ON NORMALISER PRESERVING LATTICE ISOMORPHISMS BETWEEN NILPOTENT GROUPS

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Introduction

Let $\mathscr{L}(G)$ denote the lattice of all subgroups of a group G. By an *$\mathscr{L}$-isomorphism* (lattice isomorphism) of G onto a group H, we mean an isomorphism of $\mathscr{L}(G)$ onto $\mathscr{L}(H)$. By an *$\mathscr{N}\mathscr{L}$-isomorphism* (normaliser preserving $\mathscr{L}$-isomorphism) of G onto H, we mean an $\mathscr{L}$-isomorphism ϕ such that $\mathscr{N}(A^{\phi}) = \mathscr{N}(A)^{\phi}$ for all $A \in \mathscr{L}(G)$. In this paper, we study certain properties of groups which remain invariant under $\mathscr{N}\mathscr{L}$-isomorphisms.

In § 1, we show that $\mathscr{N}\mathscr{L}$-isomorphisms can be characterised both as *commutator preserving* $\mathscr{L}$-isomorphisms and as *mixed commutator preserving* $\mathscr{L}$-isomorphisms. This result is closely related to recent work of Spring [11] on $\mathscr{L}$-isomorphisms between finite p-groups of exponent p. Spring proved, amongst other things, that every $\mathscr{L}$-isomorphism between such groups is an $\mathscr{N}\mathscr{L}$-isomorphism and that $\mathscr{L}$-isomorphic k-generator p-groups of exponent p and class 2 are necessarily isomorphic if $k \leqq 4$. We give a simple derivation of Spring's first result and of an analogous one of Pekelis [7] for locally nilpotent, torsion-free groups.

Rottländer [9] has given an example of an $\mathscr{N}\mathscr{L}$-isomorphism between non-isomorphic finite groups of the same order. In § 2, we give examples of the same phenomenon between finite p-groups [1] ($p > 2$). The groups in the simplest examples are of order p^4, exponent p^2 and class 3 ($p > 3$), and there are somewhat more complicated examples in which the groups have exponent p ($p > 5$).

Theorem 2 (§ 4) deals with the effect of an $\mathscr{N}\mathscr{L}$-isomorphism $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ on the central sections of G. (A section of G means a factor group A/B, where $B \trianglelefteq A \leqq G$. The section is called central if $B \trianglelefteq G$ and $A/B \leqq \mathscr{Z}(G/B)$.) A typical case of the theorem states that if the restriction of ϕ to the factor commutator group G/G' is induced by an

[1] It may be mentioned that the well known $\mathscr{L}$-isomorphisms between non-isomorphic modular p-groups are *not* $\mathscr{N}\mathscr{L}$-isomorphisms. Indeed, it can be deduced from Iwasawa's work ([4], [5]) that $\mathscr{N}\mathscr{L}$-isomorphic locally finite, modular p-groups (and in particular $\mathscr{L}$-isomorphic abelian p-groups) are necessarily isomorphic.

isomorphism, then the same is true for the restriction of ϕ to every factor group $G_{(i)}/G_{(i+1)}$ of the descending central series. The proof uses the calculus of commutators and a general lemma (lemma 2, § 3) about multilinear mappings. This lemma is a wide generalisation of the fact that bilinear forms $f(x, y)$, $g(x, y)$ which vanish for the same pairs of vectors x, y are scalar multiples of one another.

Theorem 2 is applied to prove that, in certain cases, $\mathcal{N}\mathcal{L}$-isomorphic groups must be isomorphic. Theorem 3 affirms [2] that this is so when the groups belong to a nilpotent variety V and one of them is an almost free group of V. Here, an almost free group of V means a factor group F/M, where F is a free group of V and M is properly contained in the penultimate member of the descending central series of F.

A consequence of theorem 3 is that (contrary to Spring's [3] assertion) $\mathcal{L}$-isomorphic k-generator p-groups of exponent p and class 2 are necessarily isomorphic, whatever the value of k. The examples cited above show that this result does not generalise to groups of exponent p and class > 2.

Because of the analogy between commutators in nilpotent groups and products in nilpotent Lie algebras [4], analogous theorems about lattice isomorphisms between nilpotent Lie algebras are to be expected. Let $\mathcal{F}$ be a commutative ring with 1 and L a Lie algebra over $\mathcal{F}$. We denote by $\mathcal{L}(L)$ the lattice of subalgebras of L. By an $\mathcal{L}$-isomorphism of L onto a Lie algebra M *over the same ring* $\mathcal{F}$, we mean an isomorphism of $\mathcal{L}(L)$ onto $\mathcal{L}(M)$. We shall merely state the principal results for Lie algebras during the course of the paper, indicating points at which the proofs differ significantly from those for groups.

Notation. Let $x, y, x_1, x_2, \cdots$ be elements of a group G. $\langle x_1, x_2, \cdots \rangle$ denotes the subgroup generated by $x_1, x_2, \cdots$. $[x_1, x_2, \cdots, x_n]$ is the simple n-fold commutator defined inductively by: $[x, y] = x^{-1}y^{-1}xy$, $[x_1, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$ $(n \geqq 3)$.

Let $X, Y, X_1, X_2, \cdots$ be subgroups of G. $X \leqq Y$ $(X \trianglelefteq Y)$ means that X is a subgroup (normal subgroup) of Y; $X < Y$ $(X \lhd Y)$ excludes equality. $\mathcal{N}_G(X)$ (or $\mathcal{N}(X)$ if no confusion can arise) is the normaliser of X in G. $\langle X_1, X_2, \cdots \rangle$ or $X_1 \cup X_2 \cup \cdots$ denotes the subgroup generated by $X_1, X_2, \cdots$. $[X_1, X_2, \cdots, X_n]$ is the simple n-fold commutator defined inductively by: $[X, Y] = \langle [x, y] | x \in X, y \in Y \rangle$, $[X_1, \cdots, X_n] = [[X_1, \cdots, X_{n-1}], X_n]$ $(n \geqq 3)$.

$G = G_{(1)} \geqq G_{(2)} \geqq \cdots$ denotes the descending central series of G. $G' = G_{(2)} = [G, G]$ is the commutator group of G. $\Phi(G)$ is the Frattini

[2] A similar result for *free* polynilpotent groups has been given by Šmel'kin [10] (theorem 10).

[3] Spring [11] cites a 5-generator counterexample without carrying through the proof.

[4] See G. Higman [3], M. Lazard [6].

subgroup, $\mathscr{Z}(G)$ the centre, of G. G^m denotes the subgroup generated by the m-th powers of the elements of G.

