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THE STATIONARY PHASE METHOD FOR CERTAIN DEGENERATE CRITICAL POINTS I

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1. Introduction. We consider, in this work, the asymptotic behaviour for large λ, of a Fourier integral

$$(1.1) \qquad I(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda\varphi(x)} a(x)dx$$

where $\varphi(x)$ is in general a C^∞ function and $a(x)$ a C^∞ function with compact support. It is well known that the asymptotic behaviour of this integral is controlled by the behaviour of φ at its critical points (i.e., points where $\partial\varphi/\partial x_j(x) = 0$) and is given by local contributions at these points ([**1**], [**3**], [**7**], [**9**]).

In general, one assumes the hypothesis of non degenerate isolated critical point, namely that the determinant of the second derivative at the critical point is non zero. The contribution of a non degenerate critical point to the asymptotic expansion of (1.1) can be derived by various methods ([**3**], [**7**], [**9**]). In the case where the critical point is isolated but degenerate, the asymptotic expansion is known in one variable ([**7**]) but almost nothing is known in several variables except the mere existence of the expansion which is an easy consequence of the resolution of singularities ([**2**], [**8**]) and some results about the exponent of λ in the first term of the expansion and also the possibility of logarithmic term in certain cases ([**1**], [**10**]). In certain special cases, the first term of the expansion was obtained (see for example [**3**], [**6**]) and in two variables, the complete expansion was derived in [**4**] assuming that the Newton polygon has only two sides.

In the case of non degenerate critical point, the asymptotic expansion of (1.1) can be derived by reducing φ to a sum of squares using Morse lemma and a change of coordinates. No such result is known in the case of a degenerate critical point and we are obliged to use holomorphic functions theory to reduce the oscillating integral (1.1) to an absolutely convergent integral of the Laplace type

$$(1.2) \qquad J(\lambda) = \int_0^\infty \cdots \int_0^\infty e^{\lambda F(x)} dx$$

under certain hypothesis on φ (with $a \equiv 1$). We also suppose that φ (or F) is a polynomial and we shall derive the first terms of the asymptotic expansion of (1.2) under the hypothesis that at most two faces of the Newton polyhedron of

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F intersect the bissectrice $X_1 = \cdots = X_n$, and without any change of variables or reduction to a normal type (which is a non constructive process in general). When there is only one face intersecting the bissectrice, the result is very simple and is given in Theorem 3. In that case there is no logarithmic term. When two faces intersect the bissectrice the result is more complicated (see Theorem 4) and a logarithmic term appears. As a consequence, we are obliged to derive the first two terms of the expansion, because it is rather intuitive that two terms in $\lambda^\alpha(\log\lambda)^\beta$ for a fixed α and distinct β (where $\beta = 1, 0$) cannot really be separated in asymptotic expansions (both for practical and for pure mathematical reasons). The case where 3 or more faces of the Newton polyhedron intersect the bissectrice will be treated elsewhere and is much more intricate. Applications to Green's function and to Fourier transforms of convex domains will appear in further work (see [**5**]).

2. Reduction of oscillating integrals to absolutely convergent integrals. Let $\varphi(x_1, \ldots, x_n)$ be a polynomial in n variables with real coefficients. We define the Fourier integral

$$I(R) = \int_{C_R} e^{i\varphi(x)} dx_1 \ldots dx_n \tag{2.1}$$

where $C_R = [0, R]^n$. We decompose φ into its homogeneous parts of degree k $\varphi_0, \varphi_1, \ldots, \varphi_d$

$$\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_d. \tag{2.2}$$

Then we can prove:

THEOREM 1. *Assume that $d > n$ and that the coefficients of the homogeneous part φ_d of maximal degree d are positive or zero and that the coefficient of $X_1^d, \ldots, X_n^d$ are positive. Then*

$$\lim_{R \to \infty} I(R)$$

exists and we have

$$\lim_{R\to\infty} I(R) = -e^{in\pi/2d} \int_0^\infty \cdots \int_0^\infty e^{\Phi(x)} dx_1, \ldots, dx_n \tag{2.3}$$

where $\Phi(x) = i\varphi(e^{i\pi/2d}x)$ and the integral on the right hand side is absolutely convergent.

Proof of Theorem 1. We introduce the $(n+1)$-dimensional manifold in $\mathbf{C}^n$

$$M_R = \left\{ z \in \mathbf{C}^n / z_j = |z_j| e^{i\theta} \quad 0 < |z_j| < R, \quad 0 < \theta < \frac{\pi}{2d} \right\}.$$

Its boundary is

$$\partial M_R = C_R \cup D_R \cup \bigcup_{j=1}^{n} E_{j,R} \cup F_R$$

where

$$C_R = [0, R]^n$$

(2.4) $$D_R = \{z \in \mathbf{C}^n / 0 < |z_j| < R, \quad z_j = |z_j| e^{i\pi/2d} \forall j\}$$

(2.5) $$E_{j,R} = \left\{ \begin{array}{ll} z \in \mathbf{C}^n / z_k = |z_k| e^{i\theta}, & 0 < |z_k| < R \text{ for } k \neq j \\ & |z_j| = R \quad 0 < \theta < \dfrac{\pi}{2d} \end{array} \right\}$$

F_R is of dimension less than n. We consider the n-form

$$\omega = e^{i\varphi(z)} dz_1 \Lambda \cdots \Lambda dz_n.$$

Because φ is a holomorphic function, ω is closed, so that by Stokes theorem

$$\int_{\partial M_R} \omega = 0$$

which gives

(2.6) $$I(R) = \int_{C_R} \omega = -\int_{D_R} \omega - \sum_{j=1}^{n} \int_{E_{j,R}} \omega.$$

Now we prove the following two assertions:

1°) $$\lim_{R \to \infty} \int_{D_R} \omega \quad \text{exists and is} \quad e^{in\pi/2d} \int_0^{\infty} \cdots \int_0^{\infty} e^{\Phi(x)} dx_1 \dots dx_n.$$

In fact

$$\Phi(x) = i \left[e^{\frac{i\pi}{2}} \varphi_d(x) + e^{\frac{i\pi}{2}\left(\frac{d-1}{d}\right)} \varphi_{d-1}(x) + \cdots + \varphi_0 \right].$$

Now,

$$\varphi_d(x) \geqq \sum a_k x_k^d \quad \text{for some } a_k > 0,$$

on the domain $([0, +\infty[)^n$, so that $\exp \Phi(x)$ is integrable on $([0, +\infty[)^n$ and we have proved the first assertion.

