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CENTRAL LIMIT THEOREMS FOR FUNCTIONALS OF STATIONARY GERM–GRAIN MODELS

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Abstract

Conditions are derived for the asymptotic normality of a general class of vector-valued functionals of stationary Boolean models in the d-dimensional Euclidean space, where a Lindeberg-type central limit theorem for m-dependent random fields, $m \in \mathbb{N}$, is applied. These functionals can be used to construct joint estimators for the vector of specific intrinsic volumes of the underlying Boolean model. Extensions to functionals of more general germ–grain models satisfying some mixing and integrability conditions are also discussed.

Keywords: Random closed set; Boolean model; stationary random field; m-dependent random field; valuation; asymptotic normality; β-mixing; specific intrinsic volume; Euler number

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1. Introduction

Consider a stationary random closed set $\Xi \subset \mathbb{R}^d$ such that $\Xi \cap K$ belongs to the convex ring $\mathcal{R}$ with probability 1 for any convex and compact test set $K \subset \mathbb{R}^d$. Assume that Ξ can be (indirectly) observed within a bounded observation window $W \subset \mathbb{R}^d$. Suppose that this indirect observation is made by measuring some 'local' geometric features,

$$Y(x) = f((\Xi - x) \cap K), \qquad x \in W \ominus \check{K}, \tag{1.1}$$

of Ξ within a small scanning window $K \subset \mathbb{R}^d$, where '$\ominus$' denotes Minkowski difference, $\check{K}$ is the reflection of K, and $f: \mathcal{R} \to \mathbb{R}$ is some real-valued functional possessing the properties of a valuation (see, e.g. [13, p. 184]). If f is invariant with respect to translations, then $Y(x) = f(\Xi \cap (K + x))$ holds, where $K + x$ can be interpreted as local neighborhood of the measurement point x. A natural unbiased estimator for the mean $\mu = \mathrm{E}(Y(x))$ of the stationary random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ is the weighted average

$$\hat{\mu} = \int_W Y(x) G(W, x)\, \mathrm{d}x, \tag{1.2}$$

where $G(W, x)$ is a weighting kernel that integrates to 1 over W and vanishes for those x for which $Y(x)$ is not observable.

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The question to be answered is, what asymptotic properties does the estimator $\hat{\mu}$ have for an unboundedly increasing sequence of observation windows $W_n \uparrow \mathbb{R}^d$, $n \in \mathbb{N}$? It is well known from the general theory of stationary random fields (see, e.g. Section 1.7 of [8, pp. 35–42]) that the estimator given in (1.2), properly normalized, is asymptotically normally distributed under the assumption that $\mathrm{E}(|Y(x)|^{2+\delta}) < \infty$ for some $\delta > 0$ and if additional Rosenblatt-type mixing conditions on Y are satisfied. Roughly speaking, these conditions ensure that various mixing rates of Y expressing the dependence between $Y(x)$ and $Y(x+t)$ decrease in order of $|t|^{-d-\varepsilon}$, as $|t| \to \infty$ and for some $\varepsilon > 0$. Notice that these assumptions are dictated by the sectioning technique of Bernstein and the classical Lyapunov-form central limit theorem used in the proofs.

However, in the context of random fields Y as defined in (1.1) and generated by random closed sets of the form $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$, where $\{X_i\}$ is a point process of 'germs' and $\{M_i\}$ is a sequence of random compact 'grains', a mixing condition on $\{X_i\}$ and an integrability condition on $\{M_i\}$ can be used to show the asymptotic normality of the estimator $\hat{\mu}$ given in (1.2). In particular, if $\{X_i\}$ is a Poisson process or a 'Poisson-like' point process with finite range of correlation, a Lindeberg-type central limit theorem developed in [4] for so-called m-dependent random fields, $m \in \mathbb{N}$, is applicable.

We emphasize that this technique can be used to prove the asymptotic normality of $\hat{\mu}$ for any conditionally bounded valuation f. Related results for another general class of functionals of germ–grain models have been derived in [7]. Furthermore, there exist various results of this sort for particular functionals f, such as the empirical volume fraction, boundary length, and convexity number; see, e.g. [1], [5], [9], and the references in [10, pp. 30–43].

The paper is organized as follows. Section 2 contains preliminary results. In Section 2.1, we recall some basic notions from stochastic geometry, such as random closed sets, germ–grain models, and, in particular, the Boolean model. Then, in Section 2.2, a quite general class of functionals of stationary random fields is introduced and an upper bound is derived for the moments of stationary random fields associated with these functionals. In Section 2.3, conditions for the mean-square consistency of the mean-value estimator $\hat{\mu}$ are given. Some examples of valuations are discussed in Section 3. The corresponding random fields can be used to construct joint estimators for the vector of specific intrinsic volumes of stationary random sets; see [11] and [14]. In Section 4, we consider a Boolean model $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$ with convex and compact grains. In particular, we show in Section 4.1 that the covariance function $\mathrm{cov}_Y(x)$ admits an integrable upper bound provided that

$$\mathrm{E}(|M_i \oplus \check{K}|^2) < \infty, \tag{1.3}$$

where '$\oplus$' denotes Minkowski sum and $|\cdot|$ is the d-dimensional Lebesgue measure. This bound depends on the distribution of the grains M_i; the dependence is monotone with respect to inclusion. Using a truncation technique and the Lindeberg-type central limit theorem for m-dependent random fields, we show in Sections 4.2 and 4.3 that the weighted average $\hat{\mu}$ of Y over W is asymptotically normally distributed for any unboundedly increasing sequence of observation windows $W_n \uparrow \mathbb{R}^d$ that satisfies certain additional regularity conditions. Using the well-known Cramér–Wold device, this result can be easily extended to a multidimensional setting.

Conditions for the asymptotic normality of the estimator $\hat{\mu}$ for more general germ–grain models are discussed in Section 5. Proceeding as in [7], a central limit theorem for β-mixing random fields given in [6] is applied, together with an upper bound for the β-mixing coefficient of random measures associated with the germ–grain models. For this theorem, a stronger integrability condition is needed, namely $\mathrm{E}(D^{2d(1+\delta')}(M_i)) < \infty$ for some $\delta' > 0$, where $D(M_i) = \sup\{|x|,\ x \in M_i\}$ is the 'radius' of the grains. Notice that condition (1.3) is fulfilled if $\mathrm{E}(D^{2d}(M_i)) < \infty$.

2. Mean-value estimators for stationary random fields

We first recall some basic notions from stochastic geometry that will be used in the paper. Further details can be found in, e.g. [13] and [15]. In the second part of this section, we consider a class of unbiased and consistent estimators for the mean value of certain stationary random fields.

2.1. Germ–grain models

Let $d \geq 2$ be an arbitrary, fixed integer. For any two sets $B, B' \subset \mathbb{R}^d$, let $B \oplus B' = \{x + y, x \in B, y \in B'\}$ be the *Minkowski sum* of B and B' and write $B + x = B \oplus \{x\}$ for the *translation* of B by the vector $x \in \mathbb{R}^d$. Furthermore, consider the *reflection* $\check{B} = \{-x, x \in B\}$ of B at the origin and denote the *Minkowski difference* of B and B' by $B \ominus B' = \{x : \check{B}' + x \subseteq B\}$.

Let $\mathcal{B}(\mathbb{R}^d)$ be the σ-algebra of Borel sets in $\mathbb{R}^d$ and let $\mathcal{B}_0(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ be the family of all bounded Borel sets. Furthermore, let $\mathcal{F} \subset \mathcal{B}(\mathbb{R}^d)$ denote the family of all closed sets and $\mathcal{K} \subset \mathcal{F}$ the family of all convex bodies, i.e. convex and compact sets in $\mathbb{R}^d$. For the *convex ring* we shall write $\mathcal{R}$. It is the family of all finite unions of sets in $\mathcal{K}$, which are sometimes also called *polyconvex sets*. The *extended convex ring* $\mathcal{S}$ is the family of Borel sets $B \in \mathcal{B}(\mathbb{R}^d)$ such that $B \cap K \in \mathcal{R}$ holds for any convex body $K \in \mathcal{K}$. A *random closed set* (RACS) Ξ in $\mathbb{R}^d$ is an $(\mathcal{A}, \sigma_{\mathcal{F}})$-measurable mapping from some probability space $(\Omega, \mathcal{A}, \mathrm{P})$ into $\mathcal{F}$ equipped with the σ-algebra $\sigma_{\mathcal{F}}$, which is generated by the events $\{F \in \mathcal{F}, F \cap K \neq \varnothing\}$, $K \in \mathcal{F}$, with K compact.

We say that Ξ is *stationary* if the distribution of the translated RACS $\Xi + x$ is equal to the distribution of Ξ for any $x \in \mathbb{R}^d$. In the following, we consider stationary RACSs Ξ with realizations from the extended convex ring $\mathcal{S}$, i.e. with $\Xi \cap K \in \mathcal{R}$ almost surely for any $K \in \mathcal{K}$. The RACS Ξ is said to be an (independently marked) *germ–grain model* if it can be represented in the form

$$\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i), \tag{2.1}$$

where the so-called *germs* X_i form a simple point process $X = \{X_i\}$ in $\mathbb{R}^d$ and the sequence $M = \{M_i\}$ of *grains* M_i is independent of $\{X_i\}$ and consists of independent copies of a nonempty compact RACS M_0. Notice that the infinite union of RACSs $M_i + X_i$ on the right-hand side of (2.1) is almost surely closed and different from $\mathbb{R}^d$ if the point process X is stationary with finite intensity λ and if

$$\mathrm{E}(|M_0 \oplus \check{K}|) < \infty \tag{2.2}$$

for each $K \in \mathcal{K}$. This condition holds, for instance, if $\mathrm{E}(D^d(M_0)) < \infty$, where $D(B) = \sup\{|x|, x \in B\}$ denotes the radius (or norm) of a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ and $|x|$ is the length of the vector $x \in \mathbb{R}^d$. Condition (2.2) and the stationarity of X imply that only finitely many translated grains $M_i + X_i$ have a nonempty intersection $(M_i + X_i) \cap K$ with any fixed convex body $K \in \mathcal{K}$. In other words, the random variable

$$N(\Xi \cap K) = \mathrm{card}\{i : (M_i + X_i) \cap K \neq \varnothing\} \tag{2.3}$$

is finite with probability 1 for each $K \in \mathcal{K}$, where $\mathrm{card}(B)$ denotes the cardinality of the set B. Let $g_{N(\Xi \cap K)}(s) = \mathrm{E}(s^{N(\Xi \cap K)})$, $s \in \mathbb{R}$, be the generating function of $N(\Xi \cap K)$.

