

MONOMIAL REPRESENTATIONS AND METABELIAN GROUPS¹⁾

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1. Introduction. In this paper we develop a method to find all the irreducible and inequivalent representations over the complex field C of a family of finite groups that includes the metabelian groups. The outline of the paper is as follows: In §2 we let P be a one-dimensional representation of a subgroup H of a finite group G and find a maximal subgroup K , $H \subseteq K \subseteq G$, such that an extension \bar{P} of P to K exists. We show that the induced representation \bar{P}^G is irreducible if K is normal in G . In §3 we give all the irreducible and inequivalent representations of the “generalized metabelian group” G , and in particular of the metabelian group, and decompose the group ring CG into its simple components. The convenience of this method is shown in §4 where we determine the representations of the metacyclic group and a metabelian group of order $2sp^2$, p an odd prime and $s|p-1$. The algorithm in §5 is supplementary and can be used to find representations of more general groups than the two in §4.

In this paper we make use of theorems 45.2 and 45.6 of Curtis-Reiner [3, §45], in the special case (due to Shoda [6]) where L , L_1 , L_2 afford one-dimensional representations.

2. Induced representations. Let G be a finite group and H , D , and K be subgroups of G such that $G \supseteq K \supseteq H \supseteq D$ with the following conditions:

- (i) D is normal in H and H/D is cyclic.
- (ii) $K' \cap H \subseteq D$ where K' is the commutator group of K .

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- (iii) K is maximal in the sense that if $K_1 \supseteq K$ and $K'_1 \cap H \subseteq D$, then $K_1 = K$.
- (iv) K is normal in G .

Let H and D be fixed. Then K depends on H and D . Sometimes, where confusion is likely to occur, we let $K = K(H, D)$. Note that we can always find a K with conditions (ii) and (iii), but not necessarily with conditions (ii), (iii), and (iv).

Now let $A = \{H, K'\}$ and $B = \{D, K'\}$. Then if $H = D + aD + \dots + a^{s-1}D$, we may set $A = B + aB + \dots + a^{s-1}B$. Since K/A is abelian we let z_1A, z_2A, \dots, z_nA be a basis for K/A . Then for each z_i there exists a smallest positive integer t_i such that $z_i^{t_i} \in A$.

Let ζ be a primitive s -th root of unity and P be a one-dimensional representation of A with kernel B such that $P(a) = \zeta$. There exist $\varphi(s) = \varphi([H: 1])$ distinct such representations of A with kernel B . Let $\theta_i, i = 1, \dots, n$, be a solution of

$$\theta^{t_i} = P(z_i^{t_i}).$$

Each $k \in K$ is uniquely expressed as $k = z_1^{e_1} \dots z_n^{e_n} h, h \in A, 0 \leq e_i \leq t_i - 1$. Set

$$\bar{P}(k) = \theta_1^{e_1} \dots \theta_n^{e_n} P(h).$$

Then \bar{P} is a one-dimensional representation of K and is afforded by the minimal left ideal $(CK)\bar{e}$ where \bar{e} is given by

$$\bar{e} = [K : 1]^{-1} \sum_{k \in K} \bar{P}(k^{-1})k.$$

For each P there exist $[K : A]$ distinct such extensions \bar{P} .

For $x \in G$, the left ideal $(CK)\bar{e}_x$, where

$$\bar{e}_x = [K : 1]^{-1} \sum_{k \in K} \bar{P}(x^{-1}k^{-1}x)k,$$

affords the representation \bar{P}_x where $\bar{P}_x(k) = \bar{P}(x^{-1}kx)$ for all $k \in K$. We call \bar{P} and \bar{P}_x G -conjugate representations and \bar{e} and \bar{e}_x G -conjugate idempotents.

Let $R(H, D, K)$ be the set of all representations \bar{P} for every possible P with kernel D and their G -conjugates and $I(H, D, K)$ be the set of all idempotents \bar{e} and their G -conjugates. Note that $K' \cap x^{-1}Hx \subseteq x^{-1}Dx$ for all

$x \in G$ and is maximal in the sense that $K \supseteq x^{-1}Hx$ and $K' \cap x^{-1}Hx \subseteq x^{-1}Dx$. Hence $R(x^{-1}Hx, x^{-1}Dx, K) = R(H, D, K)$ and $I(x^{-1}Hx, x^{-1}Dx, K) = I(H, D, K)$.

LEMMA. (a) If $\bar{P} \in R(H, D, K)$, then \bar{P}^G is an irreducible representation of G afforded by the minimal left ideal $(CG)\bar{e}$ of CG . Moreover $(CG)\bar{e}$ is a direct summand of the simple component $(CG)\bar{e}$ of CG where \bar{e} is the sum of the distinct G -conjugates of \bar{e} .

