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ON THE NUMBER OF MAXIMAL ELEMENTS IN A PARTIALLY ORDERED SET

BY

JOHN GINSBURG

ABSTRACT. Let P be a partially ordered set. For an element $x \in P$, a subset C of P is called a cutset for x in P if every element of C is noncomparable to x and every maximal chain in P meets $\{x\} \cup C$. The following result is established: if every element of P has a cutset having n or fewer elements, then P has at most 2^n maximal elements. It follows that, if some element of P covers k elements of P then there is an element $x \in P$ such that every cutset for x in P has at least $\log_2 k$ elements.

Let $(P, \leqslant)$ be a partially ordered set. For an element x of P, a subset S of P is called a *cutset for* x in P if

(i) every element of S is noncomparable to x, and

(ii) every maximal chain in P meets $\{x\} \cup S$.

Let n be a cardinal number. If every element of P has a cutset containing n or fewer elements we say that P has the *n-cutset property*.

Although our primary interest here is in finite partially ordered sets and the n-cutset property where n is a non-negative integer, our main result is valid for all partially ordered sets and for any cardinal number n, finite or infinite.

To illustrate the definition, we refer to the partially ordered sets shown in Figures 1 and 2 below. In Figure 1, the set $S = \{a, b\}$ is a cutset for x, and furthermore this partially ordered set has the 2-cutset property. In Figure 2, the set $S = \{a, b, c\}$ is a cutset for x and here P has the 3-cutset property.

For any partially ordered set P and for any $x \in P$, the set S consisting of *all* elements of P which are noncomparable to x obviously is a cutset for x in P. Two less trivial examples of cutsets in finite partially ordered sets (discussed in [3] and [4]) are the following. For $x \in P$, let $U(x) = \{p \in P$: p is noncomparable to x and either p is a maximal element or there is an element $u \in P$ such that $x < u$ and u covers $p\}$. Then $U(x)$, as well as its dual, is a cutset for x in P. (Here the phrase "u covers p" means, as usual, that $p < u$ and there is no element $q \in P$ with $p < q < u$.) As a second example, let P be a finite partially ordered set in which all maximal chains have the same number of elements. Then, for any $x \in P$, the set of all elements having the same

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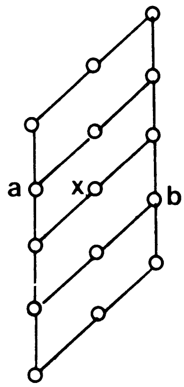

FIGURE 1

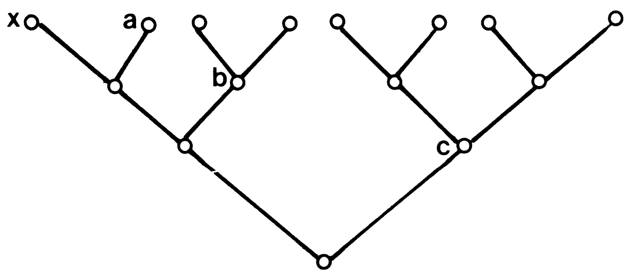

FIGURE 2

"height" as x forms a cutset for x in P. The cutsets illustrated in Figures 1 and 2 are of these types.

This notion of cutset has been investigated by several authors and many interesting results have been obtained. For example, partially ordered sets in which every element has a cutset which is an antichain are characterized in [4] as those which contain no alternating cover cycles. In [6] it is shown that if P has the 2-cutset property then every element of P is contained in a maximal antichain having 4 or fewer elements. In [2] and [7] it is shown that every partially ordered set with the 2-cutset property satisfies $w \leqslant \ell + 2$, where ℓ and w denote the length and width of P respectively. Cutsets in $P(n)$, the set of all subsets of an n-element set ordered by inclusion, are investigated in [3], where it is shown that, except for a few small exceptions, for $x \in P(n)$ the smallest size of a cutset for x is either that of $U(x)$ or its dual.