Homomorphisms $\phi, \psi \cdots$ of groups or lattices will be written exponentially: $x^{\phi\psi} = (x^\phi)^\psi$. If $\phi : G \to H$ is a (group) epimorphism and B/A a section of G, the equations $(xA)^{\phi'} = (xA)^\phi$ $(x \in B)$ define an epimorphism $\phi' : B/A \to B^\phi/A^\phi$; similarly, if ψ is an $\mathscr{N}\mathscr{L}$-isomorphism of F onto H, the equations $(X/A)^{\psi'} = X^\psi/A^\psi$ $(X/A \in \mathscr{L}(B/A))$ define an $\mathscr{N}\mathscr{L}$-isomorphism ψ' of B/A onto B^ψ/A^ψ. It is convenient to call ϕ', ψ' the *restrictions* of ϕ, ψ to B/A. We say that an $\mathscr{L}$-isomorphism ψ of G onto H is *induced by an isomorphism* if there exists a group isomorphism $\phi : G \to H$ such that $X^\phi = X^\psi$ whenever $X \in \mathscr{L}(G)$.

1. $\mathscr{N}\mathscr{L}$-isomorphisms

An $\mathscr{L}$-isomorphism $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ is called

(a) normaliser preserving if $\mathscr{N}(A)^\phi = \mathscr{N}(A^\phi)$ for all $A \leqq G$,

(b) commutator preserving if $(A')^\phi = (A^\phi)'$ for all $A \leqq G$,

(c) mixed commutator preserving if $[A, B]^\phi = [A^\phi, B^\phi]$ for all $A, B \leqq G$.

THEOREM 1. *The properties* (a), (b), (c) *of an $\mathscr{L}$-isomorphism are all equivalent.*

PROOF. (a) $\Rightarrow$ (b). If $A' \leqq B \leqq A \leqq G$, then $\mathscr{N}(B) \geqq A$ and so $\mathscr{N}(B)^\phi = \mathscr{N}(B)^\phi \geqq A^\phi$. Thus every subgroup of $A^\phi/(A')^\phi$ is normal. Hence $A^\phi/(A')^\phi$ is abelian, or $A^\phi/(A')^\phi$ has a subgroup isomorphic to the quaternion group, which is impossible since the latter is not $\mathscr{L}$-isomorphic to an abelian group. Therefore $(A')^\phi \geqq (A^\phi)'$. Similarly

$$\left((A^\phi)'\right)^{\phi^{-1}} \geqq (A^{\phi\phi^{-1}})' = A',$$

and so, by applying ϕ to this we have $(A^\phi)' \geqq (A')^\phi$. Hence $(A')^\phi = (A^\phi)'$.

(b) $\Rightarrow$ (c). Since

$$[A, B] = \cup\{[X, Y] \mid X \leqq A, Y \leqq B, X, Y \text{ cyclic}\},$$

it is sufficient to consider the case A, B cyclic. But if A, B are cyclic, the formula

$$x^{-\alpha}[x^\lambda, y^\mu]x^\alpha = [x^{\lambda+\alpha}, y^\mu][x^\alpha, y^\mu]^{-1}$$

shows that $[A, B] \trianglelefteq A \cup B$ and thus that $[A, B] = (A \cup B)'$.

(c) $\Rightarrow$ (a). $B \leqq \mathscr{N}(A)$ if and only if $[A, B] \leqq A$.

COROLLARY 1. *$\mathscr{N}\mathscr{L}$-isomorphisms preserve central and derived series (and thus solubility, derived length, nilpotency, class of nilpotency, etc.).*

The next corollary is a known result, due to Pekelis [7] in the torsion-free case and to Spring [11] in the prime exponent case. Our theorem gives a simple unified proof.

COROLLARY 2. *Let G be a locally nilpotent group, and ϕ an $\mathscr{L}$-isomorphism of G onto a group H. If*

(1.1) *either* (a) *G is torsion-free*
or (b) *G has prime exponent p and is non-abelian*

then ϕ is an $\mathscr{N}\mathscr{L}$-isomorphism.

PROOF. By the proof of the theorem, it is sufficient to show that $(M')^\phi = (M^\phi)'$ for each 2-generator subgroup M of G. Notice that M is nilpotent. Set $N = M^\phi$.

Suppose first that G has exponent p. Then M is a finite p-group. Hence, if M is non-abelian, N must be a p-group (Suzuki [12], thm. 12, p. 12). The same conclusion holds when M is abelian for M is contained in a non-abelian finitely generated subgroup of G. It is clear that N is finite and has exponent p. Hence $(M')^\phi = (\Phi(M))^\phi = \Phi(N) = N'$.

Suppose secondly that G is torsion-free. Then H is torsion-free. A torsion-free group is abelian if and only if its subgroup lattice is modular (Suzuki [12], prop. 1.12, p. 19). Thus $\mathscr{Z}(M)^\phi = \mathscr{Z}(N)$. Let $1 < \mathscr{Z}_1(M) < \cdots < \mathscr{Z}_c(M) = M$ be the ascending central series of M. Since the factor groups $M/\mathscr{Z}_i(M)$ are torsion-free the same argument shows that $\mathscr{Z}_i(M)^\phi = \mathscr{Z}_i(N)$ for all i. In particular, $N = \mathscr{Z}_c(N)$ whence H is nilpotent.

If M is abelian, then N is also abelian because $\mathscr{Z}(M) = \mathscr{Z}(N)$; in this case, $(M')^\phi = N' = 1$. If M is non-abelian, then $M/\mathscr{Z}_{c-2}(M)$ is also non-abelian and so $M/\mathscr{Z}_{c-1}(M)$ is not cyclic. On the other hand, $M/\mathscr{Z}_{c-1}(M)$ is torsion-free and a homomorphic image of the 2-generator group M/M'. Hence $M' = \mathscr{Z}_{c-1}(M)$. Similarly $N' = \mathscr{Z}_{c-1}(N)$. Hence $(M')^\phi = N'$, as required.

The following inductive principle is useful for the construction of $\mathscr{N}\mathscr{L}$-isomorphisms.

LEMMA 1. *Let G, H be finite*[5] *groups and α a mapping of $\mathscr{L}(G)$ into $\mathscr{L}(H)$. Then α is an $\mathscr{N}\mathscr{L}$-isomorphism if and only if*

(i) *α maps the interval of $\mathscr{L}(G)$ with end points $\Phi(G)$, G lattice-isomorphically onto the interval of $\mathscr{L}(H)$ with end points $\Phi(H)$, H;*

(ii) *for each maximal subgroup M of G, the restriction of α to $\mathscr{L}(M)$ is an $\mathscr{N}\mathscr{L}$-isomorphism of M onto M^α;*

(iii) $(G')^\alpha = H'$.

[5] It is sufficient to assume that each subgroup of G, H is contained in a maximal subgroup.