(b) If \bar{P} and \bar{P}' are in $R(H, D, K)$, then \bar{P}^G and \bar{P}'^G are equivalent if and only if \bar{P} and \bar{P}' are G -conjugate.

(c) Assume that H is normal in G , D is not conjugate to $D_1(\subseteq H)$, $\bar{P} \in R(H, D, K)$ and $\bar{P}_1 \in R(H, D_1, K_1)$, where K_1 is similarly defined. Then \bar{P}^G and \bar{P}_1^G are inequivalent.

Proof. (a) Let $x \notin K$ and assume that $\bar{P}(k) = \bar{P}(x^{-1}kx)$ for all $k \in K$. Then $k^{-1}x^{-1}kx$ is in the kernel \bar{D} of \bar{P} . Let $K_1 = \{K, x\}$, then $K'_1 \subseteq \bar{D}$. But $\bar{D} \cap H = D$ or $K_1 \supsetneq K \supseteq H$ and $K'_1 \cap H \subseteq D$, a contradiction since K is maximal. Hence $\bar{P}(k) \neq \bar{P}(x^{-1}kx)$ for at least one $k \in K$. From [3, (45. 2)], \bar{P}^G is irreducible. Now \bar{P}^G is of degree $[G: K]$. Let $\bar{\chi}^G$ be the character of \bar{P}^G then $\bar{\chi}^G(g) = 0$ for all $g \notin K$ and $\bar{\chi}^G(k) = \sum \bar{P}_x(k)$ where x runs over a complete set of coset representatives of K by G . The central idempotent \bar{e} is given by

$$\bar{e} = \frac{[G: K]}{[G: 1]} \sum_{g \in G} \overline{\bar{\chi}^G(g)} g.$$

Hence it can be easily shown that $\bar{e} = \sum_x \bar{e}_x$ where x runs over a complete set of coset representatives of G by K . For $x \notin K$, we have $\sum_{k \in K} P(k^{-1}) P(x^{-1}kx) = 0$, and hence $\bar{e} \cdot \bar{e}_x = 0$. Hence $(CG)\bar{e}$ is the direct sum of the $[G: K]$ left ideals $(CG)\bar{e}_x$ or each $(CG)\bar{e}_x$ is a minimal left ideal and $(CG)\bar{e}$ affords \bar{P}^G .

Parts (b) and (c) follow using (a) above and [3, (45. 6)] which completes the proof.

Let $R_c(H, D, K)$ and $I_c(H, D, K)$ be complete sets of representatives of the conjugate classes of $R(H, D, K)$ and $I(H, D, K)$ respectively. From the above lemma any two elements of $R_c(H, D, K)$ induce inequivalent representations of G and any element of $R(H, D, K)$ induces a representation equivalent to some representation induced by an element of $R_c(H, D, K)$. Let H be normal in G and $\nu(D) = [G: N_G(D)]$. Then the number of elements of $R_c(H, D, K)$ is

$$\nu(D)\varphi([H: D])[K: A]/[G: K].$$

In case $K' \subseteq D$ we have $A = H$.

3. The generalized metabelian groups. We first give a

DEFINITION. A finite group G is called a *generalized metabelian group* if G has a normal (invariant) series $E = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_{n+1} = G$, where $E = \{1\}$, satisfying the following conditions:

- (i) A_i/A_{i-1} is abelian for $i = 1, 2, \dots, n + 1$.
- (ii) For any subgroup D_i of G such that $A_{i-1} \subseteq D_i \subsetneq A_i$ when $1 \leq i \leq n-1$ and $A_{n-1} \subseteq D_n \subsetneq A_n$, with A_i/D_i cyclic, there exists a subgroup $K_i = K(A_i, D_i) \supseteq A_i$ of G such that $K'_i \subseteq D_i$ and K_i satisfies (iii) and (iv) of §2.

