) + 2xf(x) - x^2 f'(x))$.

THEOREM 2. *Suppose $f \in C^2$, and there are numbers $0 \leq s_0 < s_1 < s_2 < \cdots < s_{2m-1} < s_{2m} (m \geq 2)$ such that the following conditions are satisfied:*

$$f(s_n) = 0 \quad \textit{for } n = 0, 1, 2, \ldots, 2m, \tag{3.2}$$

$$\begin{aligned} &f''(x) > 0 \quad \textit{for } x \in (0, s_1), \\ &f''(x) < 0 \quad \textit{for } x \in (s_{2n-1}, s_{2n}),\ n = 1, 2, \ldots, m, \\ &f''(x) > 0 \quad \textit{for } x \in (s_{2n-2}, s_{2n-1}),\ n = 1, 2, \ldots, m, \end{aligned} \tag{3.3}$$

(3.4) $\int_{s_{2n-2}}^{s_{2n}} f(s)\, ds > 0$, $\quad n = 1, 2, \ldots, m$, *and there exists a $\gamma_n$ in $(s_{2n-1}, s_{2n})$ defined by $\int_{s_{2n-2}}^{\gamma_n} f(s)\, ds = 0$ and such that $\theta(\gamma_n) = 2F(\gamma_n) - \gamma_n f(\gamma_n) < 0$, $n = 1, 2, \ldots, m$,*

(3.5) *there exists $p_{2n}$, a root of $f(x) - xf'(x) = 0$ in $(s_{2n-1}, s_{2n})$, for $n = 1, 2, \ldots, m$ such that the following two conditions are satisfied:*

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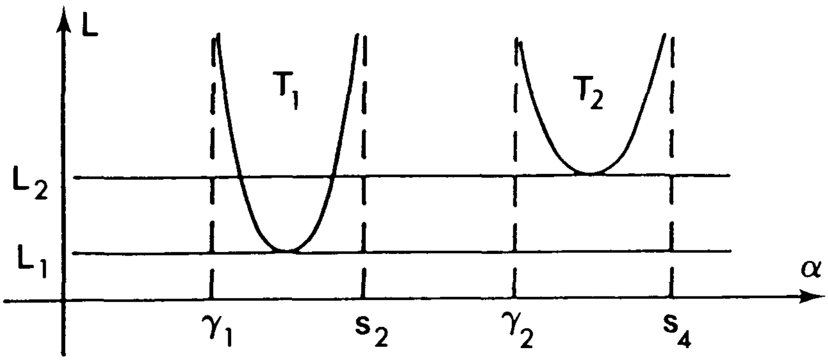
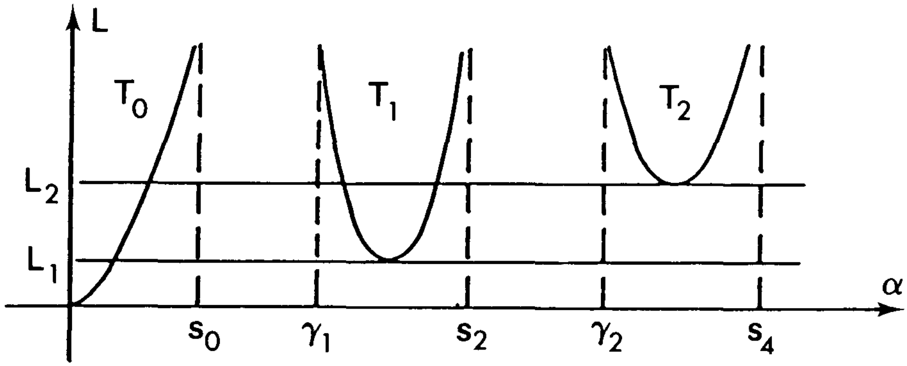



(a) $m = 2$ and $s_0 = 0$




(b) $m = 2$ and $s_0 > 0$

FIGURE 5

(1) *either* $p_{2n} \leq \gamma_n$ *for* $n = 2, 3, \ldots, m$
*or if* $p_{2n} > \gamma_n$ *for* $n = 2, 3, \ldots, m$ *then* $\theta(\gamma_n) \leq \theta(p_{2n-2})$

*and*

(2) $\phi(s_{2n-2}) \leq \phi(p_{2n})$ *for* $n = 2, 3, \ldots, m$;
*i.e.,* $-2F(s_{2n-2}) - s_{2n-2}^2 f'(s_{2n-2}) \leq -2F(p_{2n}) + 2p_{2n} f(p_{2n})$.