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PROOF. The conditions are clearly necessary. Let us prove that they are sufficient.

(a) α *is surjective.* Let $A \in \mathscr{L}(H)$. If $A = H$, then $A = G^\alpha$ by (i). If $A < H$, let B be a maximal subgroup of H containing A. By (i), $B = C^\alpha$ for some maximal subgroup C of G. Then, by (ii), $A = D^\alpha$ for some $D \in \mathscr{L}(C)$.

(b) α *is injective.* Suppose $A^\alpha = B^\alpha$, where $A, B \in \mathscr{L}(G)$. We may assume that $A < G$. Let C be a maximal subgroup of G containing A. By (ii) and (i), $A^\alpha \leqq C^\alpha < G^\alpha = H$. Since $B^\alpha = A^\alpha$, $B < G$. Let D be a maximal subgroup of G containing B. By (ii), $B^\alpha \leqq D^\alpha$, so that $B^\alpha \leqq C^\alpha \cap D^\alpha$. By (i), $C^\alpha \cap D^\alpha = (C \cap D)^\alpha$ and so $B^\alpha \leqq (C \cap D)^\alpha$. By (ii) (applied to $\mathscr{L}(D)$), $B \leqq C \cap D$. By (ii) (applied to $\mathscr{L}(C)$), $B = A$.

(c) α *is an $\mathscr{L}$-isomorphism.* It is sufficient to show that $A \leqq B \Leftrightarrow A^\alpha \leqq B^\alpha$ for $A, B \in \mathscr{L}(G)$. Since α^{-1} exists (by (a), (b)) and clearly satisfies the conditions of the theorem, it is sufficient to show that $A \leqq B \Rightarrow A^\alpha \leqq B^\alpha$. If $B = G$, then $A^\alpha \leqq H = B^\alpha$. If $B < G$, let C be a maximal subgroup of G containing B. Then $A^\alpha \leqq B^\alpha$ by (ii) applied to $\mathscr{L}(C)$.

(d) α *is an $\mathscr{N}\mathscr{L}$-isomorphism.* By theorem 1, it suffices to prove that $(A')^\alpha = (A^\alpha)'$ for $A \in \mathscr{L}(G)$. If $A = G$, this follows from (iii). If $A < G$, let B be a maximal subgroup of G containing A. Then $(A')^\alpha = (A^\alpha)'$ by theorem 1 and (ii) applied to $\mathscr{L}(B)$.

The following is the analogue of theorem 1 for Lie algebras.

THEOREM 1'. *Let L, M be Lie algebras over the same principal domain R. Let $\phi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an isomorphism of the lattice $\mathscr{L}(L)$ of all subalgebras of L onto $\mathscr{L}(M)$. Then the following are equivalent:*

(a) $\mathscr{N}(A)^\phi = \mathscr{N}(A^\phi)$ *for all subalgebras* $A \leqq L$,

(b) $(A')^\phi = (A^\phi)'$ *for all* $A \leqq L$.

PROOF. (a) $\Rightarrow$ (b). This follows as in theorem 1 once we have proved: *if every subalgebra of L is an ideal, then L is abelian.*

To prove this, let $a, b \in L$. Then a, b belong to an R-submodule $Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$. Each Rx_i is a subalgebra of L and is therefore an ideal. For $i \neq j$, $x_i x_j \in Rx_i \cap Rx_j = 0$. Thus

$$(\alpha_1 x_1 + \cdots + \alpha_n x_n)(\beta_1 x_1 + \cdots + \beta_n x_n) = 0 \quad \text{for all} \quad \alpha_i, \beta_i \in R$$

and in particular, $ab = 0$.

(b) $\Rightarrow$ (a). This is obvious because, for $A \leqq L$, $x \in L$,

$$Rx \leqq \mathscr{N}(A) \Leftrightarrow (Rx \cup A)' \leqq A.$$

COROLLARY 1'. *$\mathscr{N}\mathscr{L}$-isomorphisms of Lie algebras preserve central and derived series.*

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COROLLARY 2'. *Let L, M be nilpotent Lie algebras over a field. Then every $\mathscr{L}$-isomorphism $\phi : \mathscr{L}(L) \to \mathscr{L}(M)$ is $\mathscr{N}\mathscr{L}$.*

There is no equivalent for Lie algebras of part (c) of theorem 1, for $(Rx \cup Ry)'$ does not in general coincide with Rxy $(x, y \in L)$. In corollary 2', it is essential to assume both algebras nilpotent, as it is possible for a nilpotent algebra to be $\mathscr{L}$-isomorphic to a non-nilpotent algebra (for example, the two 2-dimensional Lie algebras are $\mathscr{L}$-isomorphic).

2. $\mathscr{N}\mathscr{L}$-isomorphisms between non-isomorphic p-groups

Let p be a prime > 2. We give examples of $\mathscr{N}\mathscr{L}$-isomorphisms between non-isomorphic finite p-groups. By means of Lazard's theory [6], these examples can be turned into examples of the same phenomenon for Lie rings; and when the groups have exponent p the Lie rings become Lie algebras over the field of p elements.

Wiman [13] and Blackburn[6] [2] have determined those p-groups of maximal class[7] which contain an abelian subgroup of index p. Such a group G, of order p^n $(n \geqq 3)$, is isomorphic to an abstract group $A_n(\gamma, \delta)$ defined by generators $s, s_1, s_2, \cdots$ and relations

$$(2.1)\qquad \begin{cases} [s_i, s] = s_{i+1} & (i \geqq 1;\ s_n = s_{n+1} = \cdots = 1), \\ [s_i, s_j] = 1, & (i, j \geqq 1), \\ s^p = s_{n-1}^{\delta}, & \\ s_1^p s_2^{\binom{p}{2}} \cdots s_p = s_{n-1}^{\gamma}, & \\ s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1} = 1 & (i \geqq 2). \end{cases}$$

Set $B_n(\gamma) = A_n(\gamma, 0)$. Then $B_n(\gamma) \cong B_n(\gamma')$ if and only if the congruence $\gamma x^{n-2} \equiv \gamma' \pmod{p}$ has a solution $x \not\equiv 0 \pmod{p}$ (Blackburn, l.c.). On the other hand, we now prove that if (a) $\gamma\gamma' \not\equiv 0 \pmod{p}$ *and* (b) *when* $p = n-1$, $(\gamma-1)(\gamma'-1) \not\equiv 0 \pmod{p}$, *then there exists an $\mathscr{N}\mathscr{L}$-isomorphism of $B_n(\gamma)$ onto $B_n(\gamma')$.*

Since the result is trivial for $n = 3$, we assume $n \geqq 4$. Let $s, s_1, \cdots$ and $s', s_1', \cdots$ be the generators of $G = B_n(\gamma)$ and $H = B_n(\gamma')$ respectively. Choose an integer k such that $\gamma \equiv k\gamma' \pmod{p}$. Then the maximal subgroups of G, H are

$$G_\lambda = \langle ss_1^{\lambda}, s_2, \cdots, s_{n-1} \rangle \qquad (0 \leqq \lambda < p),$$
$$G_p = \langle s_1, s_2, \cdots, s_{n-1} \rangle,$$

and

[6] The reader is referred to section 4 of Blackburn's paper for the properties of maximal class groups used here.