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If the point process $\{X_i\}$ of germs is a stationary Poisson process, then the stationary RACS Ξ defined by formula (2.1) is called a *Boolean model*. For Boolean models, it is not difficult to show that the random variable $N(\Xi \cap K)$ is Poisson distributed with parameter $\lambda\,\mathrm{E}(|M_0 \oplus \check{K}|)$; see, e.g. Section 4.1 of [3, p. 201]. Thus, in this case, the generating function $g_{N(\Xi\cap K)}$ is given by

$$g_{N(\Xi\cap K)}(s) = \mathrm{e}^{(s-1)\lambda\,\mathrm{E}(|M_0\oplus\check{K}|)}, \qquad s \in \mathbb{R}.$$

This means in particular that $g_{N(\Xi\cap K)}(s) < \infty$ for any $s \in \mathbb{R}$ if (2.2) is satisfied. Furthermore, a Boolean model Ξ with nonempty polyconvex grains M_i can be represented as the union set of a Poisson (particle) process $\tilde{M} = \{\tilde{M}_i\}$ on $\mathcal{R}$, where $\tilde{M}_i = M_i + X_i$; in other words, we have $\Xi = \bigcup_{i=1}^{\infty} \tilde{M}_i$ (see, e.g. [13, Section 4.4, p. 151]).

2.2. Random fields associated with germ–grain models

Let the functional $f\colon \mathcal{R} \to \mathbb{R}$ be a *valuation* on the convex ring $\mathcal{R}$. This means that $f(\varnothing) = 0$ and that f is measurable and additive, i.e.

$$f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2)$$

for any $K_1, K_2 \in \mathcal{R}$. Regarding the value $f(K_1 \cup \cdots \cup K_k)$ for the union of $k \geq 2$ sets $K_1, \ldots, K_k$ from $\mathcal{R}$, the general *inclusion-exclusion formula*

$$f(K_1 \cup \cdots \cup K_k) = \sum_{i=1}^{k} (-1)^{i-1} \sum_{j_1 < \cdots < j_i} f(K_{j_1} \cap \cdots \cap K_{j_i}) \tag{2.4}$$

easily follows from the additivity of f. Furthermore, we assume that f is *conditionally bounded* on $\mathcal{K}$, that is, for any pair $K, K' \in \mathcal{K}$ with $K' \subseteq K$, the inequality

$$|f(K')| \leq c(K)$$

holds for some finite bound $c(K)$. For any fixed convex body $K \in \mathcal{K}$ and for any RACS Ξ, consider the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ given by

$$Y(x) = f((\Xi - x) \cap K), \qquad x \in \mathbb{R}^d. \tag{2.5}$$

If Ξ is stationary then the random field Y is stationary, i.e. its finite-dimensional distributions are invariant with respect to translations. In particular, we have $Y(x) \overset{\mathrm{D}}{=} Y(o)$ for any $x \in \mathbb{R}^d$, where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution and $o \in \mathbb{R}^d$ is the origin. Throughout this paper, we assume that the field Y given by (2.5) is of second order, which means that

$$\mathrm{E}(Y^2(x)) < \infty, \qquad x \in \mathbb{R}^d.$$

This condition implies that the covariance $\mathrm{cov}_Y(x) = \mathrm{cov}(Y(o), Y(x))$ is well defined for any $x \in \mathbb{R}^d$. Notice that a sufficient condition for the existence of the second moment of Y can be provided in terms of the generating function $g_{N(\Xi\cap K)}(s)$ of the random variable $N(\Xi \cap K)$ defined in (2.3).

Lemma 2.1. *Let Ξ be a germ–grain model with $M_0 \in \mathcal{R}$ such that the minimal number of convex components of M_0 is bounded by some constant $n_0 < \infty$. Then*

$$\mathrm{E}(|Y^p(x)|) \leq c^p(K) g_{N(\Xi\cap K)}(2^{n_0 p})$$

for any $p > 0$ and $x \in \mathbb{R}^d$, where $c(K)$ is an upper bound for $|f(K')|$ for all $K' \in \mathcal{K}$ with $K' \subseteq K$.

Proof. We prove the assertion only for the special case $n_0 = 1$, i.e. we assume that $M_0 \in \mathcal{K}$. For any integer $m \geq 0$, we let $I_m(x) = \{N((\Xi - x) \cap K) = m\}$ and $p_{N(\Xi\cap K)}(m) = \mathrm{P}(I_m(o))$. Then, using the properties of valuations, we obtain

$$\begin{aligned}
\mathrm{E}|Y^p(x)| &= \sum_{m=1}^{\infty} \mathrm{E}\bigg(\bigg|f\bigg(\bigcup_{i=1}^{m}(M_i + X_i - x) \cap K\bigg)\bigg|^p \;\bigg|\; I_m(x)\bigg) p_{N(\Xi\cap K)}(m) \\
&= \sum_{m=1}^{\infty} \mathrm{E}\bigg(\bigg|\sum_{k=1}^{m}(-1)^{k-1} \sum_{i_1<\cdots<i_k} f((M_{i_1} + X_{i_1} - x) \cap \cdots \\
&\qquad\qquad \cap (M_{i_k} + X_{i_k} - x) \cap K)\bigg|^p \;\bigg|\; I_m(x)\bigg) p_{N(\Xi\cap K)}(m) \\
&\leq \sum_{m=0}^{\infty} \bigg(\sum_{k=0}^{m} \binom{m}{k} c(K)\bigg)^p p_{N(\Xi\cap K)}(m) \\
&= c^p(K) \sum_{m=0}^{\infty} 2^{mp} p_{N(\Xi\cap K)}(m) \\
&= c^p(K) g_{N(\Xi\cap K)}(2^p),
\end{aligned}$$

where the inequality is due to the conditional boundedness of f. The proof of the general case is similar and, therefore, omitted.

2.3. Unbiased and consistent estimation of the mean

Consider an unboundedly increasing sequence $\{W_n\}$ of bounded Borel sets $W_n \subset \mathbb{R}^d$ with

$$\lim_{n\to\infty} |W_n| = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} = 0 \quad \text{for any } r > 0. \tag{2.6}$$

Here, $B_r(x) = \{y \in \mathbb{R}^d : |y - x| \leq r\}$ is the closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r > 0$, and ∂B is the boundary of a Borel set B. Notice that (2.6) implies

$$\lim_{n\to\infty} \frac{|W_n \oplus B_r(o)|}{|W_n|} = \lim_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} = 1 \quad \text{for any } r > 0. \tag{2.7}$$

Thus, without loss of generality, we can assume that $|W_n \ominus \check{K}| > 0$ for each $n \geq 1$. Furthermore, let $G\colon \mathcal{B}_0(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty)$ be some nonnegative function Borel measurable in the second component such that, for each $n \geq 1$,

$$G(W_n, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus (W_n \ominus \check{K}) \qquad \text{and} \qquad \int_{W_n} G(W_n, x)\,\mathrm{d}x = 1. \tag{2.8}$$

Now assume that the RACS Ξ is stationary. It then follows from Fubini's theorem that

$$\hat{\mu}_n = \int_{W_n} Y(x) G(W_n, x)\,\mathrm{d}x$$

is an unbiased estimator for the expectation $\mu = \mathrm{E}(Y(o))$, where $Y(x)$ is given by (2.5). Moreover, the estimation variance $\mathrm{var}(\hat{\mu}_n)$ can be determined as follows.

Lemma 2.2. *For any $n \geq 1$,*

$$\operatorname{var}(\hat{\mu}_n) = \int_{\mathbb{R}^d} \operatorname{cov}_Y(x) R_{W_n}(x)\, \mathrm{d}x,$$

where $R_{W_n}(x) = \int_{\mathbb{R}^d} G(W_n, y) G(W_n, x + y)\, \mathrm{d}y$.

Proof. We have

$$\begin{aligned}
\operatorname{var}(\hat{\mu}_n) &= \mathrm{E}\bigg(\int_{W_n} (Y(u) - \mu) G(W_n, u)\, \mathrm{d}u \int_{W_n} (Y(v) - \mu) G(W_n, v)\, \mathrm{d}v\bigg) \\
&= \int_{W_n} \int_{W_n} \mathrm{E}((Y(o) - \mu)(Y(v - u) - \mu)) G(W_n, u) G(W_n, v)\, \mathrm{d}v\, \mathrm{d}u \\
&= \int_{W_n \oplus \check{W}_n} \operatorname{cov}_Y(x) \int_{W_n \cap (W_n - x)} G(W_n, y) G(W_n, x + y)\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \operatorname{cov}_Y(x) R_{W_n}(x)\, \mathrm{d}x,
\end{aligned}$$

where the last equality follows from the fact that $W_n \cap (W_n - x) = \varnothing$ for any $x \notin W_n \oplus \check{W}_n$.