In this paper we are interested in the number of maximal elements in a partially ordered set, and in particular how this number is related to the sizes of cutsets. We show that if P has the n-cutset property than P can contain no more than 2^n maximal elements. This has an immediate corollary concerning the number of elements of P covered by an element of P.

Before we proceed to the proof, we note that this result is best possible. For, let P be a binary tree of height n. Then P has 2^n maximal elements and it is easy to see that

P has the n-cutset property. (Figure 2 shows a binary tree of height 3).

We will find the following terminology useful as in [6]. If C is a chain in P and if p is an element of P such that $\{p\} \cup C$ is a chain, we say that p *extends* C. Note that condition (ii) in the above definition of cutset is equivalent to the following: for every chain C in P, either x extends C or some element p of S extends C.

THEOREM. *Let n be a cardinal number. If P is a partially ordered set having the n-cutset property then P has at most* 2^n *maximal elements.*

PROOF. CASE 1. *n is finite* (i.e. n is a non-negative integer).

We will actually establish a slightly stronger statement by induction on n, namely the the following: (*) Let k be a positive integer and let $a_1, a_2, \ldots, a_k$ be distinct maximal elements in a partially ordered set P. Suppose that for each $i \leq k$ there is a subset S_i of P with the following properties:

(i) every element of S_i is noncomparable to a_i,

(ii) $|S_i| \leq n$, and

(iii) for all $i \leq k$ and $j \leq k$ with $i \neq j$, every chain in P containing a_i is extended by some element of S_j. Then $k \leq 2^n$.

In the case $n = 0$, (*) obviously is true, since in this case, we have $S_i = \phi$ for all $i = 1, 2, \ldots, k$, and so (ii) implies $k = 1$.

Now suppose (*) is true for all integers $<n$ and we prove it for n. So, let $a_1, a_2, \ldots, a_k$ and $S_1, S_2, \ldots, S_k$ satisfy the conditions in (*). We wish to prove that $k \leq 2^n$. Now, by (iii), there are elements $b_i \in S_i$ for $i = 2, 3, \ldots, k$, such that $\{a_1\} \cup \{b_2, b_3, \ldots, b_k\}$ is a chain. Since a_1 is maximal, we may assume (relabeling if necessary) that

$$a_1 \geq b_2 \geq b_3 \geq \ldots \geq b_k.$$

Now for each $i = 3, 4, \ldots, k$, let

$$A_i = \{a_j : 2 \leq j \leq k \text{ and } a_j \geq b_i \text{ and } a_j \not\geq b_{i-1}\}.$$

Also, let

$$B = \{a_j : 2 \leq j \leq k \text{ and } a_j \not\geq b_k\}.$$

We note that

$$\{a_1, a_2, \ldots, a_k\} = \{a_1\} \leqq \left(\bigcup_{i=3}^{k} A_i\right) \cup B.$$

For, let $j \geq 2$. Then either a_j is comparable to none of the elements $b_2, b_3, \ldots, b_k$ (in which case $a_j \in B$) or there is a smallest $i \in \{2, 3, \ldots, k\}$ such that a_j is comparable to b_i. In this latter case, we must have $a_j \geq b_i$ because a_j is maximal. And since $b_2 \geq b_j$ we cannot have $i = 2$, as a_j is not comparable to b_j. (By condition (i), since $b_j \in S_j$). Therefore $i \geq 3$ and we have $a_j \in A_i$ in this case.

Now, some of the sets A_i may be empty. Let $\{i_1, i_2, \ldots, i_T\}$ enumerate the elements

