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SOME APPLICATIONS OF ARTAMONOV-QUILLEN-SUSLIN THEOREMS TO METABELIAN INNER RANK AND PRIMITIVITY

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ABSTRACT. For any variety $\mathcal{V}$ of groups, the relative inner rank of a given group G is defined to be the maximal rank of the $\mathcal{V}$-free homomorphic images of G. In this paper we explore metabelian inner ranks of certain one-relator groups. Using the well-known Quillen-Suslin Theorem, in conjunction with an elegant result of Artamonov, we prove that if r is any "Δ-modular" element of the free metabelian group M_n of rank $n \geq 2$ then the metabelian inner rank of the quotient group $M_n/\langle r\rangle$ is at most $[n/2]$. As a corollary we deduce that the metabelian inner rank of the (orientable) surface group of genus k is precisely k. This extends the corresponding result of Zieschang about the absolute inner ranks of these surface groups. In continuation of some further applications of the Quillen-Suslin Theorem we give necessary and sufficient conditions for a system $g = (g_1, \ldots, g_k)$ of k elements of a free metabelian group M_n, $k \leq n$, to be a part of a basis of M_n. This extends results of Bachmuth and Timoshenko who considered the cases $k = n$ and $k \leq n-3$ respectively.

1. **Introduction.** The *inner rank* $\mathrm{Ir}(G)$ of an aribtrary group G is defined to be the maximal rank of the free homomorphic images of G. This concept is dual to the *outer rank* $d(G)$ of G which is the minimal rank of free groups which have G as their homomorphic image, and one has the inequality $\mathrm{Ir}(G) \leq d(G)$ (see, Lyndon and Schupp [11, Chapter I]). Computation of the inner rank of a given group is, in general, a very difficult problem. Among the most general results is the following theorem due to Jaco [8]: $\mathrm{Ir}(G_1 * G_2) = \mathrm{Ir}(G_1) + \mathrm{Ir}(G_2)$. Restricting to the study of one-relator groups, some of what is known about the inner rank of $G = \langle x_1, \ldots, x_n; r\rangle$ may be summarized as follows (see [11] for proofs): (i) if $r = [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$, $n = 2k$, then $\mathrm{Ir}(G) = k$ (Zieschang [23]); (ii) if $r = x_1^N \cdots x_n^N$, $N \geq 2$, then $\mathrm{Ir}(G) = [n/2]$, the greatest integer value of $n/2$ (Lyndon [10], see Zieschang [23] for the case $N = 2$); (iii) $\mathrm{Ir}(G) = n-1$ if and only if r lies in the normal closure of a primitive element of $F = \langle x_1, \ldots x_n\rangle$ (Steinberg [16], [17]); (iv) if $r = s(x_1, \ldots, x_{n-1})x_n^k$, $k \geq 2$, is such that s is neither a proper power nor a primitive in $F = \langle x_1, \ldots, x_n\rangle$ then $\mathrm{Ir}(G) < n-1$ (Baumslag and Steinberg [4]); (v) if $r = r(x_1, \ldots, x_n) = \prod_{i<j}[x_i^{a_{ij}}, x_j^{a_{ij}}]^{b_{ij}}$, with a_{ij} all distinct 2-powers and $a_{ij}b_{ij} = N$, for some sufficiently large 2-power N, then $\mathrm{Ir}(G) = 1$ (Stallings [15]). Examples of n-generator one-relator groups with the prescribed inner rank $k = 1, \ldots, n-1$, are easily found. For instance the group $G = \langle x_1, \ldots, x_n; r\rangle$ with the Stallings'relator

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$r = r(x_1, \ldots, x_{n-k+1})$ on the first $n-k+1$ generators, is the free product $\langle x_1, \ldots, x_{n-k+1}; r\rangle * \langle x_{n-k+2}, \ldots, x_n\rangle$ and hence, using results of Stallings and Jaco above, $\mathrm{Ir}(G) = 1+(k-1) = k$.

If a group G maps onto F_n then it maps onto F_n/V for any fully invariant subgroup V of F_n. Thus, for any group G and any variety $\mathcal{V}$ of groups, we can define its *relative inner rank* $\mathrm{Ir}_{\mathcal{V}}(G)$ to be the maximal rank of the $\mathcal{V}$-free homomorphic images of G. It follows that $\mathrm{Ir}(G) \leq \mathrm{Ir}_{\mathcal{V}}(G)$ for any variety $\mathcal{V}$. Using the well-known Quillen-Suslin Theorem ([13], [18]) in conjunction with a result of Artamonov ([1], [2]), in this paper we explore metabelian inner ranks of certain one-relator groups. Specifically, we prove that if r is any "Δ-modular" element of the free metabelian group M_n, then the metabelian inner rank of the quotient group $M_n/\langle r\rangle$ is at most $[n/2]$ (Theorem 4.2). We deduce that the metabelian inner rank of the (orientable) surface group of genus n is precisely n (Corollary 4.3). The corresponding result of Zieschang ((i) above) about the absolute inner ranks of these surface groups follows as a consequence (Corollary 4.4).