Using the derived series, $E = G'' \subseteq G' \subseteq G$, of a finite metabelian group G , we can easily show that G is a generalized metabelian group. We give an example of a group G which is not metabelian but is generalized metabelian. In Blackburn [2], the existence of a p -group (p odd) $G = \{x, y, z\}$ whose derived group $G' = \{a, b\}$ is metacyclic is proved. The defining relations of a special case is as follows:

$$z^{p^3} = y^{p^3} = x^{p^2} = a^{p^2} = b^{p^3} = 1$$

with $x^{-1}y^{-1}xy = [x, y] = a$, $[y, z] = b$, $[a, z] = a^p$, $[b, z] = b^{2p}$, $[a, y] = b^p$, $[b, x] = a^{-p}b^{p^2}$, $[a, b] = b^{p^2}$, and $[b, y] = [a, x] = [z, x] = 1$. The derived series of G is $\{1\} \subseteq \{b^{p^2}\} \subseteq \{a, b\} \subseteq G$. Condition (i) in the definition is trivially satisfied. Let $K = \{a, b^p, x, y^{p^2}, z^p\}$. Then K is normal in G , $K' = \{1\}$, $K' \cap G'' = \{1\}$, and K is maximal in the sense that $K \supseteq G''$ and $K' \cap G'' \subseteq D = \{1\}$. Hence (ii) of the definition above is satisfied for $\{1\} \subseteq D \subsetneq \{b^{p^2}\}$. The proof for the remaining cases follows since $G/\{b^{p^2}\}$ is metabelian. Hence G is a generalized metabelian group.

Let D_1, \dots, D_t form the set of all subgroups of a finite abelian group A such that A/D_i is cyclic. Then, from the self-duality of A , all the cyclic subgroups of A are C_1, \dots, C_t with $C_i \cong A/D_i$, $i = 1, \dots, t$. If T_i is the set of generators of C_i , then the set theoretic union $\bigcup_{i=1}^t T_i = A$ and therefore

$$\sum_{i=1}^t \varphi([C_i: 1]) = \sum_{i=1}^t \varphi([A: D_i]) = [A: 1].$$

Using the above result and the notations of the definition we prove,

THEOREM. *Let G be a generalized metabelian group with the normal series given as in the above definition. Then all the irreducible inequivalent representations of G are given by the set of all \bar{P}^G with $\bar{P} \in \bigcup_{i=1}^n \{ \cup^{(i)} R_c(A_i, D_i, K_i) \}$ and the union $\cup^{(i)}$ is over all D_i which are non-conjugate in G .*

Proof. Applying the Lemma \bar{P}^G is irreducible and any \bar{P}^G and \bar{P}'^G , with $\bar{P} \neq \bar{P}'$, are inequivalent. The number of elements of $R_c(A_i, D_i, K_i)$ is

$$\nu(D_i)\varphi([A_i : D_i]) [K_i : A_i] [G : K_i].$$

Each \bar{P}^G for $\bar{P} \in R_c(A_i, D_i, K_i)$ is of degree $[G : K_i]$. Hence multiplying the above number by $[G : K_i]^2$ and summing over all possible D_i 's given in the definition we have for $i = 1, 2, \dots, n - 1$

$$\begin{aligned} & \Sigma' \nu(D_i)\varphi([A_i : D_i]) [G : A_i] \\ &= [\Sigma \varphi([A_i : D_i])] [G : A_i] \\ &= ([A_i : A_{i-1}] - 1) [G : A_i] = [G : A_{i-1}] - [G : A_i], \end{aligned}$$

where the first summation Σ' is over all non-conjugate D_i 's such that $A_{i-1} \subseteq D_i \subseteq A_i$, A_i/D_i cyclic, and the second summation Σ is over all D_i 's such that $A_{i-1} \subseteq D_i \subsetneq A_i$ and A_i/D_i cyclic. For $i = n$ the sum is $[G : A_n]$. Hence adding all the above quantities for $1 \leq i \leq n$ we get $[G : 1]$, the order of G . This completes the proof.

COROLLARY. *Let G be as in the above theorem. Then $CG = \Sigma(CG)\bar{e}$ is a decomposition of CG as a direct sum of its minimal left ideals where \bar{e} runs over all elements of the set $\bigcup_{i=1}^n \{ \cup^{(i)} I(A_i, D_i, K_i) \}$ where the union $\cup^{(i)}$ is over all D_i , for fixed i , which are non-conjugate in G .*

Using the lemma, we may set \bar{e} as the sum of distinct conjugates of \bar{e} , and hence we have $CG = \Sigma(CG)\bar{e}$, a decomposition of CG into its simple components.

COROLLARY. *Let G be a finite metabelian group, A a normal abelian subgroup, and G/A abelian. Then all the irreducible inequivalent representations of G are given by the set of all \bar{P}^G where $P \in \cup R_c(A, D, K)$ and the union is taken over all non-conjugate subgroups D of A such that A/D is cyclic.*

The above result for the case where G splits over A was proved by Berman [1].

COROLLARY. *Every irreducible representation of a generalized metabelian group G is equivalent to a monomial representation.*

Another proof for the above corollary for the case where G is metabelian is given in [3, §52]. Still another proof can be given by applying a result on faithful representations of metabelian groups proved by Shoda [6], which was also differently proved by Itô [4]. Below we generalize this result of Shoda to the generalized metabelian group.

