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A RATIO LIMIT THEOREM FOR APPROXIMATE MARTINGALES

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1. Introduction. It has been proved [**3**, p. 630] that the martingale convergence theorem obtained by Andersen and Jessen [**1**, p. 5] follows from the classical theory developed by Doob. By using some results of Yosida and Hewitt [**9**] on finitely additive set functions, Johansen and Karush [**7**] proved that the identification of the limit function as a derivative in the approach of Andersen and Jessen can be obtained in the general case. In this paper we sharpen the methods of Andersen and Jessen to obtain a ratio limit theorem for "approximate martingales". For related results obtained by using a different approach based on maximal inequalities the reader is referred to a recent paper of Chatterji [**2**].

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2. Derivatives. Let $(S, \mathscr{B})$ be a measurable space and let $\phi[\mu]$ be a bounded signed (nonnegative) measure on $(S, \mathscr{B})$. Write the Lebesgue decomposition of ϕ with respect to μ as $\phi = \phi_c + \phi_s$ where ϕ_c is absolutely continuous with respect to μ and ϕ_s is singular with respect to μ. Following Andersen and Jessen [**1**, p. 4], we will say that an extended real valued function f on S is an AJ-derivative of ϕ with respect to μ if f is $\mathscr{B}$-measurable, μ-integrable and for every $A \in \mathscr{B}$,

$$\phi_c(A) = \int_A f d\mu$$

$$\phi_s^+(A) = \phi\{A \cap (f = +\infty)\} \tag{2.1}$$

$$-\phi_s^-(A) = \phi\{A \cap (f = -\infty)\}$$

where ϕ_s^+ and ϕ_s^- denote the positive and negative parts of ϕ_s ($\phi_s = \phi_s^+ - \phi_s^-$). We will say that an extended real valued function f on S is a RN-derivative (RN for Radon-Nikodym) of ϕ with respect to μ if f is $\mathscr{B}$-measurable, μ-integrable and the first equation in (2.1) holds for all $A \in \mathscr{B}$. The following characterization of an AJ-derivative is given in [**1**, p. 4].

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PROPOSITION 2.1 *A $\mathscr{B}$-measurable function f is an AJ-derivative of ϕ with respect to μ if and only if for every $a \in (-\infty, \infty)$ and $A \in \mathscr{B}$,*

$$\phi\{A \cap (f \leqq a)\} \leqq a\, \mu\{A \cap (f \leqq a)\},$$
$$\phi\{A \cap (f \geqq a)\} \geqq a\, \mu\{A \cap (f \geqq a)\}.$$

3. Finitely additive set functions. Let $\mathscr{A} \subset \mathscr{B}$ be a field and let $\tilde{\phi}$ be a finitely additive nonnegative set function on $(S, \mathscr{A})$ with $\tilde{\phi}(S) < \infty$. If $A \subset S$, let

$$\phi(A) = \inf \sum_{k=1}^{\infty} \tilde{\phi}(A_k)$$

where the infimum is taken over all sequences (or disjoint sequences) $\{A_k\}$ of sets in $\mathscr{A}$ whose union covers A. The set function ϕ is called the Caratheodory measure on S generated by $\tilde{\phi}$ and ϕ is a σ-additive measure on the smallest σ-field $\sigma(\mathscr{A})$ containing $\mathscr{A}$ [**8**, p. 67, Theorem 5.4]. In fact ϕ is the largest measure on $\sigma(\mathscr{A})$ which is dominated on $\mathscr{A}$ by $\tilde{\phi}$. The following lemma was pointed out to the author by Professor M. Sion.

LEMMA 3.1. *Let $\epsilon > 0$ and $A \in \sigma(\mathscr{A})$ be given. There exists a set $B \in \mathscr{A}$ such that*

$$|\phi(A \cap C) - \tilde{\phi}(B \cap C)| < \epsilon$$

for every $C \in \mathscr{A}$.