In continuation of some further applications of the Quillen-Suslin Theorem, in Section 5 we give necessary and sufficient conditions for a system $\mathbf{g} = (g_1, \ldots, g_k)$ of k elements of a free metabelian group M_n, $k \leq n$, to be a part of a basis of M_n (Theorem 5.3). This extends results of Bachmuth [3] and Timoshenko [22] for the cases $k = n$ and $k \leq n-3$ respectively (see also Roman'kov [14]).

2. **Some results of Suslin and Artamonov.** Let Λ be a commutative ring with 1. A vector $\nu = (v_1, \ldots, v_m) \in \Lambda^m$ is said to be *unimodular* if $\mathrm{id}(\nu) = \mathrm{ideal}\{v_1, \ldots, v_m\} = \Lambda$. A well-known result of Suslin is the following:

THEOREM 2.1 (SUSLIN [18]). *If* $\Lambda = \Lambda_n = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $n \geq 1$, *is the Laurent polynomial ring then* $\mathrm{GL}_m(\Lambda)$, $m \geq 1$, *acts transitively on the set of all unimodular vectors* $(v_1, \ldots, v_m) \in \Lambda^m$. *Equivalently, when* Λ *is a Laurent polynomial ring then every vector* $\nu = (v_1, \ldots, v_m)$ *can be transformed to the base vector* $e_1 = (1, 0, \ldots, 0)$ *upon multiplication by a suitable matrix from* $\mathrm{GL}_m(\Lambda)$.

Let $\nu \in \Lambda^m$ be a vector. Following Artamonov [1] we call ν *Δ-modular* if $\mathrm{id}(\nu) = \Delta = \Delta_n = \mathrm{id}\{(x_1 - 1), \ldots, (x_n - 1)\}$, the fundamental ideal of the Laurent polynomial ring Λ. The standard Δ-modular vector is, of course, $X = (x_1 - 1, \ldots, x_n - 1, 0, \ldots, 0)$. Using Suslin's theorem, Artamanov has proved the following result (see also [6], [9]):

THEOREM 2.2 (ARTAMANOV [1]). *The group* $\mathrm{GL}_m(\Lambda_n)$ *acts transitively on all Δ-modular vectors* $(v_1, \ldots, v_m) \in \Lambda_n^m$.

Let $k \in \{1, \ldots, n\}$ be arbitrary but fixed and denote by I_k the ideal generated by $\{x_{k+1} - 1, \ldots, x_n - 1\}$. Let $G = \mathrm{GL}_m(\Lambda_n, I_k)$, $m \geq n$, be the congruence subgroup of $\mathrm{GL}_m(\Lambda)$ with respect to the ideal I_k and the subgroup

$$H = \begin{bmatrix} \mathrm{GL}_k(\Lambda) & * \\ 0 & \mathrm{GL}_{m-k}(\Lambda) \end{bmatrix}.$$

Then the above result of Artamonov is the case $k = n$ of the following extended version.

THEOREM 2.3 (ARTAMONOV [2]). *The congruence subgroup* $G = \mathrm{GL}_m(\Lambda_n, I_k)$ *acts transitively on the set of all* Δ*-modular vectors* $\nu = (v_1, \ldots, v_m, 0, \ldots, 0)$, *such that* $\nu \equiv X \pmod{I_k}$ *where* $X = (x_1 - 1, \ldots, x_n - 1, 0, \ldots, 0)$ *is the standard* Δ*-modular vector.*

Viewing Λ_n^n as the free Λ-module with an arbitrary but fixed basis $\{e_1, \ldots, e_n\}$, let $l\colon \Lambda_n^n \longrightarrow \Delta_n$ be the Λ-linear functional defined by: $l(e_i) = x_i - 1$. Then, by the well-known Magnus embedding (see, for instance, [7] Chapter I), the free metabelian group M_n of rank n is freely generated by the matrices

$$X_i = \begin{bmatrix} x_i & e_i \\ 0 & 1 \end{bmatrix}, \quad 1 \le i \le n. \tag{1}$$

Moreover, the matrix

$$X = \begin{bmatrix} x & \nu \\ 0 & 1 \end{bmatrix}, \quad \nu \in \Lambda^n,$$

belongs to M_n if and only if x belongs to $U = U_n$, the multiplicative subgroup in Λ generated by x_i, $1 \le i \le n$, and ν satisfies the *fundamental relation*: $l(\nu) = x - 1$.