2°) $$\lim_{R \to \infty} \int_{E_{j,R}} \omega = 0.$$

First we compute

$$\varphi(z)|_{E_{j,R}} = \varphi(|z_1|e^{i\theta}, \dots, |z_{j-1}|e^{i\theta}, Re^{i\theta}|z_{j+1}|e^{i\theta}, \dots, |z_n|e^{i\theta})$$
$$dz_1\Lambda \dots \Lambda dz_n|_{E_{j,R}} = ie^{in\theta}Rd|z_1|\Lambda \dots \Lambda d|z_{j-1}|d\theta\Lambda d|z_{j+1}|\Lambda \dots \Lambda d|z_n|$$

so that

$$\int_{E_{j,R}} \omega = iR\int_0^{\pi/2d} e^{in\theta}d\theta$$
$$\times \int_0^R \cdots \int_0^R \exp\left[i\varphi(r_1e^{i\theta}, \dots, Re^{i\theta}, \dots, r_ne^{i\theta})\right]dr_1 \dots dr_{j-1}dr_{j+1} \dots dr_n.$$

Now

$$Im\varphi(r_1e^{i\theta}, \dots, Re^{i\theta}, \dots, r_ne^{i\theta})$$
$$= \sum_{|\alpha|\leqq d} a_\alpha r_1^{\alpha_1} \dots r_{j-1}^{\alpha_{j-1}} r_{j+1}^{\alpha_{j+1}} \dots r_n^{\alpha_n} R^{\alpha_j} \sin(|\alpha|\theta).$$

But $\theta \geqq \sin\theta \geqq 2\theta/\pi$ for $0 \leqq \theta \leqq \pi/2$, so that

$$Im\varphi(r_1e^{i\theta}, \dots, Re^{i\theta}, \dots, r_ne^{i\theta}) \leqq -\frac{2}{\pi}\frac{\theta}{d}A(r, R)$$

where

$$A(r, R) = \sum_{|\alpha|=d} a_\alpha r_1^{\alpha_1} \dots r_{j-1}^{\alpha_{j-l}} r_{j+1}^{\alpha_{j+1}} \dots r_n^{\alpha_n} R^{\alpha_j}$$
$$- \frac{\pi}{2d}\sum_{p=0}^{d-1} p \sum_{|\alpha|=p} |a_\alpha| r_1^{\alpha_1} \dots r_{j-1}^{\alpha_{j-l}} r_{j+1}^{\alpha_{j+1}} \dots r_n^{\alpha_n} R^{\alpha_j}$$

and we obtain

$$\left|\int_{E_{j,R}} \omega\right| \leqq R\int_0^R \cdots \int_0^R dr_1 \dots dr_{j-1}dr_{j+1} \dots dr_n$$
$$\times \frac{|e^{-A(r,R)} - 1|}{A(r, R)} \frac{\pi}{2d}.$$

Now, on $[0, R]^{n-1}$, we obtain a lower bound of $A(r, R)$, using the hypothesis of the theorem

$$A(r, R) > bR^d - \sum_{p=0}^{d-1} \frac{p}{d} \sum_{|\alpha|=p} |a_\alpha| R^p + \sum_{k\neq j} a_k r_k^d$$

so that for R large enough:

$$A(r,R) \geqq a\left(R^d + \sum_{k\neq j} a_k r_k^d\right)$$

and thus

$$\left|\int_{E_{j,R}} \omega\right| \leqq CR\int_0^R \cdots \int_0^R dr_1 \ldots dr_{j-1}dr_{j+1}\ldots dr_n$$
$$\times\left(R^d + \sum_{k\neq j} r_k^d\right)^{-1} \leqq CR^{n-d}$$

which tends to zero if $n < d$.

3. The Newton polyhedron of a polynomial. From now on, we shall consider an integral

$$(3.1)\qquad J(\lambda) = \int_0^\infty \cdots \int_0^\infty \exp(\lambda\Phi(x_1,\ldots,x_n))dx_1\ldots dx_n$$

where Φ is a polynomial of degree d

$$(3.1)'\qquad \Phi(x) = \sum_{|\mu|\leqq d} a_\mu x^\mu$$

having a critical point at 0 such that $\Phi(0) = 0$. We shall also assume $\operatorname{Re} a_\mu \leqq 0$ and that the integral (3.1) is absolutely convergent. We consider the Newton polyhedron Π of Φ defined by

$$(3.2)\qquad \Pi = \{\mu \in (\mathbf{Z}_+)^n / \operatorname{Re} a_\mu < 0\}.$$

We also consider the convex envelope of the set $\Pi\cup\{\infty\}$ in $(\mathbf{R}_+)^n$; the boundary $\partial\Pi$ of this convex envelope is the union of certain convex polyhedron of dimension n which lie in hyperplanes of $\mathbf{R}^n$. We shall assume that $\partial\Pi$ contains a point on each coordinate axis (so that there is no faces of $\partial\Pi$ which are parallel to one of the coordinate axis). This hypothesis implies that 0 is an isolated critical point.

The convex polyhedron of dimension n forming $\partial\Pi$ have extremal points which are necessarily in Π. Moreover there can be other points in $\partial\Pi$ which are in Π. The convex polyhedra forming $\partial\Pi$ intersect each other according to lower dimensional polyhedra common to several faces. We consider also the set of points Π'

$$(3.3)\qquad \Pi' = \{\mu \in (\mathbf{Z}_+)^n / a_\mu \neq 0\}$$

so that

$$\Pi' = \Pi \cup \{\mu \in (\mathbf{Z}_+)^n / a_\mu \text{ purely imaginary}\}. \tag{3.4}$$

We now split Φ into two pieces

$$\Phi = P + R \tag{3.5}$$

where

$$P = \sum_{\mu \in \Pi' \cap \partial \Pi} a_\mu x^\mu \tag{3.6}$$

$$R = \Phi - P.$$

P is called the *fundamental part of* P. Moreover, we shall write the equation of any face F forming $\partial\Pi$ in the following form

$$l_F(X) \equiv \sum_{j=1}^{n} \alpha_j^{(F)} X_j = 1 \tag{3.7}$$

where $\alpha_j^{(F)}$ are all positive rationnal numbers (because of convexity). We also remark that if F is a face of $\partial\Pi$ with equation $l_F(X) = 1$ and if μ is a point of Π' which does not belong to F then

$$l_F(\mu) > 1. \tag{3.8}$$

For any face F, with coefficients of the associated $l_{(F)}, \alpha_j^{(F)}$, we define a scaling transformation by

$$x \longrightarrow x_F$$

given by

$$x_{F,j} = \lambda^{\alpha_j^{(F)}} x_j. \tag{3.9}$$

We call x_F the F-variables.

If $Q(x)$ is any polynomial in x, defined by

$$Q(x) = \sum_\mu b_\mu x^\mu$$

we define $Q(x_F, \lambda)$ by the change of variables (3.9) in λQ i.e.,

$$Q(x_F, \lambda) = \lambda Q(x) \tag{3.10}$$

with x_F related to x by (3.9), so that

$$Q(x_F,\lambda)=\sum_{\mu} b_\mu x_F^\mu \lambda^{1-l_F(\lambda)} \tag{3.11}$$

and we shall say that $Q(x_F,\lambda)$ is the expression of λQ in the F-variables. In particular, we obtain the following lemma:

LEMMA 1. *If F is a face of $\partial\Pi$ and if Q is a polynomial $Q=\sum b_\mu x^\mu$, then*
(i) *if all the μ's such that $b_\mu \neq 0$ in Q belong to F, $Q(x_F,\lambda)$ is independent of λ and is*

$$Q(x_F,\lambda)=\sum b_\mu x_F^\mu = Q(x_F)$$

(ii) *if all the μ's such that $b_\mu \neq 0$ in Q are in $\Pi' - F$, then $Q(x_F,\lambda)$ has all its monomials with a negative power in λ, i.e.,*

$$Q(x_F,\lambda)=\sum_{\mu} b_\mu x_F^\mu \lambda^{1-l_F(\mu)}$$

with all $1-l_F(\mu)<0$.

