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THE FIXED POINT PROPERTY FOR SOME UNIFORMLY NONOCTAHEDRAL BANACH SPACES

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Roughly speaking, we show that a Banach space X has the fixed point property for nonexpansive mappings whenever X has the WORTH property and the unit sphere of X does not contain a triangle with sides of length larger than 2.

Introduction

Let C be a nonempty subset of a Banach space X, with norm $\|\cdot\|$. A mapping $T: C \to X$ is called nonexpansive if $\|T(x) - T(y)\| \leqslant \|x - y\|$ for all $x, y \in C$. We say that X has the weak fixed point property (w-fpp) if every nonexpansive mapping $T: C \to C$ defined on a nonempty convex and weakly compact subset C of X has a fixed point.

It is well known that the w-fpp holds for Banach spaces with nice geometric properties. In this note we are interested in one of these properties which is closely related to an open question in fixed point theory (for details, see for example [**4**]). Recall that for a Banach space X the modulus of convexity of X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\Big\{1 - \frac{1}{2} \parallel x + y \parallel : \parallel x \parallel \leqslant 1, \parallel y \parallel \leqslant 1, \parallel x - y \parallel \geqslant \varepsilon\Big\}$$

and the characteristic of convexity of X is defined by $\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta(\varepsilon) = 0\}$. It is known that the condition $\varepsilon_0(X) < 1$ implies that X has the w-fpp. In fact, those Banach spaces X with $\varepsilon_0(X) < 1$ have normal structure and then Kirk's theorem applies [**7**]. Nevertheless, it remains unknown whether the w-fpp holds for every uniformly nonsquare Banach space X (that is, $\varepsilon_0(X) < 2$). In the positive, some partial answers have been established. For instance, we mention a result of J. García who proved in [**3**] that a uniformly nonsquare Banach space X has the w-fpp provided

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that X satisfies the additional assumption of WORTH property, a concept introduced by Sims in [**10**] as follows: X has the WORTH property if

$$\lim_n \big| \| x_n - x \| - \| x_n + x \| \big| = 0$$

for all $x \in X$ and for all weakly null sequences (x_n) in X. A generalisation of the result of García can be found in [**6**] and [**11**]. In the second one Sims relaxed the hypothesis $\varepsilon_0(X) < 2$ by showing that a stronger conclusion than the w-fpp holds under the presence of WORTH property and the condition "X is ε_0-inquadrate in every direction for some $\varepsilon_0 < 2$". In this paper we improve the result of [**3**] by replacing the coefficient $\varepsilon_0(X)$, which has a "two-dimensional character" by a coefficient with a "three-dimensional character".

Main result

Let X be a Banach space with norm $\|\cdot\|$ and let B_X be its closed unit ball. Denote by $\widetilde{\delta}(X)$ the supremum of the set of numbers $\varepsilon \in [0,2]$ for which there exist points x_1, x_2, x_3 in B_X with $\min\{\|x_i - x_j\| : i \neq j\} \geqslant \varepsilon$. Define the function $\widetilde{\delta} : \big[0, \widetilde{\delta}(X)\big) \to [0,1]$ by

$$\widetilde{\delta}(\varepsilon) = \inf\left\{1 - \frac{1}{3}\|x_1 + x_2 + x_3\| : \ x_i \in B_X,\ i = 1,2,3, \text{ and } \min\{\|x_i - x_j\| : i \neq j\} \geqslant \varepsilon\right\}.$$

and let $\widetilde{\varepsilon}_0(X)$ be the number $\widetilde{\varepsilon}_0(X) = \sup\Big\{\varepsilon \in \big[0, \widetilde{\delta}(X)\big) : \widetilde{\delta}(\varepsilon) = 0\Big\}$.

We remark that for any $\varepsilon \in \big[0, \widetilde{\delta}(X)\big)$ we have that $\widetilde{\delta}(\varepsilon) \geqslant \delta(\varepsilon)$ since for any x_1, x_2, x_3 in B_X with $\min\{\|x_i - x_j\| : i \neq j\} \geqslant \varepsilon$ we have that

$$\left\|\frac{x_1 + x_2 + x_3}{3}\right\| \leqslant \frac{1}{3}\left(\left\|\frac{x_1 + x_2}{2}\right\| + \left\|\frac{x_1 + x_3}{2}\right\| + \left\|\frac{x_2 + x_3}{2}\right\|\right) \leqslant 1 - \delta(\varepsilon).$$

Hence $\widetilde{\varepsilon}_0(X) \leqslant \varepsilon_0(X)$ and, in some cases, the inequality is strict. To see this we consider the following example.

Example. Consider the classical real sequence space ℓ_2 endowed with its usual Euclidean norm $\|\cdot\|$. Let $|\cdot|$ be a norm on ℓ_2 such that

$$\|x\| \leqslant |x| \leqslant b\|x\| \qquad (x \in \ell_2)$$

for some $b \geqslant 1$ and let X be the Banach space $(\ell_2, |\cdot|)$.