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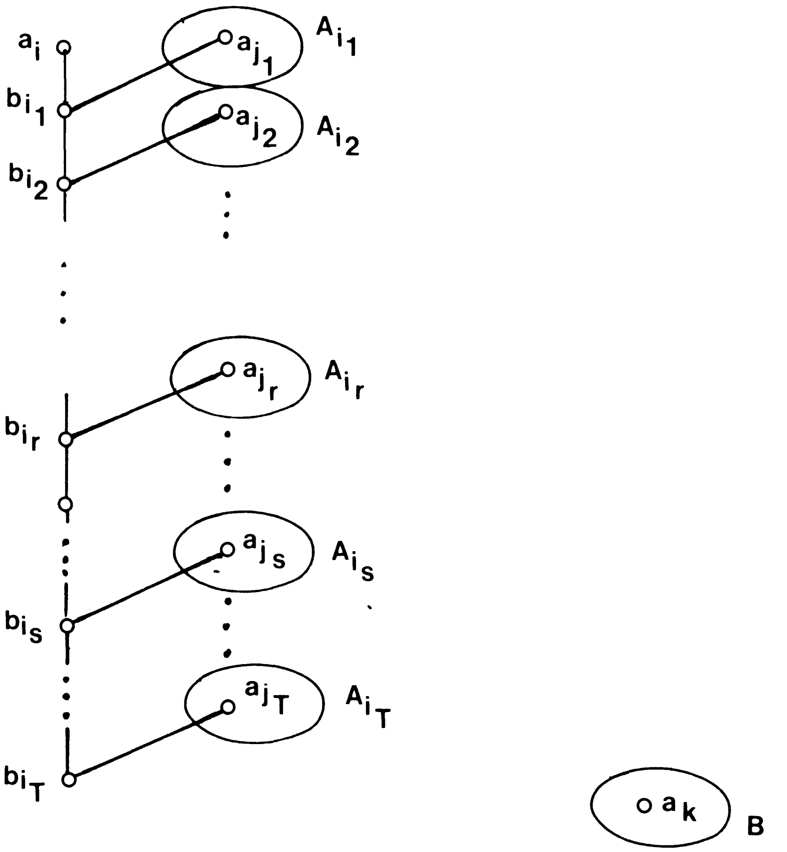

FIGURE 3

of the set $\{i: 3 \leqslant i \leqslant k \text{ and } A_i \neq \phi\}$ where we assume that $i_1 < i_2 < \dots < i_T$. Then by the above remarks, we have that

$$\{a_1, a_2, \dots, a_k\} = \{a_1\} \cup \left(\bigcup_{r=1}^{T} A_{i_r} \right) \cup B.$$

We also note that $B \neq \phi$. In fact, $a_k \in B$ because $b_k \in S_k$.

Next we estimate the size of T and of the sets A_{i_r} and B. The situation is represented in Figure 3.

Now, for each $r = 1, 2, \dots, T$ choose an element $a_{j_r} \in A_{i_r}$. For each $r = 1, 2, \dots, T$ there is an element $c_r \in S_1$ such that c_r extends the chain $\{b_{i_r}, a_{j_r}\}$. Since c_r is noncomparable to a_1 we must have $c_r \geqslant b_{i_r}$, and since a_{j_r} is maximal, we have $c_r \leqslant a_{j_r}$. Thus $b_{i_r} \leqslant c_r \leqslant a_{j_r}$ for $r = 1, 2, \dots, T$. From this we see that $r < s \rightarrow c_r \neq c_s$. For $c_r = c_s$ would imply that $b_{i_r} \leqslant c_r = c_s \leqslant a_{j_s}$. But $a_{j_s} \in A_{i_s}$ and so i_s is the smallest i for which $a_{j_s} \geqslant b_i$. In particular, $b_{i_r} \not\leqslant a_{j_s}$, a contradiction. This proves our claim that $r < s \rightarrow c_r \neq c_s$. So the elements $c_1, c_2, \dots, c_T$ are distinct. Also, there is some element $c \in S_1$ such that c extends the chain $\{a_k\}$. Hence $c \leqslant a_k$. For all $r = 1, 2, \dots, T$

we have $c \neq c_r$, because $c = c_r$ would imply that $b_k \leq b_{i_r} \leq c_r = c \leq a_k$ contrary to the fact that a_k is not comparable to b_k. Therefore the elements $c, c_1, c_2, \ldots, c_T$ account for $T + 1$ distinct elements from the set S_1. Since $|S_1| \leq n$ this implies $T \leq n - 1$.

