ON COMMUTATIVE CONTINUATION OF PARTIAL ENDOMORPHISMS OF GROUPS

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1. Introduction. Given a homomorphic mapping θ of a subgroup A of a group G onto another subgroup B of G, necessary and sufficient conditions for the existence of a supergroup G^* of G and an endomorphism θ^* of G^* such that θ^* coincides with θ on A were derived by B. H. Neumann and Hanna Neumann **(3)**. The homomorphism θ is called a partial endomorphism of G and θ^* is said to continue, or extend, θ. Necessary and sufficient conditions for the simultaneous continuation of two partial endomorphisms of a group G to total endomorphisms of one supergroup $G^* \supseteq G$ were derived by the author **(2)**.

The technique applied in both cases was that of forming the free product of G and its factor group modulo some normal subgroup with an amalgamated subgroup. In case G is abelian, the direct product with an amalgamated subgroup was used instead.

Given two partial endomorphisms θ and ϕ of a group G, we prove that the above technique no longer works if we require their continuation to commutative total endomorphisms even if we limit ourselves to an abelain group G. The breakdown of the technique does not exclude the possibility of another approach. In fact sufficient conditions for the continuation of θ and ϕ to θ^* and ϕ^* such that $\theta^*\phi^* = \phi^*\theta^*$ are known in case θ and ϕ are partial automorphisms (i.e. isomorphic mappings of subgroups of G) even without assuming that the group G is abelian **(1)**.

2. Two necessary conditions. Assume that G is an abelian group and θ, ϕ are two partial endomorphisms of G mapping A onto B and C onto D respectively, A, B, C, and D being subgroups of G. If $K(\theta)$ is the kernel of θ, then the first step of the construction is forming the group

$$G_1 = \{G \times G/K(\theta); B = A/K(\theta)\}$$

in which θ is continued to θ_1, the canonical mapping of G onto $G/K(\theta)$; and ϕ maps $C \subseteq G_1$ onto $D \subseteq G_1$.

The next step will be forming the group

$$G_2 = \{G_1 \times G_1/K(\phi); D = C/K(\phi)\}$$

where $K(\phi)$ is the kernel of ϕ. In G_2 the mapping ϕ is continued to ϕ_1, the canonical mapping of G_1 onto $G_1/K(\phi)$ and θ_1 maps $G \subseteq G_2$ onto $G/K(\theta) \subseteq G_2$.

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LEMMA. *Two necessary conditions for carrying out the first two steps of the construction towards continuing θ and ϕ to commutative total endomorphisms are that*

(1) $$g \in C\theta^{-1} \cap A\phi^{-1} \text{ implies } g\theta\phi = g\phi\theta,$$

(2) $$C\theta^{-1} \cap C = C\theta^{-1} \cap A\phi^{-1},$$

where $C\theta^{-1}$ denotes the set of elements mapped by θ into C.

Proof. Assume that G is embedded in the group G^* which possesses two total endomorphisms θ^*, ϕ^* continuing θ, ϕ respectively such that $\theta^*\phi^* = \phi^*\theta^*$.

If $g \in G^*$ is such that

$$g \in A \cap C \cap C\theta^{-1} \cap A\phi^{-1} = C\theta^{-1} \cap A\phi^{-1},$$

then $g\theta$, $g\phi$ are defined and $g\theta \in C$, $g\phi \in A$ and hence $(g\theta)\phi$, $(g\phi)\theta$ are also defined. Moreover

$$g\theta^* = g\theta, \qquad g\phi^* = g\phi,$$

and

$$g\theta^*\phi^* = g\theta\phi, \qquad g\phi^*\theta^* = g\phi\theta.$$

Since $g\theta^*\phi^* = g\phi^*\theta^*$, then $g\theta\phi = g\phi\theta$. This proves that condition (1) is necessary.