To determine the asymptotic behavior of the estimation variance $\operatorname{var}(\hat{\mu}_n)$, we need some further conditions on the weighting function $G\colon \mathcal{B}_0(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty)$. Besides (2.8), we additionally assume that there exist constants $c_1, c_2 < \infty$ such that

$$\begin{aligned}
\sup_{y \in W_n} G(W_n, y) &\leq \frac{c_1}{|W_n|} && \text{for any } n \geq 1, \\
\lim_{n \to \infty} |W_n| R_{W_n}(x) &= c_2 && \text{for any } x \in \mathbb{R}^d.
\end{aligned} \tag{2.9}$$

Notice that (2.8) and (2.9) hold, for example, if $G(W_n, x) = \mathbf{1}(x \in W_n \ominus \check{K}) / |W_n \ominus \check{K}|$ for any $n \geq 1$ and $x \in \mathbb{R}^d$, where $\mathbf{1}(B)$ denotes the indicator function of event B. In this case, from (2.7) we have $c_1 = 2$ and $c_2 = 1$. Furthermore, we assume that the covariance $\operatorname{cov}_Y(x)$ of the stationary random field Y is integrable, i.e. that

$$\int_{\mathbb{R}^d} |\operatorname{cov}_Y(x)|\, \mathrm{d}x < \infty. \tag{2.10}$$

Lemma 2.3. *Let the conditions (2.6), (2.8), (2.9), and (2.10) be fulfilled. Then*

$$\lim_{n \to \infty} |W_n| \operatorname{var}(\hat{\mu}_n) = c_2 \int_{\mathbb{R}^d} \operatorname{cov}_Y(x)\, \mathrm{d}x.$$

Proof. Conditions (2.8) and (2.9) immediately imply that $|W_n| R_{W_n}(x) \leq c_1$ holds for any $x \in \mathbb{R}^d$ and $n \geq 1$. Thus, using Lemma 2.2 and condition (2.10), the assertion follows from the Lebesgue dominated convergence theorem.

Since $\lim_{n \to \infty} |W_n| = \infty$, Lemma 2.3 implies, in particular, that $\lim_{n \to \infty} \operatorname{var}(\hat{\mu}_n) = 0$, i.e. that the unbiased estimator $\hat{\mu}_n$ is also mean-square consistent for μ.

3. Examples

In this section, we briefly discuss some examples of stationary random fields that belong to the general class of random fields $Y = \{Y(x), x \in \mathbb{R}^d\}$ introduced in (2.5). They can be used to construct unbiased and mean-square consistent estimators for various morphological characteristics of stationary RACS. In the sequel, we assume that $\{W_n\}$ is an arbitrary sequence of bounded Borel sets that satisfies (2.6), with $|W_n| > 0$ for any $n \geq 1$.

3.1. Volume fraction

Let Ξ be a stationary RACS in $\mathbb{R}^d$ with *volume fraction* $p = \mathrm{P}(o \in \Xi)$, and let $Z_d = \{Z_d(x), x \in \mathbb{R}^d\}$ be the random field given by $Z_d(x) = \mathbf{1}(x \in \Xi)$. Then

$$\hat{p}_n = \frac{1}{|W_n|} \int_{W_n} Z_d(x)\,\mathrm{d}x \tag{3.1}$$

is an unbiased estimator for p. Since $\mathbf{1}(x \in \Xi) = \mathbf{1}((\Xi - x) \cap \{o\} \neq \varnothing)$ for any $x \in \mathbb{R}^d$, it is easy to see that Z_d is of the form considered in (2.5) with $K = \{o\}$ and the (bounded) valuation $f : \mathcal{R} \to \mathbb{R}$ given by $f(K') = \mathbf{1}(K' \neq \varnothing)$. Clearly, the random field Z_d is of second order. If $\Xi = \bigcup_{i=1}^{\infty}(M_i + X_i)$ is a Boolean model with $\mathrm{E}(|M_0|^2) < \infty$, then it is well known that the covariance $\mathrm{cov}_{Z_d}(x)$ of Z_d is integrable; see, e.g. the remarks following Corollary 4.2 of [1]. According to Lemma 2.3, the unbiased estimator $\hat{p}_n$ is mean-square consistent for p.

3.2. Specific intrinsic volumes

Let Ξ be a stationary RACS such that $\Xi \in \mathcal{S}$ holds with probability 1. Then, for each $i = 0, \dots, d$, the intrinsic volume $V_i(\Xi \cap K)$ of $\Xi \cap K$ is well defined for any convex body $K \in \mathcal{K}$, where, for instance, $V_d(\Xi \cap K) = |\Xi \cap K|$ is the usual volume, $V_0(\Xi \cap K)$ is the *Euler number* of the set $\Xi \cap K$, which is defined by the inclusion-exclusion formula (2.4), and $V_0(M) = \mathbf{1}(M \neq \varnothing)$, $M \in \mathcal{K}$; see also [12].

Assume that $\mathrm{E}(2^{\tilde{N}(\Xi \cap [0,1]^d)}) < \infty$, where $\tilde{N}(B)$ denotes the minimal number of convex components of the polyconvex set $B \in \mathcal{R}$. Then, for any sequence $\{K_n\}$ of convex bodies $K_n = nK_0$, with $K_0 \in \mathcal{K}$ such that $|K_0| > 0$ and $o \in \mathrm{int}(K_0)$, the limits

$$\overline{V}_i(\Xi) = \lim_{n \to \infty} \frac{\mathrm{E}(V_i(\Xi \cap K_n))}{|K_n|}, \qquad i = 0, \dots, d,$$

exist and are called the *specific intrinsic volumes* of Ξ; see, e.g. [13, Section 5.1, pp. 185–187]. Estimators of several types for some specific intrinsic volumes have been considered in the literature. Two indirect estimation methods were proposed in [11] and [14], respectively. These have the advantage that *joint* estimators can be constructed for the vector $(\overline{V}_0(\Xi), \dots, \overline{V}_d(\Xi))$ of all $d + 1$ specific intrinsic volumes.

The construction principle considered in [11] (see Section 2.3 thereof) is based on Steiner's formula and makes use of the index of polyconvex sets. The random field used therein is defined as follows. For $i = 0, \dots, d - 1$, let $r_i > 0$ be any positive number and let the random field $Z_i = \{Z_i(x), x \in \mathbb{R}^d\}$ be given by

$$Z_i(x) = \sum_{q \in \partial((\Xi - x) \cap B_{r_i}(o)),\, q \neq 0} J((\Xi - x) \cap B_{r_i}(o),\ q,\ o), \tag{3.2}$$

where the functional

$$J(K, q, x) = \mathbf{1}(q \in K)\Big(1 - \lim_{\delta \to +0} \lim_{\varepsilon \to +0} V_0(K \cap B_{|x-q|-\varepsilon}(x) \cap B_\delta(q))\Big)$$

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is called the *index* of $K \in \mathcal{R}$ at $x \in \mathbb{R}^d$. It is not difficult to see that Z_i is of the form considered in (2.5) with $K = B_{r_i}(o)$ and the valuation $f \colon \mathcal{R} \to \mathbb{R}$ given by

$$f(K') = \sum_{q \in \partial K',\, q \neq 0} J(K', q, o).$$

Here, the functional f is bounded on $\mathcal{K}$, with $f(K') = \mathbf{1}(o \notin K', K' \neq \emptyset)$ for any $K' \in \mathcal{K}$. If the covariance $\mathrm{cov}_{Z_i}(x)$ of Z_i is integrable for any $i = 0, \ldots, d$, it can be concluded from Lemma 2.3 that

$$\hat{\mu}_{n,i} = |W_n \ominus B_{r_i}(o)|^{-1} \int_{W_n} Z_i(x)\, \mathbf{1}(x \in W_n \ominus B_{r_i}(o))\, \mathrm{d}x$$

is an unbiased estimator for $\mu_i = \mathrm{E}(Z_i(o))$ that is mean-square consistent provided that $\{W_n\}$ satisfies (2.6). Then, assuming that $r_i \neq r_{i'}$ for any $i \neq i'$, the random vector $\hat{v}_n = A^{-1}_{r_0,\ldots,r_{d-1}}(\hat{\mu}_{n,0}, \ldots, \hat{\mu}_{n,d-1}, \hat{p}_n)^\top$, where $\hat{p}_n$ is the empirical volume fraction introduced in (3.1) and $A_{r_0,\ldots,r_{d-1}}$ is a regular matrix of Vandermond type (see [11]), provides an unbiased, mean-square consistent estimator for $(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))$.

The estimator proposed in [14] employs the principal kinematic formula. Here, the construction principle is as follows. For $i = 0, \ldots, d$, introduce the random fields $\tilde{Z}_i = \{\tilde{Z}_i(x), x \in \mathbb{R}^d\}$ with $\tilde{Z}_i(x) = V_0((\Xi - x) \cap B_{r_i}(o))$. Each $\tilde{Z}_i$ is of the form (2.5) with $f(K') = V_0(K')$, $K' \in \mathcal{R}$, and $K = B_{r_i}(0)$, where f is bounded on $\mathcal{K}$ with $f(K') = \mathbf{1}(K' \neq \emptyset)$ for all $K' \in \mathcal{K}$. For any $d+1$ pairwise-different positive radii $r_0, \ldots, r_d$, define

$$\tilde{\mu}_{n,i} = \int_{W_n \ominus B_{r_i}(o)} \frac{\tilde{Z}_i(x)}{|W_n \ominus B_{r_i}(o)|}\, \mathrm{d}x$$

and

$$\tilde{A}_{r_0,\ldots,r_d} = \begin{pmatrix} r_0^d \kappa_d & r_0^{d-1}\kappa_{d-1} & \cdots & r_0^2 \kappa_2 & r_0 \kappa_1 & 1 \\ r_1^d \kappa_d & r_1^{d-1}\kappa_{d-1} & \cdots & r_1^2 \kappa_2 & r_1 \kappa_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ r_d^d \kappa_d & r_d^{d-1}\kappa_{d-1} & \cdots & r_d^2 \kappa_2 & r_d \kappa_1 & 1 \end{pmatrix},$$

where κ_i denotes the volume of the unit ball in $\mathbb{R}^i$, $i = 1, \ldots, d$. Then the random vector $\tilde{v}_n = \tilde{A}^{-1}_{r_0,\ldots,r_d}(\tilde{\mu}_{n,0}, \ldots, \tilde{\mu}_{n,d})^\top$ is an unbiased, mean-square consistent estimator for $(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))$.