Proof. The proof is obvious from (3.11) and (3.8).

4. Reduction of the integral $J(\lambda)$ and the case where one face intersects the bissectrice. We now come back to the integral (3.1); if E denotes a subset of $\partial\Pi$, we denote

$$P_E(x)=\sum_{\mu\in E} a_\mu x^\mu \tag{4.1}$$

the E-part of the fundamental part P of Φ. Let us denote by B the set of points μ in Π' which lie on some face F in $\partial\Pi$ which intersects the bissectrice $X_1=\cdots=X_n$, i.e.,

$$B=\Pi'\cap \bigcup_{\substack{\{F/F \text{ is a face of } \partial\Pi \text{ with} \\ F\cap\{X_1=\cdots=X_n\}\neq\emptyset\}}} F. \tag{4.2}$$

The main theorem is

THEOREM 2. *The integral $J_B(\lambda)$ given by*

$$J_B(\lambda)\equiv\int_0^\infty\cdots\int_0^\infty \exp(\lambda P_B(x))dx_1\dots dx_n \tag{4.3}$$

is absolutely convergent.

Proof. We know that B is the set of monomials in Φ which belong to one of the face touching the bissectrice $X_1 = \cdots = X_n$ in $(\mathbf{R}_+)^n$. The extremal points of any of these faces are point μ with $\operatorname{Re} a_\mu < 0$, and for each face, there is a number $\geqq n$ of these points. Among all these extremal points of the faces F of $\partial\Pi$ intersecting the bissectrice, we can surely chose n of them $\mu^{(1)}, \ldots, \mu^{(n)}$ such that the convex hull $C(\mu^{(1)}, \ldots, \mu^{(n)})$ of $\{\mu^{(1)}, \ldots, \mu^{(n)}\}$ intersects the bissectrice at an interior point (this is due to the convexity of $\partial\Pi$ and the fact that $\partial\Pi$ intersects each coordinate axis). It is then clear that

$$J_B(\lambda) \leqq \int_0^\infty \cdots \int_0^\infty \exp(\lambda \operatorname{Re} P_{\{\mu^{(1)}, \ldots, \mu^{(n)}\}}(x)) dx_1 \ldots dx_n. \tag{4.4}$$

Now, we have the following lemma which will prove Theorem 1:

LEMMA 2. *If $\mu^{(1)}, \ldots, \mu^{(n)}$ are n independent integer points such that the convex hull of these points intersects the bissectrice at an interior point and if a_j are negative numbers, then:*

$$\int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{j=1}^n a_j x^{\mu^{(j)}}\right) dx_1 \ldots dx_n < +\infty. \tag{4.5}$$

Proof. We define $u_j = x^{\mu^{(j)}}$, so that

$$\frac{\partial u_j}{\partial x_l} = \mu_l^{(j)} \frac{u_j}{x_l}$$

and the jacobian of the transformation $x \longrightarrow u$ is

$$\frac{D(u)}{D(x)} = \frac{\prod_{j=1}^n u_j}{\prod_{j=1}^n x_j} \det(\mu^{(1)}, \ldots, \mu^{(n)}). \tag{4.6}$$

Now, by assumption there exists some positive number λ_0 and also numbers $p_1, \ldots, p_n$ with $0 < p_j < 1$, such that

$$\lambda_0 = \sum_{k=1}^n p_k \mu_j^{(k)} \quad \text{for all } j = 1, \ldots, n$$

because the convex hull of $\mu^{(1)}, \ldots, \mu^{(n)}$ intersects the bissectrice at an interior point. Then

$$\left(\prod_{j=1}^n x_j\right)^{\lambda_0} = \prod_{j=1}^n \left(x_j^{\sum_{k=1}^n p_k \mu_j^{(k)}}\right) = \prod_{k=1}^n u_k^{p_k}$$

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so that

$$(4.7) \qquad \frac{D(u)}{D(x)} = \left(\prod_{k=1}^{n} u_j^{1-p_j/\lambda_0}\right) \det(\mu^{(1)}, \dots, \mu^{(n)})$$

and the integral (4.5) becomes

$$C \int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{j=1}^{n} a_j u_j\right) \left(\prod_{j=1}^{n} u_j^{p_j/\lambda_0 - 1}\right) du.$$

But all the p_j are > 0 (and also λ_0) so that the integral is convergent.

As a corollary of Theorem 2, we obtain

THEOREM 3. *With the notation of Theorem 1, and if there is only one face F of $\partial\Pi$ intersecting the bissectrice*

$$J(\lambda) \sim J_F(\lambda)$$

so that

$$(4.8) \qquad J(\lambda) \sim \frac{1}{\lambda^{\Sigma\alpha_j(F)}} \int_0^\infty \cdots \int_0^\infty \exp(P_F(x))dx.$$

Proof. We start from Φ and consider the face F intersecting the bissectrice. Then it is obvious that it intersects the bissectrice at one of its interior point. We write

$$\Phi = P_F + R_F$$

where R_F consists of all the monomials of Φ which do not belong to F. Now if we do the scaling transformation $x \longrightarrow x_F$ associated to the face F, we have by (3.10) and Lemma 1

$$\lambda\Phi(x) = P_F(x_F) + R_F(x_F, \lambda)$$

and in all monomials appearing in R_F, λ appears with a negative power. Then

$$(4.9) \qquad J(\lambda) = \frac{1}{\lambda^{\Sigma\alpha_j(F)}} \int_0^\infty \cdots \int_0^\infty \exp(P_F(x_F))dx_F + \frac{1}{\lambda^{\Sigma\alpha_j(F)}} \int_0^\infty \cdots \int_0^\infty \exp(P_F(x_F))[\exp(R_F(x_F, \lambda)) - 1]dx_F$$

and this splitting of the integral can be done because the first integral of the right hand side of (4.9) is absolutely convergent by Theorem 2. Moreover

$$|\exp(R_F(x_F, \lambda)) - 1| \quad \text{tends to } 0 \text{ if } \lambda \longrightarrow \infty$$

and the integrand in the last integral of (4.9) is dominated by $2\exp(P_F(x_F))$ which is integrable, so that the last term of the right hand of (4.9) is

$$O\left(\lambda^{-\Sigma\alpha_j^{(F)}}\right)$$

and the asymptotics of $J(\lambda)$ is given by the first term.