Now let $\varphi\colon M_n \longrightarrow M_k$, $k \le n$, be a homomorphism between free metabelian groups M_n and M_k with basis $\{X_1, \ldots, X_n\}$ and $\{X_1, \ldots, X_k\}$ respectively. Then it is easy to see (see [3], [1], [2]) that φ defines a ring homomorphism "$\lambda \longrightarrow \bar{\lambda}$" between Λ_n and Λ_k which maps U_n to U_k, and also defines a map $\tilde{\varphi}\colon \Lambda_n^n \longrightarrow \Lambda_k^n$, such that

$$\varphi \begin{bmatrix} x & \nu \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{x} & \nu\tilde{\varphi} \\ 0 & 1 \end{bmatrix}. \tag{2}$$

As an important example we consider the *standard* epimorphism $\pi\colon M_n \longrightarrow M_k$ ($\le M_n$), which fixes $X_1, \ldots, X_k$ and maps other generators to 1. Then $\tilde{\pi}$ acts as follows:

$$\nu\tilde{\pi} = (v_1, \ldots, v_n)\tilde{\pi} = (\bar{v}_1, \ldots, \bar{v}_k), \quad \nu \in \Lambda_n^n.$$

An easy computation based on matrix multiplication (see [2]) shows that $\tilde{\varphi}$ defined above is semi-linear:

$$(\nu_1 + \nu_2)\tilde{\varphi} = (\nu_1)\tilde{\varphi} + (\nu_2)\tilde{\varphi}; \quad (\lambda\nu)\tilde{\varphi} = \bar{\lambda}(\nu\tilde{\varphi}) \tag{3}$$

for any $\nu_1, \nu_2, \nu \in \Lambda_n^n$ and any $\lambda \in \Lambda_n$. From formula (2) we get

$$l(\nu\tilde{\varphi}) = \bar{x} - 1 = \left(\overline{l(\nu)}\right). \tag{4}$$

This relation between the ring homomorphism "$-$": $\Lambda_n \longrightarrow \Lambda_k$ and the semi-linear homomorphism $\tilde{\varphi}\colon \Lambda_n^n \longrightarrow \Lambda_k^n$ suffices to guarantee that the pair $(-, \tilde{\varphi})$ *determines* the group homomorphism $\varphi\colon M_n \longrightarrow M_k$.

When $n = k$, $\varphi\colon M_n \longrightarrow M_k$ is an automorphism if and only if "$-$" is an automorphism of $\Lambda = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $\tilde{\varphi}$ is a semi-linear automorphism of Λ^n, such that

$$\varphi \begin{bmatrix} x & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{x} & e\tilde{\varphi} \\ 0 & 1 \end{bmatrix}.$$

It is clear that "$-$" preserves the multiplicative subgroup U of Λ. Now $U-1\subseteq\Delta$ implies that $\bar{\Delta}\subseteq\Delta$ and from the invertibility of "$-$" we deduce $\bar{\Delta}=\Delta$. If

$$r=\begin{bmatrix}1 & \Sigma r_i e_i\\ 0 & 1\end{bmatrix}\in M_n$$

is such that $\mathrm{id}\{r_1,\dots r_n\}=\Delta$ then we say that r is "Δ-modular". If r is a Δ-modular element of M_n, then the co-ordinate action of $\tilde{\varphi}$ is given by:

$$(r_1,\dots,r_n)\tilde{\varphi}=(\bar{r}_1,\dots,\bar{r}_n)A$$

where A is the matrix of $\tilde{\varphi}$. By Theorem 2.2 the vector $(\bar{r}_1,\dots,\bar{r}_n)$ is Δ-modular, so the co-ordinates of $\nu=(\bar{r}_1,\dots,\bar{r}_n)A$ lie in Δ. But $\nu A^{-1}=(\bar{r}_1,\dots,\bar{r}_n)$, so

$$\Delta=\bar{\Delta}=\mathrm{id}\{\bar{r}_1,\dots,\bar{r}_n\}\subseteq\mathrm{id}\{v_1,\dots,v_n\}\subseteq\Delta$$

which means that ν is Δ-modular. We have thus proved the following:

LEMMA 2.4. *If $r\in M_n$ is Δ-modular then $r\alpha$ is Δ-modular for all $\alpha\in\mathrm{Aut}(M_n)$.*

3. **Epimorphisms of free metabelian groups.** Let M_n and M_k, $k\le n$, be free metabelian groups with basis $\{X_1,\dots,X_n\}$ and $\{X_1,\dots,X_k\}$ respectively. We include a proof of the following variation of Artamonov's theorem which gives a description of an arbitrary epimorphism $\varphi\colon M_n\longrightarrow M_k$, $k\le n$, in terms of the standard epimorphism and the automorphisms of M_n and M_k.