4. Asymptotic normality for functionals of Boolean models

Let $\Xi = \bigcup_{i=1}^{\infty}(M_i + X_i)$ be a Boolean model with compact and convex typical grain $M_0 \in \mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$. The aim of this section is to prove asymptotic normality with respect to $\hat{\mu}_n = \int_{W_n} Y(x) G(W_n, x)\, \mathrm{d}x$, which is an estimator for the mean value, μ, of the random field Y introduced in (2.5). We thus assume that the conditions (2.6), (2.8), (2.9), and (2.10) are fulfilled. More precisely, by replacing (2.10) by a moment condition on M_0, we show that

$$\sqrt{|W_n|}(\hat{\mu}_n - \mu) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2), \qquad n \to \infty, \tag{4.1}$$

where '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution and $\mathcal{N}(0, \sigma^2)$ is a Gaussian random variable with mean 0 and variance $\sigma^2 = c_2 \int_{\mathbb{R}^d} \mathrm{cov}_Y(x)\, \mathrm{d}x$.

We concentrate on the case of the Boolean model for two reasons. First, the integrability of the covariance $\mathrm{cov}_Y(x)$ of random field Y is generally quite tractable in this case; see Lemma 4.1. Second, we can make use of a central limit theorem for m-dependent random fields, from [4], without imposing further conditions. A corresponding central limit theorem for more general germ–grain models is considered in Section 5.

4.1. Integrability of the covariance

The following lemma yields a simple condition sufficient for absolute integrability of the covariance $\mathrm{cov}_Y(x)$, $x \in \mathbb{R}^d$.

Lemma 4.1. *Assume that* $\mathrm{E}(|M_0 \oplus \check{K}|^2) < \infty$. *Then (2.10) holds.*

Proof. For better readability, we use the representation of Ξ as the set-theoretic union of the generating Poisson particle process $\tilde{M} = \{\tilde{M}_i\}$ in $\mathcal{K}'$ with $\tilde{M}_i = M_i + X_i$, and let Λ denote the intensity measure of $\tilde{M}$. By Campbell's theorem for independently marked point processes on $\mathbb{R}^d$ (see, e.g. Section 3 of [13, pp. 66, 93]), the following representation for Λ holds for any set $B \subseteq \mathcal{K}$ with $B \in \sigma_{\mathcal{F}}$, where $\mathbb{Q}$ denotes the distribution of the typical grain M_0:

$$\Lambda(B) = \lambda \int_{\mathcal{K}} \int_{\mathbb{R}^d} \mathbf{1}((M_0 + y) \in B)\, \mathrm{d}y\, \mathrm{d}\mathbb{Q}(M_0). \tag{4.2}$$

Now, let $\mathcal{K}_x^* = \mathcal{K}_K \cap \mathcal{K}_{K+x}$, with $\mathcal{K}_K = \{K' \in \mathcal{K} \colon K' \cap K \neq \varnothing\}$ for any set $K \in \mathcal{K}'$, and let $B \triangle B' = (B \cup B') \setminus (B \cap B')$ be the symmetric difference between any two sets B and B'. Considering the event $A = \{\tilde{M}(\mathcal{K}_x^*) > 0\}$ and its complement A^{c}, where $\tilde{M}(B)$ is the number of particles of $\tilde{M}$ in a set $B \subseteq \mathcal{K}$, we can write

$$\mathrm{cov}_Y(x) = \mathrm{E}(Y(o)(Y(x) - \mu)) = \mathrm{E}(Y(o)\, \mathbf{1}(A)(Y(x) - \mu)) + \mathrm{E}(Y(o)\, \mathbf{1}(A^{\mathrm{c}})(Y(x) - \mu)).$$

Using arguments similar to those in the proof of Lemma 2.1, upper bounds for the absolute values of the summands in the above decomposition of $\mathrm{cov}_Y(x)$ can be deduced in the following way. We have

$$\begin{aligned}
|\mathrm{E}(Y(o)\, \mathbf{1}(A)(Y(x) - \mu))| &\le c^2(K)\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K) + \tilde{M}(\mathcal{K}_{K+x})}\, \mathbf{1}(A)) + c(K)|\mu|\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K)}\, \mathbf{1}(A)) \\
&= c^2(K)\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K \triangle \mathcal{K}_{K+x})})\, \mathrm{E}(2^{2\tilde{M}(\mathcal{K}_x^*)}\, \mathbf{1}(A)) \\
&\quad + c(K)|\mu|\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K \setminus \mathcal{K}_x^*)})\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_x^*)}\, \mathbf{1}(A)) \\
&\le 2c^2(K)\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K)})^2\, \mathrm{E}(4^{\tilde{M}(\mathcal{K}_x^*)}\, \mathbf{1}(A)),
\end{aligned}$$

since the random variables $2^{\tilde{M}(\mathcal{K}_K \triangle \mathcal{K}_{K+x})}$ and $2^{\tilde{M}(\mathcal{K}_K \setminus \mathcal{K}_x^*)}$ are independent of $4^{\tilde{M}(\mathcal{K}_x^*)}\, \mathbf{1}(A)$, where we have employed Lemma 2.1 and the stationarity of $\tilde{M}$. Now consider the second summand of the representation of $\mathrm{cov}(x)$, and define $Y_B(x) = f(\bigcup_{\{i \colon \tilde{M}_i \in B\}} (\tilde{M}_i - x) \cap K)$ for any $B \subseteq \mathcal{K}$. From the properties of the valuation f, it follows that $Y(o)\, \mathbf{1}(A^{\mathrm{c}}) = Y_{\mathcal{K}_K \setminus \mathcal{K}_x^*}(o)\, \mathbf{1}(A^{\mathrm{c}})$ and $Y(x)\, \mathbf{1}(A^{\mathrm{c}}) = Y_{\mathcal{K}_{K+x} \setminus \mathcal{K}_x^*}(x)\, \mathbf{1}(A^{\mathrm{c}})$, where $Y_{\mathcal{K}_K \setminus \mathcal{K}_x^*}(o)$, $Y_{\mathcal{K}_{K+x} \setminus \mathcal{K}_x^*}(x)$, and $\mathbf{1}(A^{\mathrm{c}})$ are mutually independent. Hence, we have

$$\begin{aligned}
|\mathrm{E}(Y(o)\, \mathbf{1}(A^{\mathrm{c}})(Y(x) - \mu))| &= |\mathrm{E}(Y_{\mathcal{K}_K \setminus \mathcal{K}_x^*}(o)\, \mathbf{1}(A^{\mathrm{c}})(Y_{\mathcal{K}_{K+x} \setminus \mathcal{K}_x^*}(x) - \mu))| \\
&= |\mathrm{E}(Y(o)\, \mathbf{1}(A^{\mathrm{c}}))|\, |\mathrm{E}(Y_{\mathcal{K}_{K+x} \setminus \mathcal{K}_x^*}(x) - \mu)| \\
&= |\mathrm{E}(Y(o)\, \mathbf{1}(A^{\mathrm{c}}))|\, |\mathrm{E}((Y_{\mathcal{K}_{K+x} \setminus \mathcal{K}_x^*}(x) - Y(x))\, \mathbf{1}(A))| \\
&\le c(K)\, \mathrm{E}(|Y(o)|)\, \mathrm{E}(2^{\tilde{M}(\mathcal{K}_K)})[\mathrm{E}(\mathbf{1}(A)) + \mathrm{E}(2^{\tilde{M}(\mathcal{K}_x^*)}\, \mathbf{1}(A))],
\end{aligned}$$

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where the inequality follows as before. Notice that $\mathrm{E}(s^{\tilde{M}(\mathcal{K}_K)}) = \mathrm{e}^{(s-1)\Lambda(\mathcal{K}_K)} < \infty$ for any $s \in \mathbb{R}$, since $\Lambda(\mathcal{K}_K) = \lambda\,\mathrm{E}(|M_0 \oplus \check{K}|) < \infty$ by condition (2.2) and (4.2). Thus, it suffices to show that $\mathrm{E}(4^{\tilde{M}(\mathcal{K}_x^*)}\,\mathbf{1}(A))$ is integrable with respect to $x \in \mathbb{R}^d$. Observe that, since $\tilde{M}$ is Poisson,

$$\mathrm{E}(4^{\tilde{M}(\mathcal{K}_x^*)}\,\mathbf{1}(A)) = \mathrm{E}(4^{\tilde{M}(\mathcal{K}_x^*)}) - \mathrm{E}(\mathbf{1}(A^{\mathrm{c}})) = \mathrm{e}^{3\Lambda(\mathcal{K}_x^*)}(1-\mathrm{e}^{-4\Lambda(\mathcal{K}_x^*)}) \le 4\mathrm{e}^{3\lambda \mathrm{E}(|M_0 \oplus \check{K}|)}\Lambda(\mathcal{K}_x^*),$$

where we have used the estimate $1 - \mathrm{e}^{-s} \le s$ for any $s \ge 0$ to obtain the latter inequality. By virtue of Campbell's formula and Fubini's formula, we can finally conclude that

$$\begin{aligned}
\int_{\mathbb{R}^d} \Lambda(\mathcal{K}_x^*)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \mathrm{E}\bigg(\sum_{i=1}^{\infty} \mathbf{1}((M_i + X_i) \cap K \neq \varnothing, (M_i + X_i) \cap (K + x) \neq \varnothing)\bigg)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \lambda\,\mathrm{E}\bigg(\int_{\mathbb{R}^d} \mathbf{1}((M_0 + y) \cap K \neq \varnothing, (M_0 + y) \cap (K + x) \neq \varnothing)\,\mathrm{d}y\bigg)\,\mathrm{d}x \\
&= \lambda\,\mathrm{E}\bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}(y \in (\check{M}_0 \oplus K))\,\mathbf{1}((y - x) \in (\check{M}_0 \oplus K))\,\mathrm{d}y\,\mathrm{d}x\bigg) \\
&= \lambda\,\mathrm{E}(|\check{M}_0 \oplus K|^2) = \lambda\,\mathrm{E}(|M_0 \oplus \check{K}|^2) < \infty.
\end{aligned}$$

Note that the proof of Lemma 4.1 provides an integrable upper bound $h(x) \equiv h(x, M_0)$ for $|\mathrm{cov}_Y(x)|$ that depends on the distribution of the typical grain M_0. This dependence is monotone with respect to set inclusion. That is, if $M_0^{(1)} \subseteq M_0^{(2)}$ then with probability 1 $h(x, M_0^{(1)}) \le h(x, M_0^{(2)})$ for any $x \in \mathbb{R}^d$.