*Let* $T_n$ *be defined by* (3.1). *Then for* $n = 1, 2, \ldots, m$, $T_n$ *has exactly one critical point, a minimum.*

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REMARK 1. The proof of Theorem 2 is not very different from that of Theorem 1. Condition (3.5) is assumed to ensure the functions $\theta$ and $\phi$ have the desired properties and hence give the conclusion of Theorem 2.

REMARK 2. We can weaken the hypotheses on $f$, $\theta$, and $\phi$ in Theorem 2 as we did in (2.21) and (2.22) in Theorem 1.

REMARK 3. If $s_0 > 0$, on $(0, s_0)$ we define the "time map" $T_0$ to be the "time map" $S$ defined by (2.23). One sees that the bifurcation diagram of (1.2) if $f$ satisfies (3.2)–(3.5) takes the form given in Figure 5(a) for $m = 2$ if $s_0 = 0$ in Figure 5(b) for $m = 2$ if $s_0 > 0$.

Thus for $0 < L < \min_{n=1,\dots,m}\{L_n\}$, there is only one nonnegative solution $u$, with $0 \le \|u\|_\infty < s_0$. But for $L = L_n$ $(n = 1, 2, \dots, m)$, a positive solution $u$ with $s_{2n-1} < \|u\|_\infty < s_{2n}$ appears. While for $L > L_n$ $(n = 1, 2, \dots, m)$, this solution bifurcates into two positive distinct solutions $u_{2n-1}$, $u_{2n}$ with $s_{2n-1} < \|u_{2n-1}\|_\infty$, $\|u_{2n}\|_\infty < s_{2n}$. Therefore, for $L > \max_{n=1,\dots,m}\{L_n\}$, there are exactly $2m+1$ positive solutions if $s_0 > 0$ and $2m$ positive solutions other than the trivial solution $u \equiv 0$ if $s_0 = 0$ in the order interval $[0, s_{2m})$ for (1.2) if $f$ satisfies (3.2)–(3.5).

## 4. A Remark on Total Ordering of Multiple Steady-States Solutions

In this short section, we show the $2m+1$ $(m \ge 1)$ steady-state solutions $u_0, u_1, u_2, \dots, u_{2m-1}, u_{2m}$ of (1.2) obtained in Theorems 1 and 2 for large $L$ are totally ordered. We have

THEOREM 3.

(4.1) $$u_0 < u_1 < u_2 < \cdots < u_{2m-1} < u_{2m}.$$

Theorem 3 is an easy consequence of a special case of the following lemma, which can be shown by considering the first order system (1.6) and observing that the total energy function $H(u, v) = v^2/2 + F(u)$ is constant along orbits of (1.6). By $w < \hat{w}$, we mean $w(x) < \hat{w}(x)$, $x \in (-L, L)$.

LEMMA 1. *Let $w$ and $\hat{w}$ be any two distinct positive solutions of* (1.2) *with $0 < \|w\|_\infty < \|\hat{w}\|_\infty$. Then*

(4.2) $$w < \hat{w}.$$

NOTE 1. Lemma 1 says any two distinct positive solutions of (1.2) are ordered.

NOTE 2. In Lemma 1, $f$ is not necessary to satisfy (3.2) and the first part of (3.4). We do not require $L$ to be large enough (cf. [3]).

NOTE 3. $\|w\|_\infty = w(0)$ and $\|\hat{w}\|_\infty = \hat{w}(0)$. It is easy to see that $\|w\|_\infty \ne$

$\|\hat{w}\|_\infty$ by the existence and uniqueness theorem for autonomous system [8, p. 162].