THEOREM 3.1 (CF. ARTAMONOV [2]). *Let $\varphi\colon M_n\longrightarrow M_k$, $k\le n$, be an arbitrary epimorphism. Then there exist automorphisms $\alpha\in\mathrm{Aut}(M_n)$, $\beta\in\mathrm{Aut}(M_k)$ such that $\alpha\varphi\beta$ is the standard epimorphism $\pi\colon M_n\longrightarrow M_k$.*

PROOF. Working modulo the commutator subgroups and using automrophisms of M_n induced by the absolutely free group F_n, if necessary, we may assume that

$$\varphi\begin{bmatrix}x & \nu\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\bar{x} & \nu\tilde{\varphi}\\ 0 & 1\end{bmatrix},$$

where $\bar{x}_1=x_1,\dots,\bar{x}_k=x_k$, $\bar{x}_{k+1}=\cdots=\bar{x}_n=1$. Expressing ν as an element of the module Λ_n^n with basis $\{e_1,\dots e_n\}$, we see that $\nu\tilde{\varphi}=(\bar{\nu}_1,\dots,\bar{\nu}_n)A$ where A is an $n\times k$ matrix over the Laurent polynomial ring $\Lambda_k=Z[x_1^{\pm1},\dots,x_k^{\pm1}]$. Thus $\tilde{\varphi}$ induces a Λ_k-linear epimorphism $\tilde{\varphi}\colon\Lambda_k^n\longrightarrow\Lambda_k^k$. Evidently, $\ker\tilde{\varphi}$ is a direct summand of Λ_k^n and so is projective. Thus, by the well-known Quillen-Suslin-Swan Theorem ([13], [19], [20]), it follows that $\ker\tilde{\varphi}$ is free. Thus there exists a basis $\{\nu_1,\dots,\nu_n\}$ of Λ_k^n such that

(i) $\{\nu_1\tilde{\varphi},\dots,\nu_k\tilde{\varphi}\}$ constitutes a basis for Λ_k^k, and

(ii) $\ker\tilde{\varphi}=\Lambda_k\nu_{k+1}+\cdots+\Lambda_k\nu_n$.

In particular, the vector $\big(l(\nu_1\tilde{\varphi}),\dots,l(\nu_k\tilde{\varphi})\big)$ is Λ_k-modular which, by Artamonov's Theorem 2.2, can be transformed to the standard vector $(x_1-1,\dots,x_k-1)$ by an element of $\mathrm{GL}_k(\Lambda_k)$. After the corresponding transformation of the basis $\{\nu_i:1\le i\le n\}$ we may assume that $l(\nu_i\tilde{\varphi})=x_i-1$ for $1\le i\le k$. Using (4) this gives, $l(\nu_i\tilde{\varphi})=x_i-1=\big(\overline{l(v_i)}\big)$

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for $1 \leq i \leq k$. Clearly, ker "$-$"$=$ id$\{x_{k+1}-1,\ldots,x_n-1\} = I_k = I$. Working modulo I^2, and applying to $\{\nu_1,\ldots,\nu_n\}$ elementary transformations we can assume that $l(\nu_i) \equiv x_i - 1 \pmod{I^2}$ for all $1 \leq i \leq n$. Now, by Artamonov's Theorem 2.3, there exists a matrix $C \in H$ (see Section 2), such that

$$\big(l(\nu_1),\ldots,l(\nu_n)\big)C = X = (x_1-1,\ldots,x_n-1),$$

and for a new basis $\{w_1,\ldots,w_n \mid w_i = \Lambda\nu_i\}$, we have $l(w_i) = x_i - 1$. Furthermore, $w_{k+1},\ldots,w_n$ generate ker $\tilde{\varphi}$ modulo I^2, and $w_1\tilde{\varphi},\ldots,w_k\tilde{\varphi}$ form a basis for Λ_k^k. Thus the matrices $\begin{bmatrix} x_i & w_i \\ 0 & 1 \end{bmatrix}$, $i = 1,\ldots,n$ form a basis for M_n. The first k of them map under φ to a basis for M_k and the remainder map to 1. This completes the proof of the theorem.