5. The case of a reduced integral with two faces intersecting the bissectrice: case $n = 2$. a) We shall now study the following situation: the polynomial Φ satisfies the usual hypothesis of Section 2, but it is reduced to its fundamental part P. Moreover, there are only two faces F_1, F_2 which intersect the bissectrice.

We shall define the following three polynomials (see (4.1))

$$\begin{aligned} P_{12} &\equiv P_{F_1\cap F_2} \\ P_1 &\equiv P_{F_1 - F_1\cap F_2} \\ P_2 &\equiv P_{F_2 - F_1\cap F_2} \end{aligned} \tag{5.1}$$

so that the P_B defined in (4.1), (4.2) is

$$P_B = P_1 + P_2 + P_{12}. \tag{5.2}$$

In words, P_{12} is the sum of monomials of P belonging to $F_1 \cap F_2$, P_1 is the sum of monomials of P in F_1 but not in $F_1 \cap F_2$ and P_2 the sum of monomials of P in F_2, but not in $F_1 \cap F_2$. We also abbreviate l_1 for l_{F_1}, l_2 for l_{F_2} and $\alpha_j^{(i)}$ for $\alpha_j^{(F_i)}$ so that

$$l_i(X) = \sum_j \alpha_j^{(i)} X_j. \tag{5.3}$$

By Theorem 2, we know that the integral J_B is absolutely convergent and we want to obtain its asymptotic behaviour for large λ.

First we begin by some remarks:

(i) $F_1 \cap F_2$ is an $(n-1)$-convex polyhedron which intersects the bissectrice $X_1 = \cdots = X_n$ at an interior point.

(ii) Let us denote H_j the hyperplane on which the face F_j lies. Then the two H_j cuts the X_n-axis in two distinct points and we can assume that H_1 cuts the X_n-axis at a higher point than H_2.

(iii) Moreover we have

$$\sum_{j=1}^{n} \alpha_j^{(1)} = \sum_{j=1}^{n} \alpha_j^{(2)} \tag{5.4}$$

(because there exists a point on the bissectrice which is common to the faces F_1 and F_2) and we have

$$\alpha_n^{(1)} < \alpha_n^{(2)} \tag{5.5}$$

because H_1 cuts the x_n axis at a higher point than H_2.

We shall also denote by $x^{(1)}$ instead of x_{F_1} the F_1-variables.

$$x_j^{(1)} = \lambda^{\alpha_j^{(1)}} x_j \tag{5.6}$$

and also $x^{(2)}$ the F_2 variables

$$x_j^{(2)} = \lambda^{\alpha_j^{(2)}} x_j \tag{5.7}$$

and $dX = dx_1 \ldots dx_n$

$$dx^{(j)} = dx_1^{(j)} \ldots dx_n^{(j)}$$

$$|\alpha^{(j)}| = \sum_{k=1}^{n} \alpha_k^{(j)}, \tag{5.8}$$

so that here $|\alpha^{(1)}| = |\alpha^{(2)}|$ as a consequence of (5.4).

b) We want now to prove the following theorem:

THEOREM 4. *With the preceding notations and conventions we have the following result for the asymptotics of*

$$J(\lambda) = \int_0^\infty \cdots \int_0^\infty \exp[\lambda(P_1 + P_2 + P_{12})]dx :$$

$$\begin{aligned} J(\lambda) &= \lambda^{-|\alpha^{(2)}|} \log(\lambda^{\alpha_n^{(2)} - \alpha_n^{(1)}}) \\ &\quad \times \int_0^\infty \cdots \int_0^\infty \exp(P_{12}(x_1, \ldots, x_{n-1}, 1)) dx_1 \ldots dx_{n-1} \\ &\quad + \lambda^{-|\alpha^{(2)}|} \Bigg[\int_0^1 dx_n^{(2)} \int_0^\infty \cdots \int_0^\infty dx_{n-1}^{(2)} \ldots dx_1^{(2)} \exp[(P_2 + P_{12})(x^{(2)})] \\ &\quad + \int_1^\infty dx_n^{(2)} \int_0^\infty \cdots \int_0^\infty \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} \ldots dx_{n-1}^{(2)} \\ &\quad + \int_0^1 dx_n^{(1)} \int_0^\infty \cdots \int_0^\infty \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] dx_1^{(1)} \ldots dx_{n-1}^{(1)} \\ &\quad + \int_1^\infty dx_n^{(1)} \int_0^\infty \cdots \int_0^\infty \exp((P_{12} + P_1)(x^{(2)})) dx_1^{(1)} \ldots dx_{n-1}^{(1)} \Bigg] \\ &\quad + R(\lambda) \end{aligned} \tag{5.9}$$

where $R(\lambda)$ *is a remainder term given by:*

$$\begin{aligned} R(\lambda) &= \lambda^{-|\alpha^{(2)}|} \int_0^\infty \cdots \int_0^\infty \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] \\ &\quad \times [\exp(P_2(x^{(1)}, \lambda)) - 1] dx^{(1)} \end{aligned} \tag{5.10}$$

and $R(\lambda) = o(\lambda^{-|\alpha^{(2)}|})$ *is of lower order than the first two terms.*

c) To make the argument clearer, we shall begin to prove this result in 2 dimensions in this section, in 3 dimensions in the next section. This result, in two dimensions, is similar to the one of [**4**], except in this work, we supposed that the Newton polygon had only two faces extending up to the coordinate axis (and also we wanted to get the whole asymptotic expansion). In two dimensions, the figure is

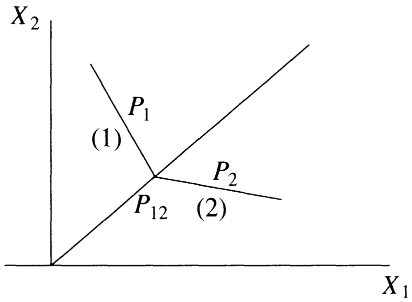

Figure 1

We start with

$$J(\lambda) = \int_0^\infty \int \exp[\lambda(P_1 + P_2 + P_{12})(x)]dx$$

and we perform the change of F_1-variables

$$x_j^{(1)} = \lambda^{\alpha_j^{(1)}} x_j$$

so that

$$J(\lambda) = \lambda^{-|\alpha_j^{(1)}|} \int_0^\infty \int \exp[(P_1 + P_{12})(x^{(1)})] \times \exp[P_2(x^{(1)}, \lambda)]dx^{(1)} \tag{5.11}$$

and we know that in the monomials appearing in $P_2(x^{(1)}, \lambda)$, λ is always at a negative power. We split (5.11) into two terms writing

$$\lambda^{|\alpha^{(1)}|} J = J_0 + J_\infty \tag{5.12}$$

where

$$\begin{aligned} J_0 &= \int_0^{\lambda^{\alpha_2^{(2)} - \alpha_2^{(1)}}} dx_1^{(1)} \int_0^\infty dx_2^{(1)} \dots \\ J_\infty &= \int_{\lambda^{\alpha_2^{(2)} - \alpha_2^{(1)}}}^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \dots \end{aligned} \tag{5.13}$$

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and we begin by studying J_0.