## 5. A Brief Remark on Stability of the Multiple Steady-State Solutions

In this section we briefly discuss the stability of these $2m+1$ $(m \geq 1)$ steady-state solutions

$$u_0, u_1, u_2, \ldots, u_{2m-1}, u_{2m} \tag{5.1}$$

obtained in Theorems 1, 2 for $L > \max_{n=1,\ldots,m}\{L_n\}$ by a powerful topological tool, the Conley index theory. We get more information about the global structure of the multiple steady-states. Much of the exposition given here is adapted from Smoller [11]. It can be shown that the Conley index of $u_{2n-1}$, $h(u_{2n-1}) = \Sigma^1$, a pointed one-sphere $(n = 1, 2, \ldots, m)$, and the Conley index of $u_{2n}$, $h(u_{2n}) = \Sigma^0$, a pointed zero-sphere $(n = 1, 2, \ldots, m)$. Then there exist solutions $v_{2n-1}$ and $v_{2n}$ of (1.1), which connect $u_{2n-1}$ to $u_{2n-2}$ and $u_{2n-1}$ to $u_{2n}$ $(n = 1, 2, \ldots, m)$ respectively; that is,

$$\begin{aligned} &\lim_{t\to-\infty} v_{2n-1}(x, t) = u_{2n-1}(x), \qquad \lim_{t\to+\infty} v_{2n-1}(x, t) = u_{2n-2}(x), \\ &\lim_{t\to-\infty} v_{2n}(x, t) = u_{2n-1}(x), \qquad \lim_{t\to+\infty} v_{2n}(x, t) = u_{2n}(x), \end{aligned} \tag{5.2}$$

uniformly for $|x| \leq L$.

We now state results about the stability of the steady-state solutions of (1.1) and (1.1').

Proposition 1. *Let $f$ satisfy (3.2)–(3.5), and $L > \max_{n=1,\ldots,m}\{L_n\}$. Then there are exactly $2m+1$ $(m \geq 1)$ steady state solutions, $u_n$ $(n = 0, 1, 2, \ldots, 2m)$ of (1.1) and (1.1') with $0 \leq \|u_0\|_\infty < s_0$, $s_{2n-2} < \|u_{2n-1}\|_\infty$, $\|u_{2n}\|_\infty < s_{2n}$ $(n = 1, 2, \ldots, m)$. Each solution $u_{2n}$ is stable and each $u_{2n-1}$ has a one-dimensional unstable manifold which consists of orbits connecting $u_{2n-1}$ to $u_{2n-2}$ and $u_{2n}$. All solutions of the problem are depicted (qualitatively) in Figure 6. Initial data $u(x, 0)$, which satisfies the condition $u_{2n-1}(x) < u(x, 0) < u_{2n}(x)$ (respectively $u_{2n-2}(x) < u(x, 0) < u_{2n-1}(x)$) on $|x| < L$ is in the stable manifold of $u_{2n}$ (respectively $u_{2n-2}$) $(n = 1, 2, \ldots, m)$.*

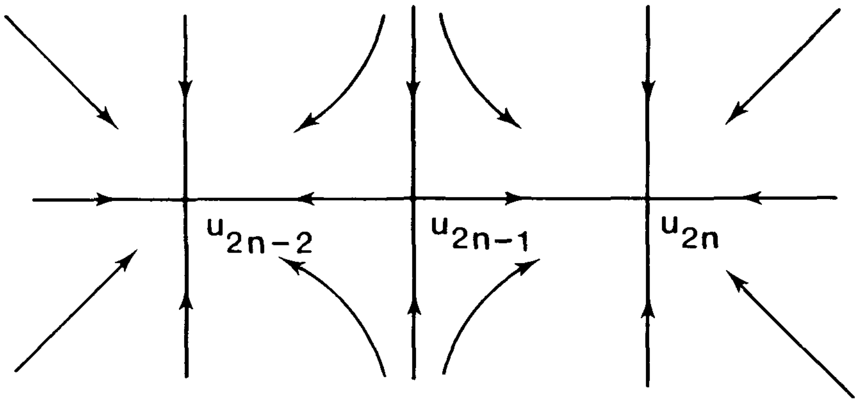


FIGURE 6

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