REMARKS. Let $F_{n,\mathcal{V}} = \langle x_1,\ldots,x_n; \mathcal{V}\rangle$ be the free group of rank n of an arbitrary variety $\mathcal{V}$, and let $\varphi: F_{n,\mathcal{V}} \longrightarrow F_{k,\mathcal{V}}$, $k \leq n$, be an arbitrary epimorphism. A natural and important question of independent interest is to ask: do there exist automorphisms $\alpha \in \mathrm{Aut}(F_{n,\mathcal{V}})$, $\beta \in \mathrm{Aut}(F_{k,\mathcal{V}})$ such that $\alpha\varphi\beta$ is the standard epimorphism $\pi: F_{n,\mathcal{V}} \longrightarrow F_{k,\mathcal{V}}$ $(x_i \longrightarrow x_i, 1 \leq i \leq k; x_i \longrightarrow 1, k+1 \leq i \leq n)$? It is easily seen to be true when $\mathcal{V}$ is assumed to be a nilpotent variety. By Theorem 3.1, it is true when $\mathcal{V}$ is the variety of metabelian groups. A combinatorial proof of this result will be much desired.

4. Metabelian inner ranks of certain one-relator groups.

In this section we prove that if the relator r of an n-generator one-relator metabelian group is Δ-modular then its metabelian inner rank is at most $[n/2]$. We shall need:

LEMMA 4.1. *Let $\pi: M_n \longrightarrow M_k$, $k \leq n$, be the standard epimorphism of free metabelian groups of matrices (given by the Magnus embedding (1)) sending $X_{k+1},\ldots,X_n$ to 1 and fixing $X_1,\ldots,X_k$. Let r be any Δ-modular element of M_n. If $2k > n$ then $r\pi \neq 1$.*

PROOF. We may identify the matrix

$$r = \begin{bmatrix} 1 & \Sigma r_i e_i \\ 0 & 1 \end{bmatrix} \in M_n$$

with its 12-entry $r_1e_1 + \cdots + r_ne_n$. Thus, regarding r as the vector $(r_1,\ldots,r_n)$, it suffices to prove that $r\pi \neq 0$. As a homomorphism of the matrix groups, our standard epimorphism π is induced by the standard ring epimorphism:

$$Z[x_1^{\pm 1},\ldots,x_n^{\pm 1},e_1,\ldots,e_n] \longrightarrow Z[x_1^{\pm 1},\ldots,x_k^{\pm 1},e_1,\ldots,e_k]$$

which sends $x_{k+1}-1,\ldots,x_n-1, e_{k+1},\ldots,e_n$ to 0 and fixes the other variables. So, π acts on $r = (r_1,\ldots,r_n)$ as follows

$$r\pi = (\bar{r}_1,\ldots,\bar{r}_k,0,\ldots,0),$$

where $\bar{r}_i$ is the image of $r_i \in Z[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ in $Z[x_1^{\pm 1},\ldots,x_k^{\pm 1}]$ under the standard ring epimorphism.

Assume to the contrary that $r\pi = 0$. Then we have $\bar{r}_1 = \cdots = \bar{r}_k = 0$. Since r is Δ-modular, it follows that $\bar{r} = (0, \ldots, 0, \bar{r}_{k+1}, \ldots, \bar{r}_n)$ is Δ_k-modular, where Δ_k is the fundamental ideal of $Z[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$. Consequently, $\bar{r}_{k+1}, \ldots, \bar{r}_n$ generate Δ_k. Since Δ_k can not be generated by fewer than k elements, we must have $n - k \geq k$, contrary to the choice of k.

We can now prove our principal result of this section.

THEOREM 4.2. *If r is a Δ-modular element of the free metabelian group M_n then the metabelian inner rank* $\mathrm{Ir}(M_n/\langle r\rangle) \leq [n/2]$.

PROOF. Assume to the contrary that there exists an epimorphism $\varphi: M_n/\langle r\rangle \longrightarrow M_k$, with $k > n/2$. Alternatively, we can consider φ as an epimorphism $\varphi: M_n \longrightarrow M_k$, which sends r to 1. By Theorem 3.1, we have automorphisms $\alpha \in \mathrm{Aut}(M_n)$ and $\beta \in \mathrm{Aut}(M_k)$ such that $\alpha\varphi\beta$ is the standard epimorphism $\pi: M_n \longrightarrow M_k$. Thus $1 = r\varphi = r\alpha^{-1}\pi\beta^{-1}$, which gives $r\alpha^{-1}\pi = 1$. On the other hand $r\alpha^{-1}$ is Δ-modular by Lemma 2.4 and consequently, by Lemma 4.1, $r\alpha^{-1}\pi \neq 1$ which gives the desired contradiction.