4.2. Truncated germ–grain models

Let the conditions (2.6), (2.8), and (2.9) be fulfilled and assume that $\mathrm{E}(|M_0 \oplus \check{K}|^2) < \infty$. To prove the central limit theorem (4.1), we approximate the random field Y corresponding to Ξ by random fields Y_n that are induced by germ–grain models Ξ_n with truncated grains. These are chosen in the following way. For any $n \ge 1$, let $A_n = [-a_n, a_n]^d$ for some $a_n > 0$ such that $\lim_{n\to\infty} a_n = \infty$. Introduce the auxiliary germ–grain model Ξ_n defined by

$$\Xi_n = \bigcup_{i=1}^{\infty} (M_{n,i} + X_i),$$

where $M_{n,i} = M_i \cap A_n \in \mathcal{K}$ for any $i, n \in \mathbb{N}$. Accordingly, define the random field $Y_n = \{Y_n(x),\ x \in \mathbb{R}^d\}$ by $Y_n(x) = f((\Xi_n - x) \cap K)$, and let $\mu_n = \mathrm{E}(Y_n(o))$ and $\hat{\mu}_n' = \int_{W_n} Y_n(x) G(W_n, x)\,\mathrm{d}x$.

Lemma 4.2. *The random fields Y and Y_n are of second order. Moreover, $Y_n(x)$ converges in mean square to $Y(x)$ as $n \to \infty$, i.e. $\lim_{n\to\infty} \mathrm{E}(|Y(x) - Y_n(x)|^2) = 0$ for all $x \in \mathbb{R}^d$.*

Proof. Due to stationarity, we can assume that $x = o$. Since $g_{N(\Xi_n \cap K)}(4) \le g_{N(\Xi \cap K)}(4) < \infty$, the random fields Y and Y_n are both of second order, by Lemma 2.1. To prove the second assertion, we let

$$N_{|A_n}(\Xi \cap K) = \mathrm{card}\{i : (M_i + X_i) \cap K \neq \varnothing, K \not\subseteq (A_n + X_i)\}.$$

Then, using arguments similar to those in the proof of Lemma 2.1, we have

$$\mathrm{E}(|Y(o) - Y_n(o)|^2) \le 4c^2(K)\,\mathrm{E}(2^{2N(\Xi \cap K)}\,\mathbf{1}(N_{|A_n}(\Xi \cap K) > 0)).$$

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Since $\mathrm{E}(2^{2N(\Xi\cap K)}) = \mathrm{e}^{3\lambda\,\mathrm{E}(|M_0\oplus \check{K}|)} < \infty$, it now suffices to show that

$$\lim_{n\to\infty} \mathrm{P}(N_{|A_n}(\Xi \cap K) > 0) = 0.$$

Campbell's formula and Fubini's theorem yield

$$\begin{aligned}\mathrm{P}(N_{|A_n}(\Xi \cap K) > 0) &\le \mathrm{E}(N_{|A_n}(\Xi \cap K)) \\ &= \lambda\,\mathrm{E}\bigg(\int_{\mathbb{R}^d} \mathbf{1}((M_0 + y) \cap K \neq \varnothing, K \not\subseteq (A_n + y))\,\mathrm{d}y\bigg) \\ &\le \lambda\,\mathrm{E}\int_{\mathbb{R}^d} \mathbf{1}(y \in (M_0 \oplus \check{K}))\,\mathbf{1}(y \notin (A_n \ominus \check{K}))\,\mathrm{d}y \\ &\le \lambda\,\mathrm{E}\int_{\mathbb{R}^d} \mathbf{1}(y \in (M_0 \oplus \check{K}))\,\mathbf{1}(|y| > a_n - D(K))\,\mathrm{d}y\end{aligned}$$

for any n large enough that $a_n - D(K) > 0$. By the dominated convergence theorem, the final expression on the right-hand side converges to 0 as $n \to \infty$, since $\lim_{n\to\infty} a_n = \infty$ and $\mathrm{E}(|M_0 \oplus \check{K}|) < \infty$.

Next, we show that the asymptotic variance of the estimator $\hat{\mu}'_n = \int_{W_n} Y_n(x)G(W_n, x)\,\mathrm{d}x$ of μ_n is equal to the asymptotic variance of the estimator $\hat{\mu}_n = \int_{W_n} Y(x)G(W_n, x)\,\mathrm{d}x$ of the mean of Y.

Lemma 4.3. *The covariance* $\mathrm{cov}_{Y_n}(x)$ *of the stationary random field* Y_n *is integrable and*

$$\lim_{n\to\infty} |W_n|\,\mathrm{var}(\hat{\mu}'_n) = c_2 \int_{\mathbb{R}^d} \mathrm{cov}_Y(x)\,\mathrm{d}x, \tag{4.3}$$

where the constant $c_2 > 0$ *is as defined in (2.9).*

Proof. The integrability of $\mathrm{cov}_{Y_n}(x)$ immediately follows from Lemma 4.1. By Lemma 2.2, we have $|W_n|\,\mathrm{var}(\hat{\mu}'_n) = \int_{\mathbb{R}^d} \mathrm{cov}_{Y_n}(x)|W_n|R_{W_n}(x)\,\mathrm{d}x$, where $\lim_{n\to\infty} |W_n|R_{W_n}(x) = c_2$ by (2.9). As mentioned above, there exists an integrable function $h\colon \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}_+$ such that $|\mathrm{cov}_{Y_n}(x)| \le h(x, M_0 \cap A_n) \le h(x, M_0)$ for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Together with (2.9), this implies that $|\mathrm{cov}_{Y_n}(x)|\,|W_n|R_{W_n}(x) \le c_1 h(x, M_0)$ for all $x \in \mathbb{R}^d$. Furthermore, we have $\lim_{n\to\infty} \mathrm{cov}_{Y_n}(x) = \mathrm{cov}_Y(x)$, since

$$\begin{aligned}|\mathrm{cov}_{Y_n}(x) - \mathrm{cov}_Y(x)| &\le |\mathrm{E}(Y_n(o)Y_n(x)) - \mathrm{E}(Y(o)Y(x))| + |\mu_n^2 - \mu^2| \\ &= |\mathrm{E}((Y_n(o) - Y(o))Y_n(x)) + \mathrm{E}((Y_n(x) - Y(x))Y(o))| + |\mu_n^2 - \mu^2| \\ &\le \mathrm{E}((Y_n(o) - Y(o))^2)^{1/2}[\mathrm{E}(Y_n^2(o))^{1/2} + \mathrm{E}(Y^2(o))^{1/2}] + |\mu_n^2 - \mu^2|,\end{aligned}$$

with $\lim_{n\to\infty} \mathrm{E}((Y_n(o)-Y(o))^2) = 0$ and $\lim_{n\to\infty} |\mu_n^2-\mu^2| = 0$ by Lemma 4.2. Furthermore, $\mathrm{E}(Y_n^2(o))$ and $\mathrm{E}(Y^2(o))$ are uniformly bounded in n. Consequently, the limit in (4.3) follows from the dominated convergence theorem.

Let $\|z\| = \max\{|z_i|, i = 1, \dots, d\}$ for any $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, and let $m > 0$ be an arbitrary integer. A family of random variables $\{Z_z,\ z \in \mathbb{Z}^d\}$ is called an *m-dependent random field* if $(Z_z)_{z\in U}$ and $(Z_z)_{z\in U'}$ are independent random vectors for any finite sets $U, U' \subset \mathbb{Z}^d$ with $\inf\{\|z - z'\|,\ z \in U,\ z' \in U'\} > m$. Any stationary random field indexed over $\mathbb{Z}^d$ that satisfies an appropriate β-mixing condition is m-dependent; see, e.g. Section 1.3.1 of [2, p. 17].

The following central limit theorem for $\hat{\mu}'_n$ is closely related to the Lindeberg-type central limit theorem for m-dependent random fields presented in Theorem 2 of [4].

Lemma 4.4. *Let $\{m_n, n \ge 1\}$ be an arbitrary sequence of positive integers such that $m_n \to \infty$, and let $\{U_n, n \ge 1\}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $\lim_{n\to\infty} \operatorname{card}(U_n) = \infty$. For each $n \ge 1$, let $\{Z_{n,z}, z \in \mathbb{Z}^d\}$ be an m_n-dependent random field with $\mathrm{E}(Z_{n,z}) = 0$ for any $z \in U_n$ and $\mathrm{E}(S_n^*)^2 = \sigma_n^2 \to \sigma^2$ as $n \to \infty$, where $S_n^* = \sum_{z\in U_n} Z_{n,z}$. If there exists a constant $c > 0$ such that*

$$\sum_{z\in U_n} \mathrm{E}(Z_{n,z}^2) \le c \tag{4.4}$$

for any $n \ge 1$, and if

$$\lim_{n\to\infty} m_n^{2d} \sum_{z\in U_n} \mathrm{E}(Z_{n,z}^2 \mathbf{1}(|Z_{nz}| \ge \varepsilon m_n^{-2d})) = 0 \tag{4.5}$$

for any $\varepsilon > 0$, then $S_n^* \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.