Proof. Let $\{A_k\}$ be a disjoint sequence of sets in $\mathscr{A}$ whose union covers A and such that

$$\sum_{k=1}^{\infty} \tilde{\phi}(A_k) < \phi(A) + \epsilon/3.$$

Choose N such that $\sum_{k=N+1}^{\infty} \tilde{\phi}(A_k) < \epsilon/3$ and let $B = \bigcup_{k=1}^{n} A_k$. If B' denotes the complement of B and $C \in \mathscr{A}$, then

$$\begin{aligned}|\phi(A \cap C) - \tilde{\phi}(B \cap C)| &\leqq |\phi(A \cap B \cap C) - \tilde{\phi}(B \cap C)| + \phi(A \cap B' \cap C)\\ &< \tilde{\phi}(B \cap C) - \phi(A \cap B \cap C) + \epsilon/3.\end{aligned}$$

If $\tilde{\phi}(B \cap C) - \phi(A \cap B \cap C) > 2\epsilon/3$, then

$$\begin{aligned}\phi(A) &= \phi(A \cap B \cap C) + \phi(A \cap B \cap C') + \phi(A \cap B')\\ &< \tilde{\phi}(B \cap C) - 2\epsilon/3 + \tilde{\phi}(B \cap C') + \epsilon/3\\ &= \tilde{\phi}(B) - \epsilon/3.\end{aligned}$$

Hence $\tilde{\phi}(B) > \phi(A) + \epsilon/3$. On the other hand,

$$\tilde{\phi}(B) = \sum_{k=1}^{N} \tilde{\phi}(A_k) \leqq \sum_{k=1}^{\infty} \tilde{\phi}(A_k) < \phi(A) + \epsilon/3.$$

This contradiction shows that

$$\tilde{\phi}(B \cap C) - \phi(A \cap B \cap C) \leqq 2\epsilon/3$$

which in turn shows that

$$|\phi(A \cap C) - \tilde{\phi}(B \cap C)| < \epsilon$$

and the proof is complete.

If we now drop the assumption that $\tilde{\phi}$ is nonnegative but assume that $\tilde{\phi}$ is bounded (there exists a constant $M > 0$ with $|\tilde{\phi}(A)| \leqq M$ for every $A \in \mathscr{A}$), then

$$\tilde{\phi}^+(A) = \sup \tilde{\phi}(B),$$

where the supremum is taken over all sets $B \in \mathscr{A}$ with $B \subset A$, defines a finitely additive nonnegative set function on $(S, \mathscr{A})$ with $\phi^+(S) < \infty$. If $\tilde{\phi}^- = (-\tilde{\phi})^+$, then $\tilde{\phi} = \tilde{\phi}^+ - \tilde{\phi}^-$ and we define the Caratheodory measure ϕ generated by $\tilde{\phi}$ to be $\phi^+ - \phi^-$ where $\phi^+[\phi^-]$ is the Caratheodory measure generated by $\tilde{\phi}^+[\tilde{\phi}^-]$. It is clear from the proof that Lemma 3.1 remains valid in the present situation.

4. The convergence theorem. Let $\{\mathscr{B}_n\}$ be an increasing sequence of σ-fields contained in $\mathscr{B}$. For each n let ϕ_n be a bounded signed measure on $(S, \mathscr{B}_n)$.