[7] The idea of looking at maximal class groups for examples was suggested to us by J. L. Alperin.

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$$H_\lambda = \langle s' s_1'^{k\lambda}, s_2', \cdots, s_{n-1}' \rangle \qquad (0 \leqq \lambda < p),$$
$$H_p = \langle s_1', s_2', \cdots, s_{n-1}' \rangle.$$

For $\lambda < p$, the generators $\sigma = s s_1^\lambda$, $\sigma_1 = s_2, \cdots, \sigma_{n-2} = s_{n-1}$ of G_λ, and the generators $\sigma' = s' s_1'^{k\lambda}$, $\sigma_1' = s_2', \cdots, \sigma_{n-2}' = s_{n-1}'$ of H_λ, satisfy the defining relations for $A_{n-1}(0, \lambda\gamma)$. Hence there is an isomorphism

$$\alpha_\lambda : G_\lambda \to H_\lambda \qquad (0 \leqq \lambda < p)$$

such that $(s s_1^\lambda)^{\alpha_\lambda} = s' s_1'^{k\lambda}$, $s_i^{\alpha_\lambda} = s_i'$ $(2 \leqq i \leqq n-1)$.

G_p is an abelian group whose defining relations are the last two lines of (2.1). H_p is an abelian group with the same defining relations, except that γ' replaces γ. We show that $H_p = \langle s_1^*, s_2' \cdots, s_{n-1}' \rangle$, where $(s_1^*)^p s_2'^{\binom{p}{2}} \cdots s_p' = s_{n-1}'^{\gamma}$. It will follow that there is an isomorphism

$$\alpha_p : G_p \to H_p$$

such that $s_1^{\alpha_p} = s_1^*$, $s_i^{\alpha_p} = s_i'$ $(2 \leqq i \leqq n-1)$.

For $p > n-1$, the defining relations of H_p reduce to $s_1'^p = s_{n-1}'^{\gamma'}$, $s_i'^p = 1$ $(i \geqq 2)$. In this case we may take $s_1^* = s_1'^k$, since $s_1^{*p} = s_1'^{kp} = s_{n-1}'^{k\gamma'} = s_{n-1}'^{\gamma}$. For $p = n-1$, the defining relations are the same, except that $s_1'^p = s_{n-1}'^{(\gamma'-1)}$. A similar choice of s_1^* is possible by our assumption (b). Finally, if $p < n-1$, then $s_{n-p}'^p s_{n-p+1}'^{\binom{p}{2}} \cdots s_{n-2}'^p = s_{n-1}'^{-1}$ so that $s_{n-1}' = \sigma^p$, where $\sigma \in \langle s_2', \cdots, s_{n-1}' \rangle$. We may therefore take $s_1^* = s_1' \sigma^{\gamma-\gamma'}$.

We now define a mapping $\alpha : \mathscr{L}(G) \to \mathscr{L}(H)$ by:

$$G^\alpha = H,$$
$$K^\alpha = K^{\alpha_\mu} \text{ if } K \leqq G_\mu \qquad (0 \leqq \mu \leqq p).$$

Since the isomorphisms α_μ all agree on $\langle s_2, \cdots, s_{n-1} \rangle$, the definition is unambiguous. All the conditions of lemma 1 are satisfied, so that α is an $\mathscr{N}\mathscr{L}$-isomorphism.

The simplest examples of non-isomorphic, $\mathscr{N}\mathscr{L}$-isomorphic pairs provided by our results are:

for $p > 3$, the groups $B_4(1)$, $B_4(k)$ of order p^4, where k is a non-quadratic residue (mod p);

for $p = 3$, the groups $B_5(1)$, $B_5(-1)$ of order 3^5.

The groups in the above examples have exponent $p^k > p$. We now give examples of $\mathscr{N}\mathscr{L}$-isomorphisms between non-isomorphic groups of exponent p. The groups in question are again groups of maximal class.

Let n be an integer, and p a prime, such that $8 \leqq n \leqq p$. Let $C_n(\varepsilon)$ denote the group of order p^n with generators $s, s_1, s_2, \cdots$ and defining relations

$$[s_i, s] = s_{i+1} \qquad (i \geqq 1;\ s_n = s_{n+1} = \cdots = 1),$$
$$[s_i, s_1] = s_{i+2}^{\varepsilon} \qquad (i \geqq 2),$$
$$[s_i, s_j] = 1 \qquad (i, j \geqq 2),$$
$$s^p = s_1^p = 1.$$

Since $C_n(\varepsilon)$ is generated by s, s_1 of order p and has class $n-1 < p$, it is a group of exponent p. We prove that $G = C_n(1)$, $H = C_n(-1)$ are $\mathscr{N}\mathscr{L}$-isomorphic but not isomorphic.

We take the generators of G to be $s, s_1, s_2, \cdots$ and those of H to be $s, s_1', s_2, \cdots$, so that the subgroup $\langle s, s_2, \cdots, \rangle$ of order p^{n-1} is common to both. Set $V = \Phi(G) = \Phi(H) = \langle s_2, \cdots \rangle$. Then, denoting the endomorphisms $v \to v^{s-1}$, $v \to v^{s_1-1}$, $v \to v^{s_1'-1}$ of V by $\sigma, \sigma_1, \sigma_1'$, we have

$$\sigma^{n-2} = 0,\ \sigma^{n-3} \neq 0,\ \sigma_1 = -\sigma_1' = \sigma^2.$$

Using these facts, it is easy to see that an $\mathscr{N}\mathscr{L}$-isomorphism α of G onto H is defined as follows:

$G^\alpha = H$;
if $K \leqq V$, $K^\alpha = K$;
if $L = K \cap V < K < G$, then K has the form

$$\langle s s_1^\lambda t, L\rangle \quad \text{or} \quad \langle s_1 t, L\rangle \qquad (t \in V)$$

and we take

$$K^\alpha = \langle s s_1'^\lambda t, L\rangle \quad \text{or} \quad \langle s_1' t, L\rangle.$$