The proof of Lemma 4.4 for $\sigma_n^2 = 1$ can be found in, e.g. Section 3 of [4] and extended easily to the case in which $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 < \infty$.

Lemma 4.5. *If the truncation sequence $\{a_n\}$ satisfies*

$$\lim_{n\to\infty} a_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{a_n^{4d(1+\delta)/\delta}}{|W_n|} = 0 \tag{4.6}$$

for some $\delta > 0$, then the random variables $S_n' = \sqrt{|W_n|}(\hat{\mu}_n' - \mu_n)$ are asymptotically normally distributed, i.e.

$$S_n' \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2) \quad \textit{as } n \to \infty,$$

where $\sigma^2 = c_2 \int_{\mathbb{R}^d} \operatorname{cov}_Y(x)\,\mathrm{d}x$ with c_2 as defined in (2.9).

Proof. Let $[z, z+e) = [z_1, z_1+1) \times \cdots \times [z_d, z_d+1)$ for $z \in \mathbb{Z}^d$, and consider the sets $U_n = \{z \in \mathbb{Z}^d : [z, z+e) \subseteq W_n\}$ and $W_n^- = \bigcup_{z\in U_n} [z, z+e)$. Furthermore, decompose S_n' into $S_n' = S_n^* + \tilde{S}_n$, with

$$S_n^* = \sqrt{|W_n|} \int_{W_n^-} (Y_n(x) - \mu_n) G(W_n, x)\,\mathrm{d}x,$$
$$\tilde{S}_n = \sqrt{|W_n|} \int_{W_n \setminus W_n^-} (Y_n(x) - \mu_n) G(W_n, x)\,\mathrm{d}x.$$

Condition (2.9) and the integrability of $\operatorname{cov}_{Y_n}(x)$ imply that

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{E}(\tilde{S}_n^2) \\
&\quad = \lim_{n\to\infty} \int_{W_n\setminus W_n^-} \int_{W_n\setminus W_n^-} |W_n|\,\mathrm{E}((Y_n(x) - \mu_n)(Y_n(y) - \mu_n)) G(W_n, x) G(W_n, y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \lim_{n\to\infty} \int_{\mathbb{R}^d} \operatorname{cov}_{Y_n}(x) |W_n| \bigg(\int_{(W_n\setminus W_n^-)\cap((W_n\setminus W_n^-)-x)} G(W_n, y)\, G(W_n, x+y)\,\mathrm{d}y \bigg)\,\mathrm{d}x \\
&\quad \le \lim_{n\to\infty} \frac{c_1^2 |W_n \setminus W_n^-|}{|W_n|} \int_{\mathbb{R}^d} h(x)\,\mathrm{d}x \\
&\quad \le c \lim_{n\to\infty} \frac{|\partial W_n \oplus B_{2\sqrt{d}}(o)|}{|W_n|} = 0
\end{aligned}$$

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for some constant $c > 0$, where the last equality follows from (2.6). Hence, the second component in the decomposition of S_n' converges to 0 in mean square. Using Slutsky's theorem, it is sufficient to show that

$$S_n^* \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty.$$

Hence, we apply Lemma 4.4 to $S_n^* = \sum_{z \in U_n} Z_{n,z}$, where

$$Z_{n,z} = \begin{cases} \sqrt{|W_n|} \displaystyle\int_{[z,z+e)} (Y_n(x) - \mu_n) G(W_n, x)\, \mathrm{d}x, & z \in U_n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

It is not difficult to see that the family of random variables $\{Z_{n,z}, z \in \mathbb{Z}^d\}$ given in (4.7) forms an m_n-dependent random field for any $m_n \geq 2(a_n + D(K))$. By the definition of $Z_{n,z}$, we have $\mathrm{E}(Z_{n,z}) = 0$ for any $z \in U_n$. Furthermore, we have

$$\lim_{n \to \infty} \mathrm{E}((S_n^*)^2) = \sigma^2$$

by Lemma 4.3 and since, as shown above, $\mathrm{E}(\tilde{S}_n^2) \to 0$. In order to complete the proof, it remains to show that conditions (4.4) and (4.5) are fulfilled. By using Fubini's theorem and (2.9), we obtain

$$\begin{aligned} \sum_{z \in U_n} \mathrm{E}(Z_{n,z}^2) &\leq \sum_{z \in U_n} \frac{c_1^2}{|W_n|} \int_{[z,z+e)} \int_{[z,z+e)} |\mathrm{cov}_{Y_n}(x-y)| \, \mathrm{d}x \, \mathrm{d}y \\ &\leq \sum_{z \in U_n} \frac{c_1^2 \, \mathrm{var}(Y_n(o))}{|W_n|} \\ &= c_1^2 \, \mathrm{var}(Y_n(o)) \frac{|W_n^-|}{|W_n|} \\ &\leq c, \end{aligned}$$

for all sufficiently large n and some constant $c < \infty$, where the uniform bound provided by the final inequality follows from the facts that $\mathrm{var}(Y_n(o)) \leq h(o) < \infty$ and $|W_n^-| / |W_n| \leq 1$ for any $n \geq 1$. Thus, (4.4) holds. Because of the stationarity of Ξ_n, the random variables

$$\tilde{Z}_{n,z} = \int_{[z,z+e)} \frac{|Y_n(x) - \mu_n|}{\sqrt{|W_n|}} \, \mathrm{d}x, \qquad z \in U_n,$$

are identically distributed. By the inequality in (2.9), we also have $|Z_{n,z}| \leq c_1 \tilde{Z}_{n,z}$. This yields the following estimates for any $\delta > 0$, where $\hat{Z}_{n,o} = \sqrt{|W_n|} \tilde{Z}_{n,o}$:

$$\begin{aligned} m_n^{2d} \sum_{z \in U_n} \mathrm{E}(Z_{n,z}^2 \mathbf{1}(m_n^{2d} |Z_{n,z}| \geq \varepsilon)) &\leq m_n^{2d} |W_n^-| c_1^2 \, \mathrm{E}\bigg(\tilde{Z}_{n,o}^2 \mathbf{1}\bigg(\tilde{Z}_{n,o}^\delta \geq \frac{\varepsilon^\delta}{c_1^\delta m_n^{2d\delta}} \bigg) \bigg) \\ &\leq \frac{c_1^{2+\delta}}{\varepsilon^\delta} m_n^{2d(1+\delta)} |W_n^-| \, \mathrm{E}(\tilde{Z}_{n,o}^{2+\delta}) \\ &\leq \frac{c_1^{2+\delta}}{\varepsilon^\delta} \bigg(\frac{m_n^{4d(1+\delta)/\delta}}{|W_n|} \bigg)^{\delta/2} \frac{|W_n^-|}{|W_n|} \, \mathrm{E}(\hat{Z}_{n,o}^{2+\delta}). \end{aligned} \tag{4.8}$$

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Since $m_n \geq 2(a_n + D(K))$, the second factor of the latter expression converges to 0 as $n \to \infty$ if (4.6) holds for some $\delta > 0$. The remaining factors of (4.8) are uniformly bounded in n, because $|W_n^-| / |W_n| \leq 1$ and

$$\begin{aligned}
\mathrm{E}(\hat{Z}_{no}^{2+\delta})^{1/(2+\delta)} &\leq \mathrm{E}\bigg(\bigg(\int_{[o,e)} (|Y_n(x)| + |\mu_n|)\,\mathrm{d}x\bigg)^{2+\delta}\bigg)^{1/(2+\delta)} \\
&\leq \mathrm{E}\bigg(\int_{[o,e)} |Y_n(x)|^{2+\delta}\,\mathrm{d}x\bigg)^{1/(2+\delta)} + |\mu_n| \\
&\leq c(K) g_{N(\Xi\cap K)}(2^{2+\delta})^{1/(2+\delta)} + c(K) g_{N(\Xi\cap K)}(2) \\
&< \infty
\end{aligned}$$

for any $n \geq 1$. Thus, condition (4.5) of Lemma 4.4 is fulfilled.

Notice that condition (4.6) of Lemma 4.5 is satisfied, for example, if the truncation sequence $\{a_n\}$ is given by $a_n = r(W_n)^\eta$, where $\eta < \delta/(4(1+\delta))$ and $r(W_n)$ denotes the radius of the largest disc that can be inscribed in W_n.

4.3. Asymptotic normality of mean-value estimators

The asymptotic normality proven in Lemma 4.5 for the mean-value estimator associated with the truncated germ–grain model implies an equivalent statement for the original functional.

Theorem 4.1. *Let conditions (2.6), (2.8), and (2.9) be fulfilled and assume that*

$$\mathrm{E}(|M_0 \oplus \check{K}|^2) < \infty.$$

Then

$$\sqrt{|W_n|}(\hat{\mu}_n - \mu) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty,$$

for $\sigma^2 = c_2 \int_{\mathbb{R}^d} \mathrm{cov}_Y(x)\,\mathrm{d}x$ *with* c_2 *as defined in (2.9).*

Proof. Under the above assumptions, Lemma 4.5 guarantees that $S_n' \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$, provided that the truncation sequence $\{a_n\}$ satisfies the imposed conditions. Moreover, by setting $S_n = \sqrt{|W_n|}\,(\hat{\mu}_n - \mu)$, we see that the sequence of random variables $S_n - S_n'$ converges to 0 in mean square as $n \to \infty$ for any truncation sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = \infty$. This assertion follows directly from Lemmas 2.3, 4.2, and 4.3 together with the fact that $\mathrm{E}((Y(o) - \mu)(Y_n(x) - \mu_n)) \leq h(x)$ for $h(x) \in L^1(\mathbb{R}^d)$, as derived in the proof of Lemma 4.1. An application of Slutsky's theorem completes the proof.