Definition 4.1. The collection $\{\phi_n, \mathscr{B}_n\}$ is called an approximate projective system of (signed) measures if for every $\epsilon > 0$ there is an integer N such that whenever $N \leqq n_1 \leqq \ldots \leqq n_M$ and $A_1, \ldots, A_M$ are disjoint sets with $A_k \in \mathscr{B}_{n_k}$, then

$$\left| \sum_{k=1}^{M} \phi_{n_k}(A_k) - \phi_{n_M}\left(\sum_{k=1}^{M} A_k \right) \right| < \epsilon. \tag{4.1}$$

LEMMA 4.2. *If $\{\phi_n, \mathscr{B}_n\}$ is approximate projective system of measures, then for $A \in \bigcup_{n=1}^{\infty} \mathscr{B}_n \equiv \mathscr{A}$,*

$$\lim_{n \to \infty} \phi_n(A) = \tilde{\phi}(A)$$

exists uniformly in the following sense: for every $\epsilon > 0$ there is an integer N such that

$$|\phi_n(A) - \tilde{\phi}(A)| < \epsilon \tag{4.2}$$

whenever $n \geqq N$ and $A \in \mathscr{B}_n$.

Proof. For $\epsilon > 0$ given, choose N as in Definition **4.1.** It follows that

$$|\phi_m(A) - \phi_n(A)| < \epsilon$$

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whenever $N \leqq m \leqq n$ and $A \in \mathscr{B}_m$. Hence $\lim_{n\to\infty} \phi_n(A)$ exists for every $A \in \mathscr{A}$ and

$$|\phi_n(A) - \tilde{\phi}(A)| \leqq \epsilon$$

if $N \leqq n$ and $A \in \mathscr{B}_n$. The proof is complete.

The set function $\tilde{\phi}$ of Lemma 4.2 is finitely additive on $\mathscr{A}$. If $\tilde{\phi}$ is bounded we let ϕ denote the Caratheodory measure generated by $\tilde{\phi}$. ϕ is a bounded signed measure on $\sigma(A) \equiv \mathscr{B}_\infty$ in this case.

THEOREM 4.3. *Let* $\{\phi_n, \mathscr{B}_n\}$ $[\{\mu_n, \mathscr{B}_n\}]$ *be an approximate projective system of bounded signed [bounded nonnegative] measures. Assume that the finitely additive set function* $\tilde{\phi}$ *of Lemma 4.2 is bounded. Let* f_n *be an AJ-derivative of* ϕ_n *with respect to* μ_n *on* $(S, \mathscr{B}_n)$. *The functions*

$$f^- = \overline{\lim_n} f_n \quad \text{and} \quad f_- = \underline{\lim_n} f_n$$

are both AJ-derivatives of ϕ *with respect to* μ *on* $(S, \mathscr{B}_\infty)$ *where* $\phi[\mu]$ *is the Caratheodory measure generated by* $\tilde{\phi}[\tilde{\mu}]$. *In particular, if* f_n *is only a RN-derivative of* ϕ_n *with respect to* μ_n *on* $(S, \mathscr{B}_n)$ *and* f *is a RN-derivative of* ϕ *with respect to* μ *on* $(S, \mathscr{B}_\infty)$, *then*

$$\lim_{n\to\infty} f_n = f$$

μ*-almost everywhere on* $(S, \mathscr{B}_\infty)$.

We will call $\{f_n, \mathscr{B}_n\}$ an approximate martingale.

Proof. The last statement concerning *RN*-derivatives follows easily from the corresponding result concerning *AJ*-derivatives. According to Proposition 2.1, we must show that if $a \in (-\infty, \infty)$ and $A \in \mathscr{B}_\infty$, then

$$\phi\{A \cap (f_- \leqq a)\} \leqq a\, \mu\{A \cap (f_- \leqq a)\}, \tag{4.3}$$

$$\phi\{A \cap (f_- \geqq a)\} \geqq a\, \mu\{A \cap (f_- \geqq a)\}, \tag{4.4}$$

$$\phi\{A \cap (f^- \leqq a)\} \leqq a\, \mu\{A \cap (f^- \leqq a)\}, \tag{4.5}$$

$$\phi\{A \cap (f^- \geqq a)\} \geqq a\, \mu\{A \cap (f^- \geqq a)\}. \tag{4.6}$$

The inequality (4.3) implies (4.5) and (4.6) implies (4.4). We prove only (4.3) since the proof of (4.6) is similar.