Suppose now that there exists an isomorphism $\theta : G \to H$. If $s^\theta V = s^\alpha s_1'^\beta V$, $s_1^\theta V = s^\gamma s_1'^\delta V$, and if ω is the restriction of θ to V, then

$$v^{\omega s^\theta} = v^{s\omega},\ v^{\omega s_1^\theta} = v^{s_1\omega} \quad \text{for} \quad v \in V,$$

so that

$$\omega^{-1}\sigma\omega = (1+\sigma)^\alpha(1-\sigma^2)^\beta - 1,$$
$$\omega^{-1}\sigma^2\omega = (1+\sigma)^\gamma(1-\sigma^2)^\delta - 1.$$

Hence

$$(1+\sigma)^\gamma(1-\sigma^2)^\delta - 1 = \{(1+\sigma)^\alpha(1-\sigma^2)^\beta - 1\}^2.$$

Comparing coefficients of $\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5$ on both sides (this is permissible as $\sigma^{n-3} \not\equiv 0$ and $n \geqq 8$), and using the fact that $\alpha\delta - \beta\gamma \not\equiv 0 \pmod p$, we get in succession the following congruences (mod p):

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$$\gamma \equiv 0, \quad -\delta \equiv \alpha^2, \quad 0 \equiv \binom{\alpha}{2} - \beta,$$
$$\binom{\delta}{2} \equiv 2\alpha \left(\binom{\alpha}{3} - \alpha\beta \right),$$
$$0 \equiv \binom{\alpha}{4} - \beta \binom{\alpha}{2} + \binom{\beta}{2}.$$

On simplification, the second and third lines yield $7\alpha^2 \equiv 1$ and $\alpha^2 \equiv 1$, giving the contradiction $7 \equiv 1$. Thus [8] $G \not\cong H$.

3. A lemma on multilinear mappings

The following lemma plays an essential part in the proof of theorem 2 in the next section.

LEMMA 2. *Let $A_1, \cdots, A_t, L, M$ be (additive) abelian groups. Let $f(x_1, \cdots, x_t)$, $g(x_1, \cdots, x_t)$ be multilinear functions of the variables $x_i \in A_i$ $(i = 1, \cdots, t)$ with values in L, M respectively. Suppose that the values of f, g generate L, M and that there exists an $\mathscr{L}$-isomorphism ϕ of L onto M such that*

$$\langle f(x_1, \cdots, x_t) \rangle^\phi = \langle g(x_1, \cdots, x_t) \rangle \text{ (all } x_1, \cdots, x_t).$$

Then there exists an isomorphism α of L onto M such that

$$f(x_1, \cdots, x_t)^\alpha = g(x_1, \cdots, x_t) \text{ (all } x_1, \cdots, x_t).$$

Remark 1. The expected generalisation to R-modules, R a principal ideal domain, is valid and can be proved by the same method. E.g., when the A_i are vector spaces over a field R and $L = M = R$, the generalisation states that multilinear forms which vanish for the same values of the variables are scalar multiples of one another.

Remark 2. The conclusion of the lemma is equivalent to the following statement:

$$f(x)+f(y)+\cdots = 0 \text{ if and only if } g(x)+g(y)+\cdots = 0.$$

(Here x denotes a row of variables $x_1, \cdots, x_t$ and so on.) In fact, if the condition holds,

$$f(x)+f(y)+\cdots \leftrightarrow g(x)+g(y)+\cdots$$

is a well defined $1-1$ correspondence; it is then clearly an isomorphism.

The lemma is proved by induction on t. If $t = 1$, f, g are homomorphisms of A_1 onto L, M with the same kernel, so that $f(x) \leftrightarrow g(x)$ is an isomorphism

[8] The same conclusion holds for $n = 7$ provided that the congruence $7\alpha^2 \equiv 1 \pmod p$ has no solution, i.e. if $(7/p) \neq 1$.

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between L, M. We assume now that $t > 1$ and that the lemma holds for a smaller number of variables.

By remark 2, and since a given sum $f(x)+f(y)+\cdots$ involves only finitely many values of the variables $x_i, y_j, \cdots$, it is sufficient to prove the lemma when the A_i are finitely generated. Then L, M are also finitely generated. Hence there exists a family of subgroups (L_i) of L such that (a) each factor group L/L_i is cyclic of prime power order and (b) $\bigcap_i L_i = 0$. Define

$$f_i(x) = f(x)+L_i \qquad (\in L/L_i)$$
$$g_i(x) = g(x)+M_i \qquad (\in M/M_i),$$

where $M_i = L_i^\phi$. Then f_i, g_i are multilinear mappings whose values generate L/L_i, M/M_i and $\langle f_i(x)\rangle^{\phi_i} = \langle g_i(x)\rangle$, where ϕ_i is the restriction of ϕ to L/L_i. Thus, if the lemma holds for each pair f_i, g_i, we have

$$f(x)+f(y)+\cdots \in L_i \Leftrightarrow g(x)+g(y)+\cdots \in M_i$$

for each i. Therefore, since $\bigcap_i L_i = 0$ and $\bigcap_i M_i = 0$,

$$f(x)+f(y)+\cdots = 0 \Leftrightarrow g(x)+g(y)+\cdots = 0,$$

so that the lemma holds for f, g. Hence it is sufficient to prove the lemma when L is cyclic of prime power order.

Suppose L cyclic of order p^m. Since ϕ is an $\mathscr{L}$-isomorphism, M is cyclic of order q^m for some prime q. Now $\langle kf(x)\rangle^\phi = \langle f(kx_1, x_2, \cdots,)\rangle^\phi = \langle g(kx_1, x_2, \cdots)\rangle = \langle kg(x)\rangle$, so that $f(x), g(x)$ have the same order. Hence $q = p$. We may therefore assume that $L = M$.

The assertion of the lemma in this case is that

$$f(x) = kg(x) \quad \text{for all } x, \tag{3.1}$$

where k is a fixed integer prime to p. Let r_x denote the order of $f(x)$. We know, since $g(x)$ also has order r_x, that

$$f(x) = k_x g(x),$$

where k_x is prime to p. It is sufficient to prove that

$$k_x \equiv k_y \pmod{r_y} \quad \text{whenever} \quad r_x \geqq r_y > 1. \tag{3.2}$$

For then (3.1) will hold with $k = k_{\bar{x}}$, where $\bar{x}$ is such that $r_{\bar{x}}$ has the largest possible value.