By using arguments similar to those in the one-dimensional setting, we can easily extend Theorem 4.1 to the multivariate case. For this, choose sets $K_i \in \mathcal{K}$, valuations $f_i : \mathcal{R} \to \mathbb{R}$, and random fields $Y_i = \{f_i((\Xi - x) \cap K_i),\ x \in \mathbb{R}^d\}$ for $i = 1, \ldots, k$, which are defined on the basis of the same stationary RACS Ξ. In addition, let $G_i(W_n, \cdot)$, $i = 1, \ldots, k$, be weight functions satisfying the conditions

$$G_i(W_n, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus (W_n \ominus \check{K}_i), \qquad \int_{W_n} G_i(W_n, x)\,\mathrm{d}x = 1, \tag{4.9}$$

$$\sup_{y \in W_n} G_i(W_n, y) \leq \frac{c_1}{|W_n|} \quad \text{for all } n \geq 1, \qquad \lim_{n\to\infty} |W_n| R_{W_n(ij)}(x) = c_{i,j} \quad \text{for all } x \in \mathbb{R}^d, \tag{4.10}$$

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where $c_1, c_{i,j} < \infty$ are some constants and

$$R_{W_n(ij)}(x) = \int_{\mathbb{R}^d} G_i(W_n, y) G_j(W_n, x+y)\,\mathrm{d}y.$$

For each $i = 1, \ldots, k$, define $\mu_i = \mathrm{E}(Y_i(o))$ and $\hat{\mu}_{n,i} = \int_{W_n} Y_i(x)\, G_i(W_n, x)\,\mathrm{d}x$. Then, as in Lemma 2.2, we see that the cross-covariances $\operatorname{cov}(\hat{\mu}_{n,i}, \hat{\mu}_{n,j})$ are given by

$$\operatorname{cov}(\hat{\mu}_{n,i}, \hat{\mu}_{n,j}) = \int_{\mathbb{R}^d} \operatorname{cov}(Y_i(o), Y_j(x))\, R_{W_n(ij)}(x)\,\mathrm{d}x.$$

As in Lemmas 2.3 and 4.1, the limits

$$\sigma_{i,j} = \lim_{n\to\infty} |W_n| \operatorname{cov}(\hat{\mu}_{n,i}, \hat{\mu}_{n,j})$$

exist and are given by

$$\sigma_{i,j} = c_{i,j} \int_{\mathbb{R}^d} \operatorname{cov}(Y_i(o), Y_j(x))\,\mathrm{d}x \tag{4.11}$$

for any $i, j = 1, \ldots, k$, provided that $\mathrm{E}(|M_0 \oplus \check{K}_i|^2) < \infty$. We are now in a position to formulate a multidimensional analogue of Theorem 4.1.

Theorem 4.2. *Let the conditions (2.6), (4.9), and (4.10) be fulfilled and assume that*

$$\mathrm{E}(|M_0 \oplus \check{K}_i|^2) < \infty \quad \textit{for each } i = 1, \ldots, k.$$

Then

$$\begin{pmatrix} \sqrt{|W_n|}(\hat{\mu}_{n,1} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}(\hat{\mu}_{n,k} - \mu_k) \end{pmatrix} \xrightarrow{\mathrm{D}} \mathcal{N}_k(o, \Sigma), \qquad n \to \infty,$$

where $\mathcal{N}_k(o, \Sigma)$ is a k-dimensional Gaussian random vector with zero vector mean and covariance matrix $\Sigma = (\sigma_{i,j})$ whose entries are defined by (4.11).

Proof. By the well-known Cramér–Wold device, the assertion is true if and only if, for all $t \in \mathbb{R}^k \setminus \{o\}$,

$$\sqrt{|W_n|} \sum_{i=1}^k t_i(\hat{\mu}_{n,i} - \mu_i) = \sqrt{|W_n|} \int_{W_n} \sum_{i=1}^k t_i(Y_i(x) - \mu_i) G_i(W_n, x)\,\mathrm{d}x \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = t^\top \Sigma\, t$. The above convergence can be proven analogously to Theorem 4.1.

5. Asymptotic normality for β-mixing random measures

In the previous section, we considered germ–grain models driven by a Poisson point process. Now we show how the above results can be extended to a more general setting in which we do not assume that the point process $\{X_i\}$ of germs is necessarily Poisson, but rather that it satisfies some mixing condition.

Let us begin by recalling some basic notions from mixing; see, e.g. [2] for further details. Consider the probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be two σ-subalgebras of $\mathcal{A}$. The *β-mixing coefficient* (also called the *absolute regularity coefficient*) of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined by

$$\beta(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2} \sup \sum_k \sum_\ell |\mathrm{P}(A_k \cap B_\ell) - \mathrm{P}(A_k)\mathrm{P}(B_\ell)|,$$

where the supremum is taken over all pairs of finite partitions $\{A_k\}$ and $\{B_\ell\}$ of Ω with $A_k \in \mathcal{A}_1$ for all k and $B_\ell \in \mathcal{A}_2$ for all ℓ. Furthermore, for any pair of bounded Borel sets $C_1, C_2 \in \mathcal{B}_0(\mathbb{R}^d)$, let $\rho(C_1, C_2) = \inf\{|x_1 - x_2| : x_1 \in C_1, x_2 \in C_2\}$ denote the distance between C_1 and C_2. For any $s > 0$, the *β-mixing rate* $\beta_X(s)$ of a point process $X = \{X_i\}$ in $\mathbb{R}^d$ is defined by

$$\beta_X(s) = \sup\{\beta(\sigma(N_X(C_1)), \sigma(N_X(C_2)))\colon\ C_1, C_2 \in \mathcal{B}_0(\mathbb{R}^d), \rho(C_1, C_2) \geq s\},$$

where $\sigma(N_X(C))$ is the σ-algebra generated by $N_X(C) = \mathrm{card}\{i\colon X_i \in C\}$.

Let Ξ be a germ–grain model of the form (2.1) for some stationary point process $X = \{X_i\}$ and convex grains, and let the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ be given by (2.5). If X is a Poisson process then $\beta_X(s) = 0$ for all $s > 0$, and Theorem 4.1 implies that $\sqrt{|W_n|}(\hat{\mu}_n - \mu)$ converges weakly to a Gaussian random variable if $\mathrm{E}(|M_0 \oplus \check{K}|^2) < \infty$. Using similar arguments, we can even consider a slightly more general case, in which $\beta_X(s) = 0$ for all s greater than or equal to some $s_0 > 0$, which holds, for example, if X is a Matérn cluster process; see, e.g. [15, p. 159] for a definition. Then, the assertions of Lemmas 4.2 and 4.3 hold if $g_{N(\Xi\cap K)}(4) < \infty$ and if there exists an integrable function $h\colon \mathbb{R}^d \to \mathbb{R}_+$ such that $|\mathrm{cov}_{Y_n}(x)| \leq h(x)$ for any $x \in \mathbb{R}^d$ and $n \geq n_0$, where $n_0 \geq 0$ is some integer. Moreover, Theorem 4.1 remains valid, with the range of dependency, $m_n \geq 2(a_n + D(K))$, in the proof of Lemma 4.5 replaced by $m'_n \geq 2(a_n + D(K)) + s_0$.

In the remaining part of this section, we briefly discuss a different technique of proving that the normal convergence (4.1) holds. (Notice that this technique has been used in [7] for another general class of functionals of germ–grain models.) Let $\{Z_z, z \in \mathbb{Z}^d\}$ be a stationary random field and let $\{U_n, n \geq 1\}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $\lim_{n\to\infty} \mathrm{card}(U_n) = \infty$. Assuming that there exist functions b^*_Z and b^{**}_Z on $[0, \infty)$ such that

$$\beta(\sigma(Z_z, |z| < p+1), \sigma(Z_z, |z| \geq p+q)) \leq \begin{cases} b^*_Z(q), & q > p = 0, \\ p^{d-1}\, b^{**}_Z(q), & p \geq q \geq 1, \end{cases}$$

the following central limit theorem for absolutely regular random fields holds; see Theorem 6.1 of [6].

Lemma 5.1. *Let $\{Z_z, z \in \mathbb{Z}^d\}$ satisfy the following conditions:*

$$\mathrm{E}(Z_0) = 0, \qquad \mathrm{E}(|Z_0|^{2+\delta}) < \infty \quad \text{for some } \delta > 0,$$
$$\sum_{q\geq 1} q^{d-1}(b^*_Z(q))^{\delta/(2+\delta)} < \infty, \qquad \lim_{q\to\infty} q^{2d-1} b^{**}_Z(q) = 0. \tag{5.1}$$

Then $\mathrm{card}(U_n)^{-1/2} S_n \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^2)$, *where* $S_n = \sum_{z\in U_n} Z_z$ *and* $\sigma^2 = \sum_{z\in\mathbb{Z}^d} \mathrm{E}(Z_0 Z_z)$ *is absolutely convergent.*

Now let W_n be the d-dimensional cube $[-n, n)^d$, set $W_n^K = [-n + D'(K), n - D'(K))^d$, where $D'(K)$ is the smallest integer greater than the norm $D(K)$, and define $G(W_n, x) = \mathbf{1}(x \in W_n^K)/|W_n^K|$ for any $x \in \mathbb{R}^d$ and all n large enough that $|W_n^K| > 0$. Furthermore, let $\Xi = \bigcup_{i=1}^\infty (M_i + X_i)$ be a germ–grain model in which the stationary point process $X = \{X_i\}$ has the following mixing property. As in [7], we assume that there exists a nonincreasing function $b_X(\cdot)$ on $[1, \infty)$ such that, for all $a, \Delta \geq 1$,

$$\beta(\sigma(N_X([-a, a]^d)), \sigma(N_X(\mathbb{R}^d \setminus [-a-\Delta, a+\Delta]^d))) \leq b_X(\Delta)\left(\frac{a}{\min\{a, \Delta\}}\right)^{d-1}. \tag{5.2}$$

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Theorem 5.1. *Let Ξ be a stationary germ–grain model satisfying (5.2) with non-empty typical grain $M_0 \in \mathcal{R}$. If there exist some $\delta, \varepsilon > 0$ such that*

$$\mathrm{E}(|Y(o)|^{2+\delta}) < \infty, \qquad \mathrm{E}(D^{2d(\delta+1)/\delta+\varepsilon}(M_0)) < \infty, \tag{5.3}$$

and

$$\sum_{n=1}^{\infty} n^{d-1} b_X(n)^{\delta/(2+\delta)} < \infty, \tag{5.4}$$

then

$$\sqrt{|W_n|}\int_{W_n} (Y(x)-\mu)G(W_n,x)\,\mathrm{d}x \xrightarrow{\mathrm{D}} \mathcal{N}(0,\sigma^2), \qquad n\to\infty,$$

where $\sigma^2 = \int_{\mathbb{R}^d} \mathrm{cov}_Y(x)\,\mathrm{d}x$.