Let $\epsilon > 0$ and $A \in \mathscr{B}_\infty$ be given. By Lemma 3.1 we may choose

$$B \in \mathscr{A} \left(= \bigcup_{n=1}^{\infty} \mathscr{B}_n\right)$$

such that

$$|\phi(A \cap C) - \tilde{\phi}(B \cap C)| < \epsilon,$$
$$|\mu(A \cap C) - \tilde{\mu}(B \cap C)| < \epsilon$$

for every $C \in \mathscr{A}$. Choose N such that $B \in \mathscr{B}_N$ and (4.1) and (4.2) hold for $(\phi_n, \tilde{\phi}, \phi)$ and $(\mu_n, \tilde{\mu}, \mu)$. Let $b > a$ and define $D_j = \{f_j < b\}$. When we write

$$\lim_i \lim_k \quad \text{or} \quad \overline{\lim_i}\, \overline{\lim_k}$$

in the calculation below it will be understood that $k \geqq i$ and $i \geqq N$. We have

$$\begin{aligned}
\phi\left\{A \cap \left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} D_j\right)\right\} &= \lim_i \lim_k \phi\left\{A \cap \left(\bigcup_{j=i}^{k} D_j\right)\right\} \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[\epsilon + \tilde{\phi}\left\{B \cap \left(\bigcup_{j=i}^{k} D_j\right)\right\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[2\epsilon + \phi_k\left\{B \cap \left(\bigcup_{j=i}^{k} D_j\right)\right\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[3\epsilon + \sum_{l=i}^{k} \phi_l\{B \cap D_i' \cap \ldots \cap D_{l-1}' \cap D_l\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[3\epsilon + b\sum_{l=i}^{k} \mu_l\{B \cap D_i' \cap \ldots \cap D_{l-1}' \cap D_l\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[3\epsilon + |b|\epsilon + b\mu_k\left\{B \cap \left(\bigcup_{j=1}^{k} D_j\right)\right\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[3\epsilon + 2|b|\epsilon + b\tilde{\mu}\left\{B \cap \left(\bigcup_{j=1}^{k} D_j\right)\right\}\right] \\
&\leqq \overline{\lim_i}\,\overline{\lim_k}\left[3\epsilon + 3|b|\epsilon + b\mu\left\{A \cap \left(\bigcup_{j=i}^{k} D_j\right)\right\}\right] \\
&= 3\epsilon(1 + |b|) + b\mu\left\{A \cap \left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} D_j\right)\right\}.
\end{aligned}$$

Letting ϵ decrease 0 and then b decrease to a, we obtain (4.3) and the proof is complete.

Doob's classical L^1-bounded submartingale convergence theorem follows from Theorem 4.3. To see this, let μ be a bounded nonnegative measure on $(S, \mathscr{B}_\infty)$, μ_n be the restriction of μ to $\mathscr{B}_n$ and

$$\phi_n(A) = \int_A f_n d\mu_n \leqq \int_A f_{n+1} d\mu_{n+1} = \phi_{n+1}(A)$$

if $A \in \mathscr{B}_n$ where

$$\sup \int |f_n| d\mu < \infty.$$

We show that $\{\phi_n, \mathscr{B}_n\}$ is an approximate projective system of measures. If $L = \lim_{n\to\infty}\phi_n(S)$ and $\epsilon > 0$ is given, choose N such that $0 \leqq L - \phi_n(S) < \epsilon$

if $n \geqq N$. We have (under the notation and assumptions of Definition 4.1)

$$0 \leqq \sum_{k=1}^{k} [\phi_{n_M}(A_k) - \phi_{n_k}(A_k)]$$
$$= \phi_{n_M}\left(\bigcup_{k=1}^{M} A_k\right) - \sum_{k=1}^{M} \phi_{n_k}(A_k)$$
$$= \sum_{k=1}^{M-1} [\phi_{n_{k+1}}(A_1 \cup \ldots \cup A_k) - \phi_{n_k}(A_1 \cup \ldots \cup A_k)]$$
$$\leqq \sum_{k=1}^{M-1} [\phi_{n_{k+1}}(S) - \phi_{n_k}(S)]$$
$$= \phi_{n_M}(S) - \phi_{n_1}(S)$$
$$< \epsilon \quad \text{if} \quad n \geqq N.$$

It remains only to verify that $\tilde{\phi}$ is bounded and this follows from the hypothesis that $\sup \int |f_n| d\mu < \infty$.