Now consider the set A_{i_r}. Let a_j be any element of A_{i_r}. Since a_j is not comparable to b_j (because $b_j \in S_j$), whereas $a_j \geq b_{i_r}$, we must have $j < i_r$. Hence for any element $a_\ell \in A_{i_r}$ we see that a_ℓ is not comparable to b_j, because i_r is the smallest i for which $a_\ell \geq b_i$.

Next, for each $s = r + 1, r + 2, \ldots, T$ choose an element $x_s \in S_j$ such that x_s extends the chain $\{b_{i_s}, a_{j_s}\}$. Arguments similar to the one above for a_1 show that $b_{i_s} \leq x_s \leq a_{j_s}$ and that $s_1 \neq s_2 \rightarrow x_{s_1} \neq x_{s_2}$. Also, let x be an element of S_j which extends the chain $\{a_k\}$. Then $x \leq a_k$, and $x \neq x_s$ for all $s = r + 1, \ldots T$. Therefore all of the elements $x, x_{r+1}, x_{r+2}, \ldots, x_T$ are distinct. Furthermore, none of these elements are $\geq b_{i_r}$. For example $x_s \geq b_{i_r}$ would imply that $a_{j_s} \geq b_{i_r}$ contrary to the definition of the set A_{i_s}.

Now, consider the partially ordered set $P' = \{p \in P : p \geq b_{i_r}\}$, with the induced ordering from P, of course. The set

$$S_j' = [S_j - \{b_j, x, x_{r+1}, x_{r+2}, \ldots, x_T\}] \cap P'$$

has at most $n - (2 + (T - r)) = n - T + r - 2$ elements. We note that, if a_ℓ is any other element of A_{i_r}, then every chain C in P' containing a_ℓ is extended by an element of S_j': for $C \cup \{b_{i_r}\}$ is a chain in P, and so some element $y \in S_j$ extends $C \cup \{b_{i_r}\}$. Therefore $y \geq b_{i_r}$ because y is not comparable to a_j. And y cannot be any of the elements $\{x, x_{r+1}, \ldots, x_T\}$ since, as shown above, none of these latter elements are $\geq b_{i_r}$. Furthermore $y \neq b_j$, because a_ℓ is not comparable to b_j by definition of A_{i_r}. Therefore $y \in S_j'$, as desired. Since A_{i_r} is a set of maximal elements of P', our inductive hypothesis implies that $|A_{i_r}| \leq 2^{n-T+r-2}$.

Also note that, for any two elements a_j, a_ℓ in B with $j \neq \ell$ we have a_ℓ is non-comparable to b_j by definition of B. It follows that any chain in P containing a_ℓ is extended by some element of $S_j - \{b_j\}$. Therefore, our inductive hypothesis implies that $|B| \leq 2^{n-1}$.

Finally, since

$$\{a_1, a_2, \ldots, a_k\} = \{a_1\} \cup \left(\bigcup_{r=1}^{T} A_{i_r}\right) \cup B,$$

we have that

$$k \leq 1 + \sum_{r=1}^{T} 2^{n-T+r-2} + 2^{n-1} = 1 + (2^{n-T-1} + 2^{n-T} + \ldots + 2^{n-2}) + 2^{n-1} \leq 1 + \left(\sum_{m=0}^{n-2} 2^m\right) + 2^{n-1} = 2^n,$$

completing the proof of Case 1.

CASE 2. *n is infinite.*

In this case we will use the partition relation $(2^n)^+ \rightarrow (n^+)^2_n$, for which we refer to