COROLLARY. *Let G be a generalized metabelian group with the normal series as in the definition and assume G has an irreducible faithful representation. Then all the irreducible faithful representations S of G have the same degree d , and all the maximal abelian subgroups K , containing A_1 , which are normal in G have the same index t , and $t = d$. Moreover S is equivalent to an induced representation \bar{P}^G where \bar{P} is some one-dimensional representation of any K .*

Proof. From the theorem, the faithfulness of S implies that there exists a subgroup $D_1, A_1/D_1$ is cyclic and $\bigcap_{x \in G} x^{-1}D_1x = \{1\}$. Hence $K = K(A_1, D_1)$ is abelian and the result follows.

4. Applications. The methods of the previous two sections are used to find all the representations of the groups below. The symbols $P, \bar{P}, D, K, \bar{e}$, and \bar{e} have the same meanings as before and sometimes a subscript is added whenever confusion is likely to occur. To avoid repetition the calculations are omitted.

(i) *Representations of the metacyclic group.* This is also done by Munkholm [5].

Let G be a metacyclic group with the defining relations

$$a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r$$

where $t|m$ and $kr - k \equiv r^t - 1 \pmod n$. Consider the normal series $\{1\} \subseteq \{a\} \subseteq \{a, b\} = G$ and let $D_s = \{a^s\}$, $s|n$. Then $K_s = K(\{a\}, D_s) = \{a, b^{t_s}\}$ where t_s is the smallest positive integer such that $r^{t_s} \equiv 1 \pmod s$. Let M_s be the multiplicative group of the reduced residues modulo s and consider $R_s = \{r\}$ as a subgroup of M_s . Let ζ_s be a primitive s -th root of unity, $P_s(a) = \zeta_s^y$, $(y, s) = 1$, and $\theta_s^{(i)}$, $i = 1, \dots, t/t_s$, set of all solutions of $\theta^{t/t_s} = \zeta_s^{y_k}$. Also, let $\bar{P}_s(a) = P_s(a)$ and $\bar{P}_s(b^{t_s}) = \theta_s^{(i)}$. Then all the irreducible inequivalent representations of G are \bar{P}_s^G where s runs over all divisors of n , y runs over

a complete set of coset representatives of M_s by R_s and $i = 1, \dots, t/t_s$. Also, we have

$$\bar{e}_s = \bar{e}_s^{(y)} = t/(nt_s) \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{t/t_s-1} \zeta_s^{-y\alpha} (\theta_s^{(t)})^{-\beta} a^\alpha b^{t-s\beta}, \text{ and } \tilde{e}_s = \sum_{j=0}^{t_s-1} \bar{e}_s^{(y r^j)}.$$

(ii) *Representations of a group of order $2sp^2$.* Let p be an odd prime, $(r, p) = 1$, $r \equiv 1 \pmod{p}$, and s the smallest positive integer such that $r^s \equiv 1 \pmod{p}$. Let $G = \{a, b, x, y\}$ be the group with the following defining relations:

$$a^p = b^p = x^s = y^2 = a^{-1}b^{-1}ab = x^{-1}y^{-1}xy = 1$$

$$x^{-1}ax = a^r, \quad x^{-1}bx = b^r, \quad y^{-1}ay = b, \quad \text{and } y^{-1}by = a.$$

Then $G' = \{a, b\}$ and $G'' = \{1\}$. Let ζ be a primitive p -th root of unity. We list below a complete set of non-conjugate subgroups D , G'/D cyclic, and for each D we find the corresponding \bar{P} , \bar{e} , and \tilde{e} .

(1) $D = \{b\}$, $D = \{ba^{g^i}\}$, $i = 1, 2, \dots, (p-1)/2 - 1$ where g is a primitive root mod p . Here $K = K(G', D) = G'$ and $P(a) = \zeta^t$, $(t, p) = 1$, and $P(d) = 1$ for $d \in D$. The $(p-1)/s$ irreducible inequivalent representations P^G of degree $2s$ are given by letting t run over a complete set of coset representatives of $M_p = \{g\}$ by $R_p = \{r\}$. The idempotents corresponding to $D_1 = \{b\}$ and $D_2 = \{ba^{g^i}\}$ are

$$\bar{e}_1 = \bar{e}_1^{(t)}(a, b) = p^{-2} \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p-1} \zeta^{-t\alpha} a^\alpha b^\beta \text{ and } \bar{e}_2 = \bar{e}_2^{(t)}(a, b) = p^{-2} \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p-1} \zeta^{t(g^i\alpha - \beta)} a^\alpha b^\beta,$$

respectively and $\tilde{e}_j = \sum_{k=0}^{s-1} [\bar{e}_j^{(tr^k)}(a, b) + \bar{e}_j^{(tr^k)}(b, a)]$, $j = 1, 2$.