The proof of Theorem 5.1 is similar to that of Theorem 6.2 of [7]. Hence, we merely sketch the main steps. We consider the set function $\eta\colon \mathcal{B}_0(\mathbb{R}^d) \to \mathbb{R}$ with

$$\eta(B) = \int_B (Y(x)-\mu)\,\mathrm{d}x, \qquad B \in \mathcal{B}_0(\mathbb{R}^d),$$

and show that $|W_n^K|^{-1/2}\eta(W_n^K) \xrightarrow{\mathrm{D}} \mathcal{N}(0,\sigma^2)$ as $n\to\infty$, using Lemma 5.1. That is, the latter expression can be written as

$$|W_n^K|^{-1/2}\eta(W_n^K) = \mathrm{card}(U_n)^{-1/2}\sum_{z\in U_n} Z_z, \qquad U_n = \{z\in\mathbb{Z}^d : [z,z+e)\subseteq W_n^K\},$$

for the stationary random field $Z = \{Z_z, z\in\mathbb{Z}^d\}$ with $Z_z = \int_{[z,z+e)}(Y(x)-\mu)\,\mathrm{d}x$. Then we have $\mathrm{E}(Z_0) = 0$ and $\mathrm{E}(|Z_0|^{2+\delta}) \le \mathrm{E}(|Y(0)-\mu|^{2+\delta})$, where the latter bound is finite by assumption (5.3), and, finally,

$$\sigma^2 = \sum_{z\in\mathbb{Z}^d}\mathrm{E}(Z_0Z_z) = \sum_{z\in\mathbb{Z}^d}\int_{[o,e)}\int_{[z,z+e)} \mathrm{cov}_Y(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d}\mathrm{cov}_Y(x)\,\mathrm{d}x.$$

To check the conditions in (5.1), let $\sigma_\eta(B)$ be the σ-algebra generated by $\{\eta(B'), B'\subseteq B, B'\in \mathcal{B}_0(\mathbb{R}^d)\}$. For any $p,q\in\mathbb{N}$, there exist $a,\Delta\ge 0$ satisfying

$$\beta(\sigma(Z_z,|z|<p+1),\sigma(Z_z,|z|\ge p+q)) \le \beta(\sigma_\eta([-a,a)^d),\sigma_\eta(\mathbb{R}^d\setminus[-a-\Delta,a+\Delta)^d)).$$

The right-hand side of the last estimate can be bounded by the β-mixing coefficient of the underlying point process X and certain moments of $D(M_0)$. This can be seen by following the proofs of Lemmas 5.1 and 5.2 of [7], whence

$$\begin{aligned}
&\beta(\sigma_\eta([-a,a)^d),\sigma_\eta(\mathbb{R}^d\setminus[-a-\Delta,a+\Delta)^d))\\
&\quad\le \beta\biggl(\sigma\biggl(N_X\biggl(\biggl[-a-\frac{\Delta}{4},a+\frac{\Delta}{4}\biggr)^d\biggr)\biggr),\sigma\biggl(N_X\biggl(\mathbb{R}^d\setminus\biggl[-a-\frac{3\Delta}{4},a+\frac{3\Delta}{4}\biggr)^d\biggr)\biggr)\biggr)\\
&\qquad+\lambda\, d2^{d+1}\biggl(\biggl(\frac{\Delta+4a}{\Delta}\biggr)^{d+1}+\biggl(\frac{\Delta+12a}{\Delta}\biggr)^{d+1}\biggr)\\
&\qquad\quad\times\mathrm{E}\biggl(D^d(M_0\oplus\check{K})\,\mathbf{1}\biggl(D(M_0\oplus\check{K})>\frac{\Delta}{4}\biggr)\biggr)\\
&\quad\le\biggl(\frac{a}{\min\{a,\Delta\}}\biggr)^{d-1}\biggl(c_1(d)b_X\biggl(\frac{\Delta}{2}\biggr)+c_2(d)\,\mathrm{E}\biggl(D^d(M_0\oplus\check{K})\,\mathbf{1}\biggl(D(M_0\oplus\check{K})>\frac{\Delta}{4}\biggr)\biggr)\biggr)
\end{aligned}$$

for some finite constants $c_1(d)$ and $c_2(d)$, employing assumption (5.2) and the fact that $\mathrm{E}(D^d(M_0 \oplus \check{K})) < \infty$. In the proofs of the above-mentioned lemmas, let

$$\Xi_B = \bigcup_{X_i \in B} (M_i + X_i)$$

denote the germ–grain model restricted to germs within $B \in \mathcal{B}(\mathbb{R}^d)$, define

$$Y_B(x) = f((\Xi_B - x) \cap K),$$

and let $\eta_B(B') = \int_{B'} Y_B(x)\,\mathrm{d}x$ for any $B' \in \mathcal{B}_0(\mathbb{R}^d)$. Next, define

$$b_\eta(\Delta) = c_1(d) b_X(\tfrac{1}{2}\Delta) + c_2(d)\,\mathrm{E}(D^d(M_0 \oplus \check{K})\,\mathbf{1}(D(M_0 \oplus \check{K}) > \tfrac{1}{4}\Delta))$$

and let $b_Z^*(\Delta) = b_\eta(\Delta)$ and $b_Z^{**}(\Delta) = b_\eta(\Delta)/\Delta^{d-1}$. Hence, the proof of Theorem 5.1 is completed by noting that (5.3) and (5.4) imply

$$\begin{aligned}
&\lim_{\Delta \to \infty} \Delta^d b_Z^{**}(\Delta) \\
&\quad \le c_1(d) \lim_{\Delta \to \infty} \mathrm{E}(D(M_0 \oplus \check{K})\,\mathbf{1}(D(M_0 \oplus \check{K}) > \Delta)) \lim_{\Delta \to \infty} \Delta^{d-1} b_X^{\delta/(2+\delta)}(\tfrac{1}{2}\Delta) \\
&\qquad + 4^d c_2(d) \lim_{\Delta \to \infty} \mathrm{E}(D^{2d}(M_0 \oplus \check{K})\,\mathbf{1}(D(M_0 \oplus \check{K}) > \tfrac{1}{4}\Delta)) = 0,
\end{aligned}$$

and that there exists some Δ_0 such that

$$\begin{aligned}
&\sum_{\Delta \ge \Delta_0} \Delta^{d-1} (b_Z^*(\Delta))^{\delta/(2+\delta)} \\
&\quad \le 2^{d-1} c_1^{\delta/(2+\delta)}(d) \sum_{\Delta \ge 1} \Delta^{d-1} (b_X(\Delta))^{\delta/(2+\delta)} \\
&\qquad + 4^{d-1} c_2^{\delta/(2+\delta)}(d) \sum_{\Delta \ge 1} \Delta^{d-1}\,\mathrm{E}(D^d(M_0 \oplus \check{K})\,\mathbf{1}(D(M_0 \oplus \check{K}) > \Delta))^{\delta/(2+\delta)} \\
&\quad \le \tilde{c}_1(d) + \tilde{c}_2(d)\,\mathrm{E}(D^{2d(\delta+1)/\delta+\varepsilon}(M_0 \oplus \check{K})) \sum_{\Delta \ge 1} \Delta^{-(1+\varepsilon')} < \infty,
\end{aligned}$$

with

$$\begin{aligned}
\tilde{c}_1(d) &= 2^{d-1} c_1^{\delta/(2+\delta)}(d) \sum_{\Delta \ge 1} \Delta^{d-1} (b_X(\Delta))^{\delta/(2+\delta)} < \infty, \\
\tilde{c}_2(d) &= 4^{d-1} c_2^{\delta/(2+\delta)}(d) < \infty.
\end{aligned}$$

For the Boolean model $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$, with $M_0 \in \mathcal{R} \setminus \{\varnothing\}$ such that the minimal number of convex components of M_0 is bounded by some finite constant, the conditions of Theorem 5.1 are fulfilled if $\mathrm{E}(D^{2d(\delta+1)/\delta+\varepsilon}(M_0)) < \infty$ holds for some $\delta, \varepsilon > 0$. Notice that this integrability condition is stronger than the assumption $\mathrm{E}(|M_0 \oplus \check{K}|^2)$ made in Theorems 4.1 and 4.2. Further examples of point processes X satisfying conditions (5.2) and (5.4) can be found in [7]. We also remark that in the special case $Y = Z_r$ considered in (3.2) with arbitrary X and $M_0 \in \mathcal{K}'$, the integrability condition $\mathrm{E}(N^{(2+\delta)d}(\Xi \cap B_1(o))) < \infty$ implies that $\mathrm{E}(|Y(x)|^{2+\delta}) < \infty$ for $\delta > 0$, provided that there is almost surely no boundary point of Ξ where more than d germs overlap; see Section 4 of [11].

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