1°) Decomposition of J_0. We write

$$(5.14)\quad J_0 = \int_0^1 dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})(x^{(1)}) + \int_1^{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}} dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})(x^{(1)}) + \int_0^{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}} dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})(x^{(1)}) \times [\exp(P_2(x^{(1)}, \lambda)) - 1].$$

This decomposition can be done provided that we can check that the integral

$$(5.15)\quad \int_0^A dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})(x^{(1)})$$

is convergent. Now, in P_1, we have at least one monomial $-|\alpha|x_1^r x_2^s$ with $r < s$. If we change the variable according to $x_2' = x_1^{r/s}\, x_2$ the integral turns out to be controlled by

$$\int_0^A \frac{dx_1}{x_1^{r/s}} \int_0^\infty \exp(-|\alpha| x_2'^{r/s}) dx_2' < +\infty$$

because $r/s < 1$.

In (5.14), the second integral can be treated as

$$(5.16)\quad \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})(x^{(1)}) = \int_0^\infty dx_2^{(1)} e^{-a(x_1^{(1)}x_2^{(1)})^r} + \int_0^\infty dx_2^{(1)} e^{-P_{12}}[e^{P_1} - 1]$$

where we have written

$$P_{12} = a(x_1^{(1)}x_2^{(1)})^r;$$

the first term of the second member of (5.16) is

$$\frac{1}{x_1^{(1)}} \int_0^\infty dy e^{ay^r}$$

and finally, we obtain from (5.14) and (5.16)

$$(5.17)\quad J_0 = \int_0^1 dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12}) + \log\left(\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}\right) \int_0^\infty dy \exp(P_{12}(1, y)) + \int_1^{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}} dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1) - 1] + \int_0^{\lambda^{\alpha_2^{(2)}-\alpha_1^{(2)}}} dx_1^{(1)} \int_0^\infty \exp(P_1 + P_{12}) \times [\exp(P_2(x_1^{(1)}, \lambda)) - 1].$$

The third term of (5.17) can be treated as follows

$$(5.18)\quad \int_1^{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}} dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1)-1]$$
$$= \int_1^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1)-1]$$
$$- \int_{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}}^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1)-1]$$

provided that the second integral on the right hand side of (5.18) is convergent. To check this last point, we change the F_1-variables $x^{(1)}$ in the F_2-variables $x^{(2)}$, so that

$$x_1^{(1)} = \lambda^{\alpha_1^{(1)}-\alpha_2^{(1)}} x_1^{(2)}$$
$$x_2^{(1)} = \lambda^{\alpha_2^{(1)}-\alpha_2^{(2)}} x_2^{(2)}.$$

Remember here that $\alpha_1^{(1)} + \alpha_2^{(1)} = \alpha_1^{(2)} + \alpha_2^{(2)}$ because the sides of the Newton polygon F_1 and F_2 cut the bissectrice at the same point say (r, r). The last integral in (5.18) becomes

$$(5.19)\quad \int_{\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}}^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12}))[\exp(P_1)-1]$$
$$= \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x^{(2)},\lambda))-1].$$

Now, $P_1(x^{(2)}, \lambda)$ is a sum of monomials of type $x_1^{(2)p} x_2^{(2)q}$ with $p < q$ and

$$P_{12} = -|a|(x_1^{(2)} x_2^{(2)})^r.$$

If we change the variable $x_2^{(2)}$ in $y = x_1^{(2)} x_2^{(2)}$ and use the estimate

$$|\exp(P_1) - 1| \leqq |P_1|$$

we see that the second integral in (5.19) is controlled by a sum of terms of the type

$$\int_1^\infty \frac{dx_1^{(2)}}{(x_1^{(2)})^{1+q-p}} \int_0^\infty \exp(-|a|y^r)dy$$

which is convergent because $q > p$.

If we put together (5.17), (5.18) and (5.19) we obtain

$$(5.20)\quad J_0 = \log(\lambda^{\alpha_2^{(2)}-\alpha_2^{(1)}}) \int_0^\infty dy \exp(P_{12}(1,\lambda))$$
$$+ \int_0^1 dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})$$
$$+ \int_1^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1) - 1]$$
$$- \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x_1^{(2)},\lambda)) - 1]$$
$$+ \int_0^1 dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12}) \exp(P_1(x_1^{(2)},\lambda))[\exp(P_2) - 1].$$

The last term of (5.20) can be written as

$$\int_0^1 dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_2) - 1]$$
$$+ \int_0^1 dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x_1^{(2)},\lambda)) - 1][\exp(P_2) - 1]$$

because we can check easily that the first integral is convergent and finally

$$(5.21)\quad J_0 = \log(\lambda^{(\alpha_2^{(2)}-\alpha_2^{(1)})}) \int_0^\infty dy \exp(P_{12}(1,y))$$
$$+ \int_0^1 dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_1 + P_{12})$$
$$+ \int_1^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12})[\exp(P_1) - 1]$$
$$+ \int_0^1 dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_2) - 1]$$
$$+ \int_0^1 dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x^{(2)},\lambda)) - 1][\exp(P_2) - 1]$$
$$- \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x^{(2)},\lambda)) - 1].$$

2°) Decomposition of J_∞. We come back to J_∞ given in (5.13)

$$J_\infty = \int_{\lambda^{\alpha_1^{(1)}-\alpha_1^{(2)}}}^\infty dx_1^{(1)} \int_0^\infty dx_2^{(1)} \exp(P_{12} + P_1) \exp(P_2(x^{(1)},\lambda))$$

and we change the F_1-variables in the F_2-variables

$$x_1^{(1)} = x_1^{(2)} \lambda^{\alpha_1^{(1)}-\alpha_1^{(2)}}$$
$$x_2^{(1)} = x_2^{(2)} \lambda^{\alpha_2^{(1)}-\alpha_2^{(2)}}$$

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and we use again $\alpha_1^{(1)} + \alpha_2^{(1)} = \alpha_1^{(2)} + \alpha_2^{(2)}$, to obtain

$$(5.22)\quad \begin{aligned} J_\infty &= \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12}) \exp(P_2) \exp(P_1(x^{(2)}, \lambda)) \\ &= \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12}) \exp(P_2) \\ &+ \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12} + P_2)[\exp(P_1(x^{(2)}, \lambda)) - 1] \end{aligned}$$

where the first integral converges by the same kind of reasoning as before. When we add $J_0 + J_\infty$ to obtain $\lambda^{|\alpha^{(1)}|} J$, we see that the last integral of (5.22) combines with the last integral of (5.21) and we obtain exactly the statement of the theorem (up to a change of name of the axis). The only remaining problem is to prove that the remainder term is of lower order; but this remainder is the sum of the two contributions.

$$(5.23)\quad -\int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12})[\exp(P_1(x^{(2)}, \lambda)) - 1]$$

and

$$(5.24)\quad \int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12} + P_2)[\exp(P_1(x^{(2)}, \lambda)) - 1].$$

The first integral (5.23) is dominated by a sum of terms of the type

$$(5.25)\quad \left(\int_1^\infty dx_1^{(2)} \int_0^\infty dx_2^{(2)} \exp(P_{12}) x_1^{(2)p} x_2^{(2)q} \right) \lambda^{1 - l_2(p,q)}$$

where (p, q) is on F_1, so that $p < q$ and $l_2(p, q) > 1$, so that we have seen above that the integral in (5.25) is convergent.