Consider the homomorphisms $\alpha, \beta : A_t \to L$ defined by

$$a^\alpha = f(x_1, \cdots, x_{t-1}, a), \quad a^\beta = f(y_1, \cdots, y_{t-1}. a).$$

Since α, β are non-zero, the inverse images $\alpha^{-1}(pA_t^\alpha)$ and $\beta^{-1}(pA_t^\beta)$ are subgroups of A_t of index p. An abelian group cannot be the set-theoretical

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union of two proper subgroups. Therefore we can choose $a \in A_t$ such that $a \notin \alpha^{-1}(pA_t^\alpha)$ and $a \notin \beta^{-1}(pA_t^\beta)$. Then $a^\alpha \notin pA_t^\alpha$ and $a^\beta \notin pA_t^\beta$, so that

$$\langle a^\alpha \rangle = A_t^\alpha, \qquad \langle a^\beta \rangle = A_t^\beta.$$

Thus, if u and v denote the rows of variables $x_1, \cdots, x_{t-1}, a$ and $y_1, \cdots, y_{t-1}, a$, we have

$$A_t^\alpha = \langle f(u) \rangle, \quad A_t^\beta = \langle f(v) \rangle$$

and therefore

$$(3.3) \qquad r_u \geqq r_x, \qquad r_v \geqq r_y.$$

Let us now apply the lemma in the case $t = 1$ to the linear mappings $\alpha, \alpha' : A_t \to L$ given by

$$\alpha(a) = f(x_1, \cdots, x_{t-1}, a), \quad \alpha'(a) = g(x_1, \cdots, x_{t-1}, a).$$

We deduce that

$$(3.4) \qquad k_u \equiv k_x \pmod{r_x},$$

since $r_u \geqq r_x$. Similarly

$$(3.5) \qquad k_v \equiv k_y \pmod{r_y}.$$

Finally, let us apply the lemma to the multilinear forms

$$\chi(z_1, \cdots, z_{t-1}) = f(z_1, \cdots, z_{t-1}, a),$$
$$\psi(z_1, \cdots, z_{t-1}) = g(z_1, \ldots, z_{t-1}, a).$$

We deduce that

$$(3.6) \qquad k_u \equiv k_v \pmod{\rho},$$

where ρ is the smaller of r_u, r_v. Putting together the results (3.3)—(3.6), we get (3.2) as required.

4. The main results

We study properties of a fixed $\mathscr{N}\mathscr{L}$-isomorphism between two groups G, H. It is convenient for this purpose to represent G, H as factor groups of an auxiliary group F. In the applications, F will be a suitable free, or relatively free, group.

Let $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ be an $\mathscr{N}\mathscr{L}$-isomorphism and $\lambda : F \to G$, $\mu : F \to H$ epimorphisms. We say that ϕ *is compatible with the pair* λ, μ if $K^{\lambda\phi} = K^\mu$ for all $K \in \mathscr{L}(F)$. When this is the case, the equations $(x^\lambda)^\theta = x^\mu$ $(x \in F)$ clearly define an isomorphism $\theta : G \to H$ and ϕ is the $\mathscr{L}$-isomorphism induced by θ.

More generally, we say that ϕ *is compatible with the pair* λ, μ *on the*

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section P/Q *of* F if $K^{\lambda\phi} = K^{\mu}$ whenever $Q \leqq K \leqq P$, i.e. if the restriction of ϕ to P^{λ}/Q^{λ} maps P^{λ}/Q^{λ} onto P^{μ}/Q^{μ} and is compatible with the restrictions λ', μ' of λ, μ to P/Q.

THEOREM 2. *Let* $\lambda : F \to G$, $\mu : F \to H$ *be epimorphisms and* $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ *an* $\mathscr{N}\mathscr{L}$-*isomorphism. Let* $P_1/Q_1, \cdots, P_t/Q_t$ *be central sections of* F *on each of which* ϕ *is compatible with the pair* λ, μ. *Let*

$$P = [P_1, \cdots, P_t]$$
$$Q = [Q_1, P_2, \cdots, P_t][P_1, Q_2, \cdots, P_t] \cdots [P_1, P_2, \cdots, Q_t].$$

Then P/Q *is a central section of* F *and* ϕ *is compatible with the pair* λ, μ *on* P/Q.

PROOF. We prove the theorem for two sections P_1/Q_1, P_2/Q_2. The theorem for t sections then follows in an obvious way by induction.

Since

$$\begin{aligned}[P, G] = [P_1, P_2, G] &\leqq [G, P_1, P_2][P_2, G, P_1] \\ &\leqq [Q_1, P_2][Q_2, P_1] = Q,\end{aligned}$$

P/Q is a central section. Let $u, v \in P_1$, $x, y \in P_2$. Since $P/Q \leqq \mathscr{Z}(G/Q)$, we have

$$[uv, x] = v^{-1}[u, x]v[v, x] \equiv [u, x][v, x] \pmod{Q}.$$

Similarly,

$$[u, xy] \equiv [u, x][u, y] \pmod{Q}.$$

Taking $v \in Q_1$ and $y \in Q_2$, we deduce that the coset of Q containing $[u, x]$ depends only on the cosets of Q_1, Q_2 containing u, x. Hence

$$f(uQ_1, xQ_2) = [u^{\lambda}, x^{\lambda}]Q^{\lambda},$$
$$g(uQ_1, xQ_2) = [u^{\mu}, x^{\mu}]Q^{\mu},$$

are bilinear functions of the variables $uQ_1 \in P_1/Q_1$ and $xQ_2 \in P_2/Q_2$ with values in P^{λ}/Q^{λ}, P^{μ}/Q^{μ} respectively. It is clear that the values of f, g generate P^{λ}/Q^{λ}, P^{μ}/Q^{μ}.

Now, since ϕ is mixed commutator preserving and is compatible with the pair λ, μ on each of P_1/Q_1, P_2/Q_2, we have

$$P^{\lambda\phi} = P^{\mu}, \quad Q^{\lambda\phi} = Q^{\mu},$$

and

$$\begin{aligned}\langle [u, x], Q\rangle^{\lambda\phi} &= \langle [\langle u, Q_1\rangle, \langle x, Q_2\rangle], Q\rangle^{\lambda\phi} \\ &= \langle [\langle u, Q_1\rangle^{\lambda\phi}, \langle x, Q_2\rangle^{\lambda\phi}], Q^{\lambda\phi}\rangle \\ &= \langle [\langle u, Q_1\rangle^{\mu}, \langle x, Q_2\rangle^{\mu}], Q^{\mu}\rangle \\ &= \langle [u, x], Q\rangle^{\mu}.\end{aligned}$$

Thus the restriction, ϕ', of ϕ to P^λ/Q^λ maps P^λ/Q^λ onto P^μ/Q^μ and satisfies $\langle f(uQ_1, xQ_2)\rangle^{\phi'} = \langle g(uQ_1, xQ_2)\rangle$. By lemma 1, there is an isomorphism $\alpha : P^\lambda/Q^\lambda \to P^\mu/Q^\mu$ such that $f(uQ_1, xQ_2)^\alpha = g(uQ_1, xQ_2)$, i.e. $([u, x]Q)^{\lambda\alpha} = ([u, x]Q)^\mu$. Hence, if $Q \leqq K \leqq P$, we have $(K^\lambda/Q^\lambda)^\alpha = (K^\mu/Q^\mu)$ and so, since α induces ϕ', $(K^\lambda/Q^\lambda)^{\phi'} = K^\mu/Q^\mu$. Thus $K^{\lambda\phi} = K^\mu$, which proves the theorem.