We have the following ratio limit theorem for approximate martingales.

THEOREM 4.4. *Let $\{\phi_n, \mathscr{B}_n\}$ and $\{\mu_n, \mathscr{B}_n\}$ be as in Theorem 4.3. For each n let ν_n be a bounded nonnegative measure on $\mathscr{B}_n$. Assume that μ_n is absolutely continuous with respect to ν_n and let $g_n[h_n]$ be a RN-derivative of $\phi_n[\mu_n]$ with respect to ν_n on $(S, \mathscr{B}_n)$. Then*

$$\lim_{n\to\infty} g_n/h_n = f$$

μ-almost everywhere where f is a RN-derivative of ϕ with respect to μ on $(S, \mathscr{B}_\infty)$.

Note that we do not make the assumption that $h_n > 0$ ν_n-almost everywhere. The theorem states that for μ-almost every $s \in S$, g_n/h_n is eventually well defined and converges to a finite limit.

Proof. This theorem will follow immediately from Theorem 4.3 as soon as we show that g_n/h_n is a *RN*-derivative of ϕ_n with respect to μ_n on $\mathscr{B}_n$. For notational simplicity we drop the subscript n on ϕ_n, μ_n, ν_n and $\mathscr{B}_n$ in the following calculation. Let

$$\phi = \phi_{c(\nu)} + \phi_{s(\nu)}$$
$$\phi = \phi_{c(\mu)} + \phi_{s(\mu)}$$

be the Lebesgue decompositions of ϕ with respect to ν and μ respectively. Let S_ν and S_μ be $\mathscr{B}$-measurable sets such that

$$\phi_{c(\nu)}(A) = \phi(A \cap S_\nu),\ \nu(S_\nu') = 0,$$
$$\phi_{c(\mu)}(A) = \phi(A \cap S_\mu),\ \mu(S_\mu') = 0.$$

Now $\mu(S_\nu') = 0$ (by absolute continuity) and hence $\mu(S_\nu' \cup S_\mu') = 0$ and $\mu(S_\nu \cap S_\mu) = \mu(S)$. Let $S_+ = \{s : h(s) > 0\}$.

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We have

$$\mu(S_+') = \int_{\{h=0\}} h d\nu = 0$$

and hence $\mu(S_\nu \cap S_\nu \cap S_+) = \mu(S)$. It suffices to show that if f is a RN-derivative of ϕ with respect to μ on $(S, \mathscr{B})$, then

$$g/h = f \quad (\text{or } g = fh) \tag{4.7}$$

μ-almost everywhere on $(S_\nu \cap S_\mu \cap S_+)$. Since μ is absolutely continuous with respect to ν it suffices to verify that (4.7) holds ν-almost everywhere on $(S_\nu \cap S_\mu \cap S_+)$. If $A \in \mathscr{B}$ and $A \subset S_\nu \cap S_\mu \cap S_+$, then

$$\int_A g d\nu = \phi_{c(\nu)}(A) = \phi_{c(\nu)}(A \cap S_\nu) = \phi(A)$$

$$= \phi_{c(\mu)}(A \cap S_\mu) = \phi_{c(\mu)}(A) = \int_A f d\mu$$

$$= \int_A f h d\nu$$

and our result follows.