The second integral (5.24) has its integrand dominated by $\exp(P_{12} + P_2)$ and because λ is at a negative power in

$$[\exp(P_1(x^{(2)}, \lambda)) - 1],$$

this quantity tends to 0, so that by Lebesgue theory, (5.24) tends to 0 if $\lambda \longrightarrow \infty$.

6. Proof of theorem 4 in 3 dimensions. The three dimensional case has a different proof from the two dimensional case because it involves other kinds of evaluation of integrals, although the final result is formally the same.

a) *Notations.* We split P in $P_1 + P_2 + P_{12}$ with the convention that the hyperplane on which the face F_1 lies, cuts the axis X_3 at a higher point than the intersection point of the hyperplane on which the face F_2 lies; this means that $\alpha_3^{(1)} < \alpha_3^{(2)}$. As before, we begin by doing the rescaling in F_2-variables, so that

$$J = \lambda^{-|\alpha^{(2)}|} \int_0^\infty \int_0^\infty \int_0^\infty \exp(P_{12} + P_2) \exp(P_1(x^{(2)}, \lambda)) dx^{(2)}$$

and we split the integral J in two pieces

$$\lambda^{|\alpha^{(2)}|} J = J_0 + J_\infty$$

(6.1)
$$J_0 = \int_0^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_2^{(2)} dx_1^{(2)} \times \exp(P_{12} + P_2) \exp(P_1(x^{(2)}, \lambda))$$

$$J_\infty = \int_{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}}^\infty dx_3^{(2)} \int_0^\infty \int \ldots .$$

b) *Splitting of* J_0.

(6.2)
$$J_0 = \int_0^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_2^{(2)} dx_2^{(1)} \exp(P_{12} + P_2) + \int_0^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_2^{(2)} dx_2^{(1)} \times \exp(P_{12} + P_2)[\exp(P_1(x^{(2)}, \lambda)) - 1]$$

provided that the first integral is convergent. Now, $P_{12} + P_2$ is the sum of at least 3 monomials of type

$$-|\alpha| x_1^{(2)r} x_2^{(2)s} x_3^{(2)t}$$

where the points (r, s, t) lie on the F_2-face. Let us define

$$x_3^{(2)} = \zeta_2^{\alpha_2^{(3)}}$$
$$x' = \zeta_2^{-\alpha_1^{(2)}} x_1^{(2)} \qquad y' = \zeta_2^{-\alpha_2^{(2)}} x_2^{(2)}$$

so that because

$$\alpha_1^{(2)} r + \alpha_2^{(2)} s + \alpha_3^{(2)} t = 1,$$

we have

$$x_1^{(2)r} x_2^{(2)s} x_3^{(2)t} = \zeta_2 x'^r y'^s$$

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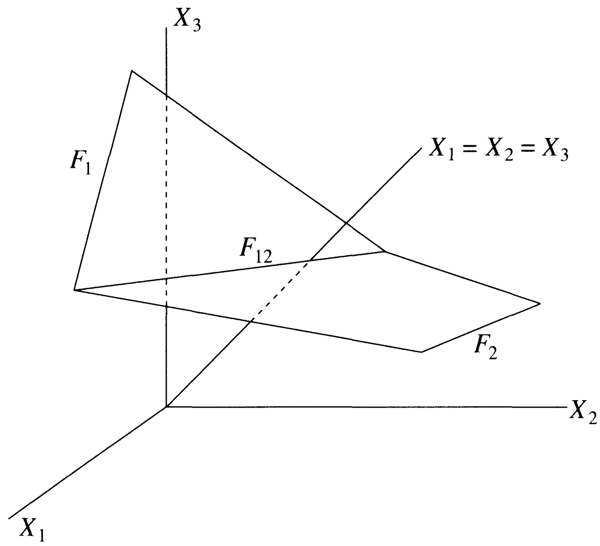

Figure 2

and the first integral of (6.2) becomes

$$\int_0^{\cdots} d\zeta_2 \zeta_2^{|\alpha^{(2)}|-1} \int_0^\infty | \exp(\zeta_2(P_2 + P_{12})(x', y', 1))dx'dy' \tag{6.3}$$

and we now need to study the asymptotic behaviour of the double integral in x', y' for ζ_2 tending to 0. Let us consider the Newton polygon of

$$(P_2 + P_{12})(x', y', 1).$$

It is given by the set of points (r, s) which are the first two exponents of monomials in

$$(P_2 + P_{12})(x^{(2)}),$$

and thus, it is just the projection on the plane X_1X_2 of the face F_2: this projection of the face F_2 is bounded by a polygon as drawn in Figure 3. It is rather clear that the asymptotic behaviour of the auxiliary integral

$$\int_0^\infty \int \exp(\zeta_2(P_2 + P_{12})(x', y', 1))dx'dy' \tag{6.4}$$

is controlled by the edge of this polygon which is as far as possible from 0 and which cuts the bissectrice $X_1 = X_2$. In fact, let

$$\rho_1 X_1 + \rho_2 X_2 = 1$$

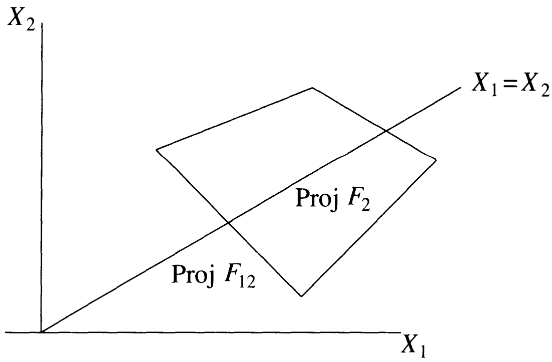

Figure 3

the equation of this edge and let us perform the change of variables

$$x'' = x'\zeta_2^{\rho_1} \quad \text{and} \quad y'' = y'\zeta_2^{\rho_2}.$$

Then the integral (6.4) is controlled by

$$\zeta_2^{-(\rho_1+\rho_2)} \quad \text{or} \quad \zeta_2^{-(\rho_1+\rho_2)} \log \zeta_2$$

(in the first case there is only one side opposite to 0 of the polygon cutting $X_1 = X_2$, in the second case there are two such sides), so that the integral (6.3) is like

$$\text{(6.5)} \qquad \int_0^{\cdots} d\zeta_2 \zeta_2^{|\alpha^{(2)}|+\rho_1+\rho_2-1}$$

with an extra logarithm $\log \zeta_2$.