Now we form

$$G_1 = \{G \times G/K(\theta);\, B = A/K(\theta)\}.$$

For the next step we replace

$$G;\, A,\, B,\, C,\, D;\, \theta \text{ and } \phi$$

by

$$G_1;\, C,\, D,\, G,\, G/K(\theta);\, \phi \text{ and } \theta_1$$

respectively. Condition (1) then translates into

(3) $$g \in C\theta_1^{-1} \cap G\phi^{-1} \text{ implies } g\theta_1\phi = g\phi\theta_1.$$

Now we prove that the relation

(4) $$C\theta_1^{-1} = C\theta^{-1}$$

holds. For if $a \in C\theta^{-1}$, then $a\theta = a\theta_1 \in C$, which means that $a \in C\theta_1^{-1}$ and hence

(4i) $$C\theta^{-1} \subseteq C\theta_1^{-1}.$$

Conversely, if $x \in C\theta_1^{-1}$, then $x\theta_1 \in C$. Since $x\theta_1 \in G/K(\theta)$ also, then $x\theta_1 \in C \cap G/K(\theta)$ and by the amalgamation made in G_1 we have

$$G \cap G/K(\theta) = B;$$

thus intersecting both sides by C, we obtain

$$C \cap G/K(\theta) = B \cap C.$$

Thus $x\theta_1 \in B$, i.e. for some $a \in A$ we have

$$x\theta_1 = a\theta = a\theta_1,$$
$$xa^{-1} \in K(\theta_1) = K(\theta) \subseteq A,$$

and hence

$$x \in A, \qquad x\theta_1 = x\theta \in C.$$

This means that $x \in C\theta^{-1}$ and hence

(4ii) $$C\theta_1^{-1} \subseteq C\theta^{-1}.$$

Relations (4i) and (4ii) together imply (4).

Since $G\phi^{-1} = C$, relation (3) becomes

$$g \in C\theta^{-1} \cap C \text{ implies } g\theta_1 \phi = g\phi\theta_1.$$

For such an element g we have

$$g\theta_1 = g\theta \in C,$$
$$g\theta_1 \phi = g\theta\phi \in D.$$

On the other hand,

$$g\phi\theta_1 \in G/K(\theta).$$

For these two elements to be equal we must have

$$g\phi\theta_1 \in G/K(\theta) \cap D = B \cap D,$$
$$g\phi \in (B \cap D)\theta_1^{-1} \subseteq A \cup K(\theta) = A,$$
$$g \in C\theta^{-1} \cap A\phi^{-1},$$

which proves that

$$C\theta^{-1} \cap C \subseteq C\theta^{-1} \cap A\phi^{-1},$$

but we obviously have

$$C\theta^{-1} \cap A\phi^{-1} \subseteq C\theta^{-1} \cap C.$$

These two relations together prove that condition (2) is necessary.

3. Completion of the construction. Assume that in the group G conditions (1) and (2) are fulfilled. In G_1 let

$$x \in C\theta_1^{-1} \cap G\phi^{-1} = C\theta^{-1} \cap C;$$

thus

$$x = a \in A,$$

(5) $$x\theta_1 \phi = a\theta\phi.$$

From relation (2), we obtain

$$x\phi = a\phi \in A,$$

(6) $$x\phi\theta_1 = a\phi\theta.$$

Relations (5) and (6) together with (1) give

$$x\theta_1 \phi = x\phi\theta_1.$$

To ensure what corresponds to relation (2) in the group G_2 which is formed in the second step of the construction, we must have

$$G\phi^{-1} \cap G = G\phi^{-1} \cap C\theta_1^{-1}$$

or

(7) $$C = C \cap C\theta^{-1}.$$

In the next step we replace

$$C, G, \phi, \theta_1$$

by

$$G, G_1, \theta_1, \phi_1$$

respectively. Then relation (7) becomes

(8) $$G = G \cap G\phi_1^{-1}.$$

Now we prove that

(9) $$C = G\phi_1^{-1}.$$

Obviously,

(9i) $$C \subseteq G\phi_1^{-1}.$$

If $x \in G\phi_1^{-1}$, then $x\phi_1 \in G$ and hence

$$x\phi_1 \in G \cap G/K(\phi) = D;$$

thus there exists an element $c \in C$ such that

$$x\phi_1 = c\phi = c\phi_1,$$

$$xc^{-1} \in K(\phi_1) = K(\phi) \subseteq C,$$

which shows that $x \in C$ and consequently

(9ii) $$G\phi_1^{-1} \subseteq C.$$

Relations (9i) and (9ii) together prove (9). Relation (8) now becomes

$$G = G \cap C = C.$$

Relation (7) also gives

$$G = G \cap G\theta^{-1} \subseteq G \cap A = A.$$

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Thus it is necessary to have

$$A = C = G.$$

This means that if we require that θ and ϕ could be continued simultaneously to two commutative total endomorphisms, they have to be themselves total endomorphisms from the start.

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