Notation. If $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ is an $\mathscr{N}\mathscr{L}$-isomorphism, let $\phi_i : \mathscr{L}(G_{(i)}/G_{(i+1)}) \to \mathscr{L}(H_{(i)}/H_{(i+1)})$ denote its restriction to the i-th factor $G_{(i)}/Q_{(i+1)}$ of the descending central series $(i = 1, 2, \cdots)$.

COROLLARY. *Let $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ be an $\mathscr{N}\mathscr{L}$-isomorphism such that $\phi_1 : \mathscr{L}(G/G') \to \mathscr{L}(H/H')$ is induced by an isomorphism $\theta : G/G' \to H/H'$. Then every ϕ_i is induced by an isomorphism.*

PROOF. We may of course assume G and H nilpotent. Represent G as a factor group F/M of a free group F. Let $\lambda : F \to F/M = G$, $\alpha : G \to G/G'$ be the canonical epimorphisms. Since F is free, the epimorphism $\lambda\alpha\theta_1 : F \to H/H'$ can be lifted to a homomorphism $\mu : F \to H$ such that $H'F^\mu = H$. Since H is nilpotent, $F^\mu = H$. Clearly, ϕ is compatible with the pair λ, μ on F/F'. By the theorem (with all sections P_j/Q_j equal to F/F'), ϕ is compatible with the pair λ, μ on $F_{(i)}/F_{(i+1)}$. Hence ϕ_i is induced by an isomorphism.

For the proof of the next theorem we require the following technical result.

LEMMA 3. *Let F be a finite, relatively free, 2-generator p-group of class 3, and $\phi : \mathscr{L}(F) \to \mathscr{L}(F)$ an $\mathscr{N}\mathscr{L}$-automorphism of F. Let p^t be the exponent of $F_{(3)}$. Then the restriction, ψ, of ϕ to $F/F'F^{p^t}$ is induced by an automorphism of $F/F'F^{p^t}$.*

PROOF. It is not difficult to see that F has generators x, y and defining relations of the form

$$x^{p^r} = y^{p^r} = z^{p^s} = u^{p^t} = v^{p^t} = 1 \qquad (r \geqq s \geqq t),$$
$$[x, u] = [y, u] = [x, v] = [y, v] = 1,$$

where

$$z = [x, y], \quad u = [x, z], \quad v = [y, z].$$

Replacing F by F/F^{p^t}, we may assume that $r = s = t$ and $F' = F'F^{p^t}$.

Let $\langle x\rangle^\phi = \langle X\rangle$, $\langle y\rangle^\phi = \langle Y\rangle$ and write

$$Z = [X, Y], \quad U = [X, Z], \quad V = [Y, Z].$$

Since ϕ is mixed commutator preserving,

$$\langle z\rangle^\phi \equiv \langle Z\rangle \ (\text{mod } F_{(3)}), \quad \langle u\rangle^\phi = \langle U\rangle, \quad \langle v\rangle^\phi = \langle V\rangle.$$

Now, since F' is the direct product of 3 cyclic groups of order p^t, the restriction of ϕ to F' is induced by an automorphism of F' (Baer [1]; cf. Suzuki [12], p. 35). Hence

$$\langle u^\alpha v^\beta \rangle^\phi = \langle U^{m\alpha} V^{n\beta} \rangle$$

for certain fixed integers m, n. On the other hand, if

$$\langle x^\alpha y^\beta \rangle^\phi \equiv \langle X^{\alpha'} Y^{\beta'} \rangle \pmod{F'},$$

then

$$\begin{aligned} \langle u^\alpha v^\beta \rangle^\phi &= \langle [x^\alpha y^\beta, z] \rangle^\phi \\ &= \langle [X^{\alpha'} Y^{\beta'}, Z] \rangle \\ &= \langle U^{\alpha'} V^{\beta'} \rangle. \end{aligned}$$

Hence $\langle x^\alpha y^\beta \rangle^\phi \equiv \langle X^{m\alpha} Y^{n\beta} \rangle \pmod{F'}$, and so ψ is induced by the automorphism $x^\alpha y^\beta F' \to X^{m\alpha} Y^{n\beta} F'$ of F/F'. This proves the lemma.

Let V be a nilpotent variety (i.e. a variety consisting of nilpotent groups) and F a free group of V. Let c be the class of F. Then a factor group

$$(3.1) \qquad G = F/M, \quad M < F_{(c)},$$

will be called an *almost free group of* V. Notice that the class of G is c and that $G/G_{(c)} \cong F/F_{(c)}$ is relatively free.

THEOREM 3. *Let V be a nilpotent variety, G a non-abelian almost free group of V. Let ϕ be an $\mathcal{N}\mathcal{L}$-isomorphism of G onto a member H of V. Then $G \cong H$.*

PROOF. Represent G in the form (3.1) and let $\lambda : F \to F/M = G$, $\alpha : F \to G/G'$ be the canonical epimorphisms. Since G is non-abelian, G/G' is relatively free and so is a (restricted) direct product of isomorphic cyclic groups. Since G is non-cyclic, the number, k, of cyclic factors is at least 2. Therefore $G/G' \cong H/H'$. Let θ be any isomorphism of G/G' onto H/H'. Since $H \in V$ and $H' \leq \Phi(H)$, the epimorphism $\lambda\alpha\theta : F \to H/H'$ can be lifted to an epimorphism $\mu : F \to H$. Let $N = \ker \mu$.

Now, it follows from results of Baer [1] (cf. Suzuki [12], p. 35) that either

(a) $\phi_1 : \mathcal{L}(G/G') \to \mathcal{L}(H/H')$ is induced by an isomorphism, or

(b) G/G' is finite and $k = 2$.

In case (a), we may suppose that ϕ_1 is induced by θ, so that ϕ is compatible with the pair λ, μ on F/F'. By theorem 2, ϕ is compatible with the pair λ, μ on each factor group $F_{(i)}/F_{(i+1)}$. Thus

$$(M \cap F_{(i)})F_{(i+1)} = (N \cap F_{(i)})F_{(i+1)} \qquad (i = 1, 2, \cdots)$$

and therefore, since $M < F_{(i)}$, $M = N$. Hence $G \cong H$.