Now, the side $\rho_1 X_1 + \rho_2 X_2 = 1$ cuts the bissectrice $X_1 = X_2$ at a point (μ_0, μ_0) where $\mu_0 = (\rho_1 + \rho_2)^{-1}$. This side is the projection on the plane X_1X_2 of a side of F_2 and the point (μ_0, μ_0) is the projection of a point (μ_0, μ_0, ν_0) where $\nu_0 < \mu_0$ for obvious reasons (see figure 2). Then

$$\alpha_1^{(2)}\mu_0 + \alpha_2^{(2)}\mu_0 + \alpha_3^{(2)}\mu_0 > (\alpha_1^{(2)} + \alpha_2^{(2)})\mu_0 + \alpha_3^{(2)}\nu_0 = 1 = (\rho_1 + \rho_2)\mu_0$$

and

$$|\alpha^{(2)}| > \rho_1 + \rho_2$$

which implies that (6.5) is convergent at 0, and thus also (6.3) and the splitting (6.2) is legitimate. Because $\alpha_3^{(2)} > \alpha_3^{(1)}$, we obtain

$$(6.6) \quad J_0 = \int_0^1 dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \exp(P_{12} + P_2) + \int_1^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \exp(P_{12} + P_2) + \int_0^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \times \exp(P_{12} + P_2)[\exp(P_1(x^{(2)}, \lambda)) - 1].$$

The first integral does not depend on λ and the second integral of (6.6) is treated as previously by the change of variables

$$(6.7) \quad x_3^{(2)} = x_3'^{\alpha_3^{(2)}}, \quad x_2^{(2)} = x_3'^{\alpha_2^{(2)}} x_2', \quad x_1^{(2)} = x_3'^{\alpha_1^{(2)}} x_1'$$

so that

$$(6.8) \quad \int_1^{\lambda^{\alpha_3^{(2)} - \alpha_3^{(2)}}} dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \exp(P_{12} + P_2) = \alpha_3^{(2)} \int_1^{\lambda^{(\alpha_3^{(2)} - \alpha_3^{(1)})/\alpha_3^{(2)}}} x_3'^{|\alpha^{(2)}| - 1} dx_3' \times \int_0^\infty \int \exp(x_3'(P_2 + P_{12})(x_1', x_2', 1)) dx_1' dx_2'.$$

Now the double integral in the right-hand side of (6.8) is controlled for large x_3' by the projection of the side $F_1 \cap F_2$ on the plane $X_1 X_2$. which has an equation $\sigma_1 X_1 + \sigma_2 X_2 = 1$ and which cuts the bissectrice $X_1 = X_2$ at an interior point. In fact we have

$$(6.9) \quad \int_0^\infty \int \exp(x_3'(P_2 + P_{12})(x_1', x_2', 1)) dx_1' dx_2' = (x_3')^{-(\sigma_1 + \sigma_2)} \int_0^\infty \int \exp(P_{12}(x_1', x_2', 1)) dx_1' dx_2' + \int_0^\infty \int \exp(x_3' P_{12}(x_1' x_2', 1))[\exp(x_3' P_2(x_2', y_2', 1)) - 1].$$

Now the edge $F_1 \cap F_2$ cuts the bissectrice $X_1 = X_2 = X_3$ at the point $(\lambda_0, \lambda_0, \lambda_0)$ and its projection on the plane $X_1 X_2$ cuts the bissectrice $X_1 = X_2$ at the point (λ_0, λ_0), so that

$$(\sigma_1 + \sigma_2)\lambda_0 = a_1^{(2)} \lambda_0 + \alpha_2^{(2)} \lambda_0 + \alpha_3^{(2)} \lambda_0 = 1$$

and $\sigma_1 + \sigma_2 = |\alpha^{(2)}|$, from which we deduce using (6.8) and (6.9), that

$$\int_1^{\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}} dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \exp(P_{12} + P_2)$$
$$= \log\left(\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}\right) \int_0^\infty \int \exp(P_{12}(x_1', x_2', 1)) dx_1' dx_2'$$
$$+ \int_1^{\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}} dx_3^{(2)}$$
$$\times \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} dx_2^{(2)}$$
$$= \log\left(\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}\right) \int_0^\infty \int \exp(P_{12}(x_1', x_2', 1)) dx_1' dx_2'$$
$$+ \int_1^\infty dx_3^{(2)} \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} dx_2^{(2)}$$
$$- \int_{\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}}^\infty dx_3^{(2)}$$
$$\times \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} dx_2^{(2)}$$

provided that the second integral of the last member of (6.10) is convergent. Again this integral is treated by the change of variables (6.7) and is

$$(6.11) \quad \int_1^\infty dx_3^{(2)} \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} dx_2^{(2)}$$
$$= \alpha_3^{(2)} \int_1^\infty x_3'^{|\alpha^{(2)}|-1} dx_3'$$
$$\times \int_0^\infty \int \exp(x_3' P_{12}(x_1', x_2', 1))[\exp(x_3' P_2(x_1', x_2', 1)) - 1] dx_1' dx_2'.$$

We want the asymptotic behaviour of the double integral on the right hand side of (6.11) for large x_3'. This is controlled by the scaling associated to the projection of $F_1 \cap F_2$ on X_1X_2 with the equation

$$\sigma_1 X_1 + \sigma_2 X_2 = 1.$$

Call

$$x_1'' = x_1' x_3'^{\sigma_1}$$
$$x_2'' = x_2' x_3'^{\sigma_2}$$

so that (6.11) becomes (remembering that $\sigma_1 + \sigma_2 = |\alpha^{(2)}|$ as we have already seen)

$$(6.12) \quad \int_1^\infty \frac{dx_3'}{x_3'} \int_0^\infty \int \exp(P_{12}(x_1'', x_2'', 1))$$
$$\times [\exp(P_2(x_1'', x_2'' | x_3')) - 1] dx_1'' dx_2''$$

where $P_2(x_1'', x_2''|x_3')$ denotes a sum of monomials of the type

$$x_1''^r x_2''^s x_3'^{1-(\sigma_1 r+\sigma_2 s)}$$

where (r, s) are the first two coordinates of points of P_2. In particular we can dominate

$$\left|[\exp(P_2(x_1'', x_2''|x_3')) - 1]\right| \leqq \frac{x_1''^r x_2''^s}{x_3'^{\sigma_1 r+\sigma_2 s-1}}$$

and $\sigma_1 r + \sigma_2 s - 1 > 0$ and so the integral (6.12) is convergent.