We claim that $\widetilde{\varepsilon}_0(X) < 2$ for $b < \sqrt{7/3}$. Indeed, if this were not true there would exist sequences (x_n^1), (x_n^2) and (x_n^3) in B_X such that

$$\lim_n |x_n^1 + x_n^2 + x_n^3| = 3 \quad \text{and} \quad \lim_n |x_n^i - x_n^j| = 2 \quad \text{for all } i \neq j.$$

Since $\|\cdot\|$ is an Euclidean norm we have that

$$\|x_n^1 + x_n^2 + x_n^3\|^2 = 3\sum_{i=1}^{3} \|x_n^i\|^2 - \sum_{i\neq j} \|x_n^i - x_n^j\|^2$$

$$\leqslant 3\sum_{i=1}^{3} |x_n^i|^2 - \frac{1}{b^2}\sum_{i\neq j} |x_n^i - x_n^j|^2 \leqslant 9 - \frac{1}{b^2}\sum_{i\neq j} |x_n^i - x_n^j|^2 .$$

Then it follows that

$$9 = \lim_n |x_n^1 + x_n^2 + x_n^3|^2 \leqslant b^2\left(9 - \frac{12}{b^2}\right),$$

which is impossible unless $b^2 \geqslant 7/3$.

Now consider the space $E_\beta = (\ell_2, |\cdot|_\beta)$, where $|x|_\beta = \max\{\|x\|, \beta\|x\|_\infty\}$. Since $\|x\| \leqslant |x|_\beta \leqslant \beta\|x\|_2$ for all $x \in \ell_2$ then $\widetilde{\varepsilon}_0(E_\beta) < 2$ for $\beta < \sqrt{7/3}$. On the other hand, for $\beta \geqslant \sqrt{2}$ we have $\varepsilon_0(E_\beta) = 2$ and, since E_β has the WORTH property, this shows that the fixed point result of [**3**] is strictly included in the following theorem.

THEOREM 1. *If X is a Banach space with WORTH property such that $\widetilde{\varepsilon}_0(X) < 2$ then X has the w-fpp.*

Before proving our theorem, we shall state some basic results which will be used in the proof.

Suppose that C is a nonempty, convex and weakly compact subset of the Banach space X and that $T: C \to C$ is nonexpansive. Standard arguments show that C contains a subset K which is minimal for the properties of being nonempty, convex, weakly compact and T-invariant, and that K contains an approximate fixed point sequence (afps) for T (that is, a sequence (x_n) in K such that $\|x_n - T(x_n)\| \to 0$). The well known Goebel-Karlovitz lemma (see [**4**]) ensures that if K is minimal for T and (x_n) is an afps for T in K, then (x_n) is diametral, that is, $\|x_n - x\| \to \operatorname{diam}(K)$ for every $x \in K$.

We denote by $[X]$ the quotient space $(l_\infty(X))/(C_0(X))$ endowed with the norm $\|[z_n]\| = \limsup \|z_n\|$ where $[z_n]$ denotes the equivalence class of $(z_n) \in l_\infty(X)$. Any $x \in X$ will be considered as an element of $[X]$ by identifying x with the class which contains the constant sequence $(x, x, \dots)$. If C is a subset of X and $T: C \to C$ is a nonexpansive mapping, we denote by $[C]$ the set

$$[C] = \{[c_n]: (c_n) \text{ is a sequence in } C\}$$

and by $[T]$ the mapping $[T]: [C] \to [C]$ defined by $[T]([c_n]) = [T(c_n)]$. With these notations we can state the following version of the Goebel-Karlovitz lemma (see [**1, 8**]):

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Lemma. *Let K be a nonempty, convex and weakly compact subset of X which is minimal for the nonexpansive mapping $T: K \to K$. If W is any nonempty, closed and convex subset of $[K]$ which is invariant under $[T]$, then*

$$\sup\left\{\left\|[w_n] - x\right\| : [w_n] \in W\right\} = \operatorname{diam}(K)$$

for every $x \in K$.

Now we proceed to the proof of the theorem 1.

Proof of Theorem 1: We shall argue by contradiction. Hence, suppose that X is as in Theorem 1 but lacks the w-fpp. Then there exist a nonempty, convex and weakly compact set $K \subset X$ and a fixed point free $\|\cdot\|$-nonexpansive mapping $T: K \to K$ such that K is minimal for T. Since T has no fixed point in K, then $d = \operatorname{diam}(K) > 0$ and we can suppose that $d = 1$. Let (x_n) be an afps for T in K. Since every subsequence of (x_n) is again an afps for T and K is weakly compact, we may assume that (x_n) is itself a weakly convergent afps. By translating K if necessary, we may also assume that $0 \in K$, K is minimal for T and (x_n) is weakly null.