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[1].* For the sake of contradiction, suppose P has the n-cutset property and has more than 2^n maximal elements. Then there exists a set of distinct maximal elements $\{a_i : i < m\}$, where $m = (2^n)^+$, the first cardinal larger than 2^n. For each $i < m$, let S_i be a cutset for a_i in P with $|S_i| \leq n$. For each i, list the elements of S_i as $S_i = \{b^i_\alpha : \alpha < n\}$. For each $i < m$, let C_i be a maximal chain in P with $a_i \in C_i$. Now, for any $i, j < m$ with $i \neq j$, C_j meets S_i and hence there is some $\alpha < n$ such that $b^i_\alpha \in C_j$. Thus we have a partition of the set of all pairs $\{i, j\}$, where $i, j < m$ and $i \neq j$, into blocks $B_{\alpha\beta}$, for $\alpha, \beta < n$, where, for any pair $\{i, j\}$ with $i < j$, we set $\{i, j\} \in B_{\alpha\beta} \leftrightarrow b^i_\alpha \in C_j$ and $b^j_\beta \in C_i$. Since there are $n \cdot n = n$ such blocks, the partition relation stated above implies that there exists an $\alpha < n$ and a $\beta < n$, and elements i, j, k with $i < j < k < m$ such that all three of the pairs $\{i, j\}$, $\{i, k\}$, $\{j, k\}$ belong to $B_{\alpha\beta}$. This means that $b^i_\alpha \in C_j \cap C_k$, $b^j_\alpha \in C_k$, and that $b^j_\beta \in C_i$ and $b^k_\beta \in C_i \cap C_j$. Now, since b^i_α and a_j are both in the chain C_j, and since a_j is maximal, we have $b^i_\alpha \leq a_j$. Similarly we have $b^i_\alpha \leq a_k$. Furthermore, b^k_β is in the chain C_j along with a_j and b^i_α. We cannot have $b^k_\beta \leq b^i_\alpha$ because this would imply $b^k_\beta \leq a_k$, contrary to the fact that b^k_β belongs to the cutset S_k for a_k, and so is noncomparable to a_k. So we must have (again using the maximality of a_j) $b^i_\alpha \leq b^k_\beta \leq a_j$. Since both a_j and b^i_α are noncomparable to a_i, these latter relations imply that b^k_β is noncomparable to a_i. But this contradicts the fact that b^k_β is in the chain C_i along with a_i. This completes the proof of Case 2. □

We note that the proof of our theorem above actually establishes a somewhat stronger statement, namely that, if every maximal element of P has a cutset containing n or fewer elements then P has at most 2^n maximal elements. This can be applied, for finite partially ordered sets, to the cutsets $U(x)$ and their duals discussed above, and so we have the following corollary.

COROLLARY 1. *Let P be a finite partially ordered set and suppose that P has k maximal elements. Then for some maximal element x in P, the set $L(x) = \{p \in P : p$ is noncomparable to x and either p is minimal in P or there is an element $u \in P$ such that $u < x$ and p covers $u\}$ contains at least* $\log_2 k$ *elements.*

A second corollary concerns the number of elements of P covered by an element of P.

COROLLARY 2. *Let P be a partially ordered set, and suppose that some element of P covers k elements of P. Then there is an element x in P such that every cutset for x in P contains at least* $\log_2 k$ *elements.*

PROOF. This follows directly from the theorem using the following two observations: for any element $p \in P$, the (sub) partially ordered set $P' = \{x \in P : x < p\}$ has the

*Here we are using the standard notation k^+ to denote the first cardinal number larger than k. The partition relation $(2^n)^+ \to (n^+)^2_n$ has the following meaning (see [1]): Let X be a set of cardinality $(2^n)^+$. We let $[X]^2$ denote the set of all pairs $\{x, y\}$ of elements of X. Suppose $\{B_i : i < n\}$ is a family of sets such that $[X]^2 = \cup_{i<n} B_i$. Then there is a subset Y of X having cardinality n^+, and an element $i < n$ such that $[Y]^2 \subset B_i$.

n-cutset property if P does, and the maximal elements of P' are just the elements in P covered by p. □

The author has learned that, while investigating the relationship between length, width and cutset size, N. Sauer [5] has obtained the bound $((n + 1)!)$ on the number of maximal elements in a partially ordered set with the n-cutset property.

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THE UNIVERSITY OF WINNIPEG
WINNIPEG, MANITOBA
CANADA R3B 2E9