Moreover, we can treat the last integral in (6.10) going to F_1-variables

$$(6.13)\quad -\int_{\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}}^{\infty} dx_3^{(2)} \times \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1]dx_1^{(2)}dx_2^{(2)}$$
$$= -\int_1^\infty dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_2 x^{(1)}, \lambda)) - 1]dx_1^{(1)}dx_2^{(1)}$$

and we can also treat the last integral in (6.6) by the same rescaling $x^{(2)} \longrightarrow x^{(1)}$

$$(6.14)\quad \int_0^{\lambda^{\alpha_3^{(2)}-\alpha_3^{(1)}}} dx_3^{(2)} \times \int_0^\infty \int \exp(P_{12} + P_2)[\exp(P_1(x^{(2)}, \lambda)) - 1]dx_1^{(2)}dx_2^{(2)}$$
$$= \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] \times \exp(P_2(x^{(1)}, \lambda))dx_1^{(1)}dx_2^{(1)}$$
$$= \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1]dx_1^{(1)}dx_2^{(1)}$$
$$+ \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] \times [\exp(P_2(x^{(1)}, \lambda)) - 1]dx_1^{(1)}dx_2^{(1)}.$$

This last splitting is legitimate provided that the first integral of the last member of (6.14) is convergent. To check this, we define

$$(6.15)\quad x_3^{(1)} = x_3^{\alpha_3^{(1)}} \quad x_1^{(1)} = x_1 x_3^{\alpha_1^{(1)}} \quad x_2^{(1)} = x_2 x_3^{\alpha_2^{(1)}}$$

so that

$$(6.16)\quad \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)})) \times [\exp(P_1(x^{(1)})) - 1]dx_1^{(1)}dx_2^{(1)}$$
$$= \int_0^1 x_3^{|\alpha^{(1)}|-1} dx_3 \int_0^\infty \int \exp(x_3 P_{12}(x_1, x_2, 1)) \times [\exp(x_3 P_1(x_1, x_2, 1)) - 1].$$

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Let us now look at the projection of the face F_1 on the plane X_1X_2; the edge $F_1 \cap F_2$ has a projection whose equation is $\sigma_1 X_1 + \sigma_2 X_2 = 1$; it cuts the bissectrice $X_1 = X_2$ at an interior point and for any point (r, s) which is the first two coordinates of a monomials in P_1, we have

(6.17) $\quad \sigma_1 r + \sigma_2 s < 1.$

Let us now do in the double integral in the right-hand side of (6.16) the rescaling associated to the edge which is the projection of $F_1 \cap F_2$ on the plane X_1X_2. Taking into account the fact that $\sigma_1 + \sigma_2 = |\alpha^{(1)}|$, we see that the second member of (6.16) is controlled by a sum of terms like

$$\int_0^1 dx_3 x_3^{-1+(1-\sigma_1 r-\sigma_2 s)} \int_0^\infty \int \exp(P_{12}(x_1', x_2', 1)) x_1'^{r} x_2'^{s} dx_1' dx_2'$$

and by (6.17) this is convergent (see figure 4).

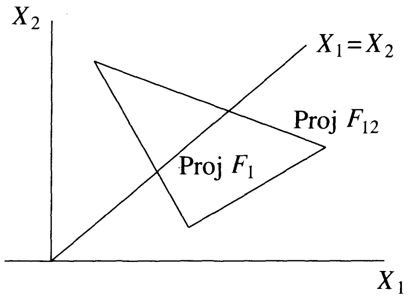

Figure 4

Finally we case combine together (6.6), (6.10), (6.13), (6.14), to get

$$\begin{aligned}
(6.18) \quad J_0 = &\int_0^1 dx_3^{(2)} \int_0^\infty \int dx_1^{(2)} dx_2^{(2)} \exp(P_{12} + P_2) \\
&+ \log\left(\lambda^{(\alpha_3^{(2)} - \alpha_3^{(1)})}\right) \int_0^\infty \int \exp(P_{12}(x_1', x_2', 1)) dx_1' dx_2' \\
&+ \int_1^\infty dx_3^{(2)} \int_0^\infty \int \exp(P_{12}(x^{(2)}))[\exp(P_2(x^{(2)})) - 1] dx_1^{(2)} dx_2^{(2)} \\
&- \int_1^\infty dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_2(x^{(1)}, \lambda)) - 1] dx_1^{(1)} dx_2^{(1)} \\
&+ \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] dx_1^{(1)} dx_2^{(1)} \\
&+ \int_0^1 dx_3^{(1)} \int_0^\infty \int \exp(P_{12}(x^{(1)})) \\
&\times [\exp(P_1(x^{(1)})) - 1][\exp(P_2(x^{(1)}, \lambda)) - 1] dx_1^{(1)} dx_2^{(1)}.
\end{aligned}$$

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We also notice that the λ-dependent terms in (6.18) (except the logarithmic term of course) tend to zero if λ tends to infinity; this can be checked easily because the fourth integral comes, via (6.13), from the last integral of (6.10) which is a remainder of an integral convergent at $x_3^{(2)} = \infty$. And the last integral in (6.18) is dominated by

$$2|\exp(P_{12})[\exp(P_1(x^{(1)})) - 1]|$$

with is convergent (because the fifth integral in (6.18) is convergent) and the integrand tends point wise to 0 if $\lambda \longrightarrow \infty$ because in

$$\exp(P_2(x^{(1)}, \lambda)) - 1$$

λ is at a negative power, and by Lebesgue theory the last integral in (6.18) tends to 0.

c) *Decomposition of* J_∞. We come back to J_∞ given in (6.1) and rescale it by changing the F_2-variables to the F_1-variables so that

$$(6.19) \quad \begin{aligned} J_\infty = & \int_1^\infty dx_3^{(1)} \int_0^\infty \int dx_1^{(1)} dx_2^{(1)} \exp(P_{12}(x^{(1)})) \\ & \times [\exp(P_1(x^{(1)})) - 1] \exp(P_2(x^{(1)}, \lambda)) \\ = & \int_1^\infty dx_3^{(1)} \int_0^\infty \int dx_1^{(1)} dx_2^{(1)} \exp(P_{12}(x^{(1)}))[\exp(P_1(x^{(1)})) - 1] \\ & + \int_1^\infty dx_3^{(1)} \int_0^\infty \int dx_1^{(1)} dx_2^{(1)} \exp(P_{12}(x^{(1)})) \\ & \times [\exp(P_1(x^{(1)})) - 1][\exp(P_2(x^{(1)}, \lambda)) - 1] \end{aligned}$$

provided that the first or the second integral is convergent. But we have already seen that

$$(6.20) \quad \int_1^\infty dx_3^{(1)} \int_0^\infty \int dx_1^{(1)} dx_2^{(1)} \exp(P_{12}(x^{(1)}))[\exp(P_2(x^{(1)}, \lambda)) - 1]$$

is convergent (this is the fourth integral in (6.18)) so that the second integral in (6.19) is dominated by the convergent integral (6.20). We have also checked previously that (6.20) tends to 0 if λ tends to infinity so that the second integral in (6.19) tends to 0 if λ tends to infinity.

d) *End of the proof of Theorem* 4 ($n = 3$). We now put together $J_0 + J_\infty$ given by (6.18) and (6.19). This gives exactly the formula given by (5.9) in the statement of Theorem 4 with the remainder (5.10) which is of lower order.

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