Let M_{2n} be the free metabelian group of rank $2n$ generated by the Magnus matrices

$$X_i = \begin{bmatrix} x_i & u_i \\ 0 & 1 \end{bmatrix}, \quad Y_i = \begin{bmatrix} y_i & v_i \\ 0 & 1 \end{bmatrix}, \quad 1 \leq i \leq n.$$

The matrix image of the element $r = [X_1, Y_1] \cdots [X_n, Y_n]$ in M_{2n} is of the form:

$$\begin{bmatrix} 1 & \Sigma_i (y_i - 1)u_i - (x_i - 1)v_i \\ 0 & 1 \end{bmatrix}.$$

There exists an obvious epimorphism of $M_{2n}/\langle r\rangle$ to M_n: $X_i \longrightarrow X_i$, $Y_i \longrightarrow 1$. We assert that $M_{2n}/\langle r\rangle$ can not be mapped epimorphically on to M_{n+1}. To see this we simply observe that as a vector, $r = (y_1 - 1, \ldots, y_n - 1, 1 - x_1, \ldots, 1 - x_n)$ is Δ-modular. By Lemmas 2.4 and 4.1, we have $r\alpha\pi \neq 1$ for any $\alpha \in \mathrm{Aut}(M_{2n})$ ($\pi: M_{2n} \longrightarrow M_{n+1}$ is standard). Thus, by Theorem 3.1, $r\varphi \neq 1$ for any epimorphism $\varphi: M_{2n} \longrightarrow M_{n+1}$. We thus have the following metabelian analogue of Zieschang's result:

COROLLARY 4.3. *Let $G_n = \langle x_1, \ldots, x_n, y_1, \ldots, y_n; [x_1, y_1] \cdots [x_n, y_n]\rangle$ be the surface group of genus n. The metabelian inner rank of G is equal to n.*

Now, let $F_{2n} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ be free of rank $2n$ and consider the element $r = [x_1, y_1] \cdots [x_n, y_n]$ in F_{2n}. Then, by Corollary 4.3, $\mathrm{Ir}(F_{2n}/\langle r\rangle) \leq \mathrm{Ir}_M(G_n) = n$. Passing on to metabelian groups we see, as above, that $r\varphi \neq 1$ for any epimorphism $\varphi: M_{2n} \longrightarrow M_{n+1}$. We thus have the following:

COROLLARY 4.4 (ZIESCHANG [23]). *The inner rank* $\mathrm{Ir}(G_n)$ *of the orientable surface group $G_n = \langle x_1, \ldots, x_n, y_1, \ldots, y_n; [x_1, y_1] \cdots [x_n, y_n]\rangle$ is precisely n.*

REMARKS. We note that for the Δ-modularity of $r \in M_n$, it is necessary that r lies in the commutator subgroup M_n'. Further, viewing r as the vector $(r_1, \ldots, r_n)$ over the integral group ring ZM^{ab} ($M^{ab} = M_n/M_n'$), it is clear that $r = (r_1, \ldots, r_n) = (\partial r/\partial x_1, \ldots, \partial r/\partial x_n)$, the Fox derivatives of the co-ordinates of r. Then the Δ-modularity of r implies that $\Delta = \mathrm{id}\{\partial r/\partial x_i; i = 1,\ldots,n\}$. We have just seen that $r = [x_1,y_1]\cdots[x_n,y_n]$ is a Δ-modular element of M_{2n} and that $r\alpha$ is also Δ-modular for every automorphism $\alpha \in \mathrm{Aut}(M_{2n})$. It seems very likely that the set $\{r\alpha; r = [x_1,y_1]\cdots[x_n,y_n], \alpha \in \mathrm{Aut}(M_{2n})\}$ characterizes all the Δ-modular elements of M_{2n}. Also, it would be of interest to know whether or not the *other* metabelian analogues of the results about the inner ranks mentioned in the Introduction are also valid. In particular, we ask the analogue of Jaco's Theorem: is the metabelian inner rank of the free product of two metabelian groups additive?

5. **Bases in free metabelian groups.** Let M_n be the free metabelian group freely generated by the Magnus matrices $X_1,\ldots,X_n$. We shall need the following variation of the well-known criterion due to Bachmuth [3] for a system of n elements of M_n to form a basis of M_n.

PROPOSITION 5.1. *The matrices* $\begin{bmatrix} y_i & \nu_i \\ 0 & 1 \end{bmatrix}$, $1 \leq i \leq n$, *form a basis of* M_n *if and only if* y_i, $1 \leq i \leq n$, *constitute a basis of the free abelian group generated by* x_i, $1 \leq i \leq n$, *and the vectors* ν_i, $1 \leq i \leq n$, *generate the Laurent polynomial ring* $\Lambda = Z[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$.