In case (b), G itself is finite. Hence G is the direct product of its Sylow subgroups $P_1, \cdots, P_m$. Suppose P_i is a Sylow p_i-subgroup. Since ϕ is $\mathscr{N}\mathscr{L}$ and P_i is noncyclic, P_i^ϕ is a p_i-group. It follows that $P_1^\phi, \cdots, P_m^\phi$ are the Sylow subgroups of H (Suzuki [12], p. 5). If P_i is abelian, it is the direct product of 2 cyclic groups of the same order and so $P_i \cong P_i^\phi$. It is therefore sufficient to prove the theorem when G, H are finite (non-abelian) 2-generator p-groups.

It is not difficult to see that, when $c = 2$ or 3, G and H are groups with generators x, y and defining relations of the form

$$(c = 2) \qquad x^{p^\alpha} = y^{p^\alpha} = z^{p^\beta} = u = v = 1,$$

$$(c = 3) \qquad x^{p^\alpha} = y^{p^\alpha} = z^{p^\beta} = u^{p^\gamma} = v^{p^\delta} = 1,$$
$$[x, u] = [y, u] = [x, v] = [y, v] = 1,$$

where $z = [x, y]$, $u = [x, z]$, $v = [y, z]$. Since corresponding factors $G_{(i)}/G_{(i+1)}$, $H_{(i)}/H_{(i+1)}$ are $\mathscr{N}\mathscr{L}$-isomorphic and so isomorphic, the indices α, β, γ, δ are the same for both G, H. Hence $G \cong H$.

Suppose finally that $c \geqq 4$. We remark that $H/H_{(c)}$ is a homomorphic image of $F/F_{(c)}$, and that H/H_c has the same order as F/F_c (since $F/F_{(c)} \cong G/G_{(c)}$ and $G/G_{(c)}$, $H/H_{(c)}$ are $\mathscr{N}\mathscr{L}$-isomorphic p-groups). Therefore $H/H_{(c)} \cong F/F_{(c)}$ and so $N < F_{(c)}$. Set $K = G'G^{p^r}$, $L = H'H^{p^r}$, where p^r is the exponent of $F_{(c)}$. Let $\phi' : \mathscr{L}(G/K) \to \mathscr{L}(H/L)$ be the restriction of ϕ to G/K. Since $c \geqq 4$ and $F/F_{(c)} \cong G/G_{(c)} \cong H/H_{(c)}$, $G/G_{(4)}$ and $H/H_{(4)}$ are relatively free groups of class 3 and the exponent of $G_{(3)}/G_{(4)}$ is $p^s \geqq p^r$. By lemma 3, ϕ' is induced by an isomorphism θ'. Thus, if we choose the isomorphism $\theta : G/G' \to H/H'$ of the first paragraph of the proof in such a way that θ' is the restriction of θ to G/K, then ϕ is compatible with the pair λ, μ on $F/F'F^{p^r}$. By theorem 2 (with each P_i/Q_i equal to $F/F'F^{p^r}$), ϕ is compatible with the pair λ, μ on $F_{(c)}$. Thus $(M \cap F_{(c)})F_{(c+1)} = (N \cap F_{(c)})F_{(c+1)}$, i.e. $M = N$. Hence $G \cong H$. This proves the theorem.

In some cases it is unnecessary to postulate that $H \in V$ since this follows from the existence of the $\mathscr{N}\mathscr{L}$-isomorphism ϕ. The following is the simplest example.

COROLLARY 1. *Let $V = V(p^\mu, c)$ be the variety of all nilpotent groups of class $c' \leqq c$ and exponent $p^{\mu'} \leqq p^\mu$. Let G be a non-abelian almost free group of V and ϕ an $\mathscr{N}\mathscr{L}$-isomorphism of G onto a group H. Then $G \cong H$.*

Recalling that every $\mathscr{L}$-isomorphism of a group of prime exponent is an $\mathscr{N}\mathscr{L}$-isomorphism, we get

COROLLARY 2. *Let G be a non-abelian, almost free group of $V(p, c)$ and ϕ an $\mathscr{L}$-isomorphism of G onto a group H. Then $G \cong H$.*

In particular:

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COROLLARY 3. *Let G be a nilpotent group of class 2 and exponent p. Let ϕ be an $\mathscr{L}$-isomorphism of G onto a group H. Then $G \cong H$.*

Lie rings. The analogue of theorem 2 for Lie algebras over a principal ideal domain holds. (The arguments involving the mixed commutator preserving property can all be rephrased in terms of the commutator preserving property). Theorem 3 has the following analogue.

THEOREM 3′. *Let V be a nilpotent variety of Lie rings, L a non-abelian almost free ring of V. Let ϕ be an $\mathscr{N}\mathscr{L}$-isomorphism of L onto a member M of V. Then $L \cong M$.*

References

[1] Baer, R., The significance of the system of subgroups for the structure of the group, Amer. J. Math. 61 (1939), 1–44.
[2] Blackburn, N., On a special class of p-groups, Acta Math. 100 (1958), 45–92.
[3] Higman, G., Lie ring methods in the theory of finite nilpotent groups, Proc. Internat. Congress of Mathematicians, Edinburgh (1958), 307–312.
[4] Iwasawa, K., Über die endlichen Gruppen und die Verbände ihrer Untergruppen, J. Univ. Tokyo 4–3 (1941), 171–199.
[5] Iwasawa, K., On the structure of infinite M-groups, Jap. J. Math. 18 (1943), 709–728.
[6] Lazard, M., Sur les groupes nilpotents et les anneaux de Lie, Ann. Sci. École Norm. Sup. (3) 71 (1954), 101–190.
[7] Pekelis, A. S., Structural isomorphisms of locally nilpotent torsion-free groups, Uspehi mat. nauk 18 (1963), pp. 187–190 of part 3(iii). (Russian).
[8] Pekelis, A. S. and Sadovskii, L. E., Projections of a metabelian torsion-free group, Doklady Akad. Nauk S.S.S.R. 151 (1963), 42–44 (English translation in Soviet Math. 4 (1963), 918–920).
[9] Rottländer, A., Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen, Math. Z. 28 (1928), 641–653.
[10] Šmel'kin, Free polynilpotent groups, Doklady Akad. Nauk S.S.S.R. 151 (1963), 73–75 (English translation in Soviet Math. 4 (1963), 950–953).
[11] Spring, R. F., Lattice isomorphisms of finite non-abelian groups of exponent p, Proc. Amer. Math. Soc. 14 (1963), 407–413.
[12] Suzuki, M., Structure of a Group and the Structure of its Lattice of Subgroups (Springer, Berlin, 1956).
[13] Wiman, A., Über mit Diedergruppen verwandte p-Gruppen, Arkiv för Mat., Astron. och Fys. 33A (1946).

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