(2) $D = \{ab\}$ where s is odd. Here $K = G'$, $P(a) = \zeta^t$ and $P(ab) = 1$ where $(t, p) = 1$. The $(p-1)/2s$ irreducible inequivalent representations P^G of degree $2s$ are given by letting t run over a complete set of coset representatives of M_p by $R_1 = \{-r\}$. Here

$$\bar{e} = \bar{e}^{(t)} = p^{-2} \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p-1} \zeta^{t(\alpha-\beta)} a^\alpha b^\beta \text{ and } \tilde{e} = \sum_{k=0}^{s-1} [\bar{e}^{(tr^k)} + \bar{e}^{(-tr^k)}].$$

(3) $D = \{ab\}$ where s is even. Here $K = \{G', yx^{s/2}\}$, $P(a) = \zeta^t$ and $P(ab) = 1$ where $(t, p) = 1$. We have $\bar{P}_i(h) = P(h)$ for $h \in G'$ and $\bar{P}_i(yx^{s/2}) = (-1)^i$, $i = 1, 2$. The $(p-1)/s$ irreducible inequivalent representations \bar{P}^G ,

$i = 1, 2$, of degree s are given by letting t run over a complete set of coset representatives of $M_p = \{g\}$ by $R_p = \{r\}$. Also

$$\bar{e}_i = \bar{e}_i^{(t)} = 1/(2p^2) \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p-1} \zeta^{t(\alpha-\beta)} a^\alpha b^\beta [1 + (-1)^i y x^{s/2}] \text{ and } \bar{e}_i = \sum_{k=0}^{s-1} \bar{e}_i^{(tr^k)}.$$

(4) $D = \{ab^{-1}\}$. Exactly as (3) with y exchanged for $yx^{s/2}$, $P(ab^{-1}) = 1$ for $P(ab) = 1$, and $\zeta^{-t(\alpha+\beta)}$ for $\zeta^{t(\beta-\alpha)}$.

(5) $D = G'$. $K = G$ and we get $2s$ one-dimensional representations of G .

5. Cyclic factor groups. In this section we give an algorithm to find all the subgroups D of a finite abelian group A , with A/D cyclic. Clearly we only need to give this for the case where A is an abelian p -group.

ALGORITHM. Let $A = \{a_1, \dots, a_n\}$ be an abelian p -group of type $(\alpha_1, \dots, \alpha_n)$, $1 \leq \alpha_1 \leq \dots \leq \alpha_n$, $a_i^{p^{\alpha_i}} = 1$. Let $B = \{a_1, \dots, a_{n-1}\}$ and assume D is a subgroup of B such that B/D is cyclic of order p^α , $0 \leq \alpha \leq \alpha_{n-1} \leq \alpha_n$, and set

$$B = D + aD + \dots + a^{p^\alpha - 1}D.$$

If $\alpha = \alpha_n$ then all the distinct subgroups \bar{D} of A such that A/\bar{D} is cyclic and $B \cap \bar{D} = D$ are given by $\bar{D} = \{a_n a^k, D\}$, $k = 0, 1, \dots, p^\alpha - 1$, and we may set $A = \bar{D} + a\bar{D} + \dots + a^{p^\alpha - 1}\bar{D}$.

If $\alpha < \alpha_n$ then, in addition to the subgroups \bar{D} above, we have $\bar{D} = \{a_n^{p^\sigma k} a, D\}$, $\sigma = 1, \dots, \alpha_n - \alpha$, $1 \leq k \leq p^\alpha$, $(k, p) = 1$, and we may set $A = \bar{D} + a_n \bar{D} + \dots + a_n^{p^{\alpha+\sigma} - 1} \bar{D}$.

Proof. Since $D = \bar{D} \cap B$, the smallest positive power of a in \bar{D} is a^{p^α} . If $d \in \bar{D}$ then $d = a_n^x a^y d'$, $d' \in D$. If $p \nmid x$, for some x , then it can be shown that $\bar{D} = \{a_n a^k, D\}$. If $p \mid x$ for every x , then $p \mid y$ for some y and $\alpha < \alpha_n$. The results follow after some calculations. Note that when D runs over all subgroups of B , B/D cyclic, \bar{D} runs over all subgroups of A , A/\bar{D} cyclic.

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