PROOF. Without loss of generality, we can assume that $y_i = x_i$, $1 \leq i \leq n$, so that the matrices $\begin{bmatrix} x_i & \nu_i \\ 0 & 1 \end{bmatrix}$, $1 \leq i \leq n$, generate a free metabelian group $\tilde{M}_n \subseteq M_n$. We need to prove the inclusion $M_n \subseteq \tilde{M}_n$. Let $\tilde{l}$ be a Λ-linear functional on $\sum_{1\leq i\leq n} \Lambda\nu_i$ defined by $\tilde{l}(\nu_i) = x_i - 1$, $i = 1,\ldots,n$. To prove that $\begin{bmatrix} x_i & e_i \\ 0 & 1 \end{bmatrix} \in \tilde{M}_n$, $1 \leq i \leq n$, we must verify the relation $\tilde{l}(e_i) = x_i - 1$, $i = 1,\ldots,n$. Indeed, let $\nu_i = \Sigma_j e_j b_{ij}$, $b_{ij} \in \Lambda$; then $x_i - 1 = \tilde{l}(\nu_i) = \Sigma_j(x_j - 1)b_{ij}$, so that the matrix $B = (b_{ij})$ stabilizes the vector $X = (x_1 - 1,\ldots,x_n - 1)$. Since the $\{\nu_1,\ldots,\nu_n\}$ is a basis of $\Sigma_i\Lambda e_i$, it follows that $B \in \mathrm{GL}_n(\Lambda)$. Consequently, $B^{-1} = (a_{ij}) \in \mathrm{GL}_n(\Lambda)$ and also stabilizes the X. This, in turn, yields

$$\tilde{l}(e_i) = \tilde{l}(\Sigma_j \nu_j a_{ij}) = \Sigma_j \tilde{l}(\nu_j)a_{ij} = \Sigma_j(x_j - 1)a_{ij} = (x_i - 1),$$

as was to be proved.

We shall also need the following criterion for projectivity of certain factor modules.

PROPOSITION 5.2 (BUCHSBAUM AND EISENBUD [5]). *Let* Λ *be an arbitrary commutative domain and let* V *be the submodule of* Λ^n *generated by the elements* $\nu_i = (a_{i1},\ldots,a_{in})$, $1 \leq i \leq t \leq n$. *For each* $m \leq t$, *let* J_m *be the ideal in* Λ *generated by all* $m \times m$*-minors of the matrix* (a_{ij}). *Then the factor module* Λ^n/V *is projective of rank* $n - k$ *if and only if* $J_k = \Lambda$ *and* $J_{k+1} = 0$.

We can now prove the following primitivity criteria.

THEOREM 5.3 (CF. ROMAN'KOV [14]). *Let $M = M_n$ be the free metabelian group with the (Magnus) basis $X_i = \begin{bmatrix} x_i & e_i \\ 0 & 1 \end{bmatrix}$, $1 \le i \le n$, and let $g_i = \begin{bmatrix} y_i & \nu_i \\ 0 & 1 \end{bmatrix}$, $1 \le i \le k$, $\nu_i = \Sigma_j e_j a_{ij}$, be certain elements of M. Define $J_m = \mathrm{id}\{m \times m$ minors of the matrix $(a_{ij})\}$, $1 \le m \le n$, an ideal of the Laurent polynomial ring $\Lambda = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following are equivalent:*

(i) $\{g_1, \ldots, g_k\}$ *is part of a basis of* M_n;

(ii) $\{y_1, \ldots, y_k\}$ *is a part of a basis of the free abelian group* U $(= U_n)$ *generated by* $x_1, \ldots, x_n$, *and* $\{\nu_1, \ldots, \nu_k\}$ *is a part of a basis of the module* $\Lambda^n = \Sigma_i \Lambda e_i$;

(iii) $\{y_1, \ldots, y_k\}$ *is a part of a basis of the free abelian group* U *and* $J_k = \Lambda$.

PROOF OF (i) $\Rightarrow$ (ii). If $\{g_1, \ldots, g_k\}$ is a part of a basis $\{g_1, \ldots, g_n\}$ of M then the matrices X_i, $1 \le i \le n$, can be expressed as words in g_i, $1 \le i \le n$. In particular, $e_i \in \Sigma_j \Lambda \nu_j$. This means that $\{\nu_1, \ldots, \nu_n\}$ generates $\Lambda^n = \Sigma_j \Lambda e_j$, and consequently $\{\nu_1, \ldots, \nu_k\}$ is part of a basis of Λ^n.