We remark that Theorem 4.4 implies as a special case that if $\{g_n\}[\{h_n\}]$ is an L^1-bounded submartingale [nonnegative submartingale] on $(S, \mathscr{B}, \nu)$, then $\lim g_n/h_n$ exists μ-almost everywhere where μ is the measure determined by the $d\mu_n = h_n \cdot d\nu$. Ratio limit theorems in the case where g_n and h_n are excessive functions composed with a Markov process have been investigated by several authors (see, for example, [**4**], [**5**] and [**6**]). Theorem 4.4 may be viewed as an abstract ratio limit theorem when no underlying Markov process is available.

5. Convergence in measure. It is possible to weaken the assumptions of Theorems 4.3 and 4.4 if we do not insist on convergence μ-almost everywhere.

THEOREM 5.1. *Let $\{\phi_n, \mathscr{B}_n\}$ $[\{\mu_n, \mathscr{B}_n\}]$ be a system of bounded signed [nonnegative measures] which satisfy the conclusions of Lemma* 4.2. *Assume that $\tilde{\phi}$ is bounded. If f_n is an AJ-derivative of ϕ_n with respect to μ_n on $(S, \mathscr{B}_n)$, then*

$$\lim_{n\to\infty} f_n = f$$

in μ-measure where f is an AJ-derivative of ϕ with respect to μ. The same statement holds if f_n and f are only RN-derivatives.

Proof. We leave the verification of the last statement to the reader. To prove the first result it suffices to show that if $-\infty < a < b < \infty$, then

(5.1) $$\lim_{n\to\infty} \mu\{(f \leqq a) \cap (b \leqq f_n)\} = 0,$$

(5.2) $$\lim_{n\to\infty} \mu\{(f_n \leqq a) \cap (b \leqq f)\} = 0.$$

We prove (5.1) only since the proof of (5.2) is similar. Let $\epsilon > 0$ be given and let $(A = (f \leqq a)$. Choose $B \in \mathscr{A}$ according to Lemma 3.1. Choose N such that $B \in \mathscr{B}_N$ and (4.2) is satisfied for $(\phi_n, \tilde{\phi})$ and $(\mu_n, \tilde{\mu})$ if $n \geqq N$. If $n \geqq N$, then

$$\begin{aligned}\phi\{(f \leqq a) \cap (b \leqq f_n)\} &\geqq \tilde{\phi}\{B \cap (b \leqq f_n)\} - \epsilon \\ &\geqq \phi_n\{B \cap (b \leqq f_n)\} - 2\epsilon \\ &\geqq b\mu_n\{B \cap (b \leqq f_n)\} - 2\epsilon \\ &\geqq b\tilde{\mu}\{B \cap (b \leqq f_n)\} - |b|\epsilon - 2\epsilon \\ &\geqq b\mu\{B \cap (b \leqq f_n)\} - 2|b|\epsilon - 2\epsilon.\end{aligned}$$

On the other hand,

$$\phi\{(f \leqq a) \cap (b \leqq f_n)\} \leqq a\mu\{(f \leqq a) \cap (b \leqq f_n)\}.$$

Hence

$$(b - a)\,\mu\{(f \leqq a) \cap (b \leqq f_n)\} \leqq 2\epsilon(|b| + 1)$$

if $n \geqq N$ and the proof is complete.

We remark that if $(S, \mathscr{B}, \mu)$ is a measure space with $\mu(S) < \infty$ and $\{f_n\}$ is a sequence of $\mathscr{B}$-measurable functions converging to 0 in $L^1(S, \mathscr{B}, \mu)$ but not μ-almost everywhere, then $\{\phi_n\}$ satisfies the assumptions of theorem 5.1 but not those of Theorem 4.3 where

$$\phi_n(A) = \int_A f_n d\mu \quad \text{if} \quad A \in \mathscr{B}$$

$(\mathscr{B}_n = \mathscr{B}$ and $\mu_n = \mu$ for $n \geqq 1)$.

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