PROOF OF (ii) $\Rightarrow$ (i). Applying Nielsen transformations to $\{g_1, \ldots g_k\}$, if necessary, we may assume that $y_i = x_i$ for $1 \le i \le k$. Let $\{\nu_1, \ldots, \nu_k\}$ be a part of a basis $\{\nu_1, \ldots, \nu_n\}$ of Λ^n. It seems natural to prove that the matrices $X_i^* = \begin{bmatrix} x_i & \nu_i \\ 0 & 1 \end{bmatrix}$, $1 \le i \le n$, form a basis of M_n. However, while the fundamental relation $l(\nu_i) = x_i - 1$ may be assumed to hold for $1 \le i \le k$, it may not be valid for $i = k+1, \ldots, n$. Consequently, the matrices X_i^*, $k+1 \le i \le n$, may not necessarily lie in M_n. We must therefore re-organize the part $\{\nu_{k+1}, \ldots, \nu_n\}$ of our basis. By hypothesis, $\big(l(\nu_1), \ldots, l(\nu_n)\big)$ is a Δ-modular vector with $l(\nu_i) = x_i - 1$ for $1 \le i \le k$. For $j \ge k+1$, we write $l(\nu_j) = \Sigma_i (x_i - 1)c_{ij} + d_j$, where d_j depends only on $\{x_{k+1}, \ldots, x_n\}$. Then $l\big(\nu_j - \Sigma_i (x_i - 1)c_{ij}\big) = d_j$, for $k+1 \le j \le n$. Thus adding to $\nu_{k+1}, \ldots, \nu_n$ appropriate linear combinations of $\nu_1, \ldots, \nu_k$, we may assume that $l(\nu_{k+1}), \ldots, l(\nu_n)$ depend only on $x_{k+1}, \ldots, x_n$. Now, $\nu_1, \ldots, \nu_n$ generate Λ^n, so $l(\nu_1) \ldots, l(\nu_n)$ generate Δ. Reducing modulo the ideal $I = \mathrm{id}\{(x_1 - 1), \ldots, (x_k - 1)\}$, we see that the vector $\nu = \big(l(\nu_{k+1}), \ldots, l(\nu_n)\big)$ is Δ_{n-k}-modular in $Z[x_{k+1}^{\pm 1} \ldots, x_n^{\pm 1}]$. By Theorem 2.2 there exists a matrix $A \in \mathrm{GL}_{n-k}(\Lambda_{n-k})$, such that $\nu A = \big((x_{k+1} - 1), \ldots, (x_n - 1)\big)$. Thus, replacing $\nu_{k+1}, \ldots, \nu_n$ by their suitable linear combinations we may assume that $\nu_1, \ldots \nu_n$ possess the property $l(\nu_j) = x_j - 1$ for all j. This, in turn, implies that the matrices X_i^*, $1 \le i \le n$, belong to M_n, and by Proposition 5.1 they generate M_n.

PROOF OF (ii) $\Rightarrow$ (iii). Since the ideals J_m remain invariant under multiplication of the matrix (a_{ij}) by invertible matrices over Λ, the proof follows from the fact that the $k \times k$-minors of the $k \times n$-matrix with respect to $\{e_1, \ldots e_k\}$ generate Λ.

PROOF OF (iii) $\Rightarrow$ (ii). As before we can assume, without loss of generality, that $y_i = x_i$ for $1 \le i \le k$. The condition that $\{\nu_1, \ldots, \nu_k\}$ is a part of a basis of Λ^n is equivalent to the condition that the factor-module $\Lambda^n / (\Lambda \nu_1 + \cdots + \Lambda \nu_k)$ is free of rank $n - k$. By Proposition 5.2, the projectivity of the factor-module is, in turn, equivalent to $J_k = \Lambda$ and $J_{k+1} = 0$ (*cf.* Noskov [12]).

This completes the proof of the theorem.

REMARKS. The case $k = n$ is much easier. The corresponding criterion, due to Bachmuth [3], is the invertibility of the Jacobian matrix $(\partial g_i/\partial x_j)$ of Fox derivatives over the Laurent polynomial ring $\Lambda = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Timoshenko [22] has the same criterion as in Theorem 5.3 for the primitivity of a given system $\mathbf{g} = (g_1, \ldots, g_k)$ in M_n but with the restriction that $k \leq n - 3$. Roman'kov [14] has also proved that the same criterion holds independent of the choice of k. Our proof uses different approach.

Theorem 5.3 yields algorithmic decidability of primitivity in the free metabelian group M_n of a given system $\mathbf{g} = (g_1, \ldots, g_k)$, $k \leq n$. This follows from the fact that the problem clearly reduces to the existence of a solution of a system of linear equations over $\Lambda = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ which, in turn, can be effectively decided (Timoshenko [21]).

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