Appealing to the Goebel-Karlovitz lemma and the WORTH property of X we obtain that

$$\lim_n \|v + x_n\| = \lim_n \|v - x_n\| = 1 \tag{1}$$

for all $v \in K$. Hence, we may assume, passing to a subsequence if necessary, that

$$\lim_n \|x_n + x_{n+1}\| = \lim_n \|x_n - x_{n+1}\| = 1. \tag{2}$$

Consider the subset W of $[X]$ defined by

$$W = \left\{[z_n] \in [K] : \max\left\{\left\|[z_n] - [x_n]\right\|, \left\|[z_n] - [x_{n+1}]\right\|, \left\|[z_n] - x\right\|\right\} \leqslant \frac{1}{2} \text{ for some } x \in K\right\}.$$

W is nonempty since $\left[(x_n + x_{n+1})/2\right] \in W$. It is also closed, convex and $[T]$-invariant. Since $0 \in K$, we have, by the lemma, that

$$\sup\left\{\left\|[w_n]\right\| : [w_n] \in W\right\} = 1. \tag{3}$$

Let $[z_n]$ be any element of W, where without loss of generality we may assume that $z_n \in K$ for all n. By definition of W, there exists $x \in K$ such that

$$\left\|[z_n] - [x_n]\right\| \leqslant \frac{1}{2}, \quad \left\|[z_n] - [x_{n+1}]\right\| \leqslant \frac{1}{2} \quad \text{and} \quad \left\|[z_n] - x\right\| \leqslant \frac{1}{2}.$$

Fix $\varepsilon > 0$ with $\widetilde{\varepsilon}_0(X) < \varepsilon < 2$ and choose b arbitrarily in $(\varepsilon/2, 1)$.

For every positive integer n define

$$v_n^1 = 2b(z_n - x_n), \quad v_n^2 = 2b(z_n - x_{n+1}), \quad v_n^3 = 2b(z_n - x).$$

Then our previous considerations lead to the following:

$$\limsup_n \|v_n^i\| \leqslant b < 1 \quad (i = 1, 2, 3) \quad \text{and} \quad \lim_n \|v_n^i - v_n^j\| = 2b > \varepsilon \quad (i \neq j).$$

Then, for n large enough we have,

$$\left\| z_n - \frac{x_n + x_{n+1} + x}{3} \right\| = \frac{1}{2b} \left\| \frac{v_n^1 + v_n^2 + v_n^3}{3} \right\| \leqslant \frac{1}{2b} \left(1 - \widetilde{\delta}(\varepsilon)\right),$$

and since b is arbitrary in $(\varepsilon/2, 1)$,

$$\limsup_n \left\| z_n - \frac{x_n + x_{n+1} + x}{3} \right\| \leqslant \frac{1}{2} \left(1 - \widetilde{\delta}(\varepsilon)\right). \tag{4}$$

On the other hand, it follows from (1) and (2) that

(5)

$$\begin{aligned}\limsup_n \|x_n + x_{n+1} + x\| &\leqslant \frac{1}{2} \left[\lim_n \|x + x_n\| + \lim_n \|x + x_{n+1}\| + \lim_n \|x_n + x_{n+1}\|\right] \\ &= \frac{1}{2} \left[\lim_n \|x - x_n\| + \lim_n \|x - x_{n+1}\| + \lim_n \|x_n - x_{n+1}\|\right] = \frac{3}{2}.\end{aligned}$$

From (4) and (5) we get that

$$\begin{aligned}\big\|[z_n]\big\| = \limsup_n \|z_n\| &\leqslant \limsup_n \left\| z_n - \frac{x_n + x_{n+1} + x}{3} \right\| + \limsup_n \left\| \frac{x_n + x_{n+1} + x}{3} \right\| \\ &\leqslant \frac{1}{2} \left(1 - \widetilde{\delta}(\varepsilon)\right) + \frac{1}{2},\end{aligned}$$

and since $[z_n]$ is arbitrary in W and $\widetilde{\varepsilon}_0(X) < \varepsilon$,

$$\sup\left\{\big\|[w_n]\big\| : [w_n] \in W\right\} \leqslant \frac{1}{2} \left(1 - \widetilde{\delta}(\varepsilon)\right) + \frac{1}{2} < 1,$$

which contradicts (3). ∎

REMARK 1. Following James [**5**] we say that a Banach space X is uniformly nonoctahedral if it does not contain ℓ_1^3 uniformly.

It is known that every uniformly nonsquare Banach space is reflexive, whereas there exists a nonreflexive uniformly nonoctahedral Banach space [**5**].

Since any Banach space X with $\widetilde{\varepsilon}_0(X) < 2$ is uniformly nonoctahedral, this suggests the following question: Does every uniformly nonoctahedral Banach space with WORTH property have the w-fpp? And also: Does $\widetilde{\varepsilon}_0(X) < 2$ imply that X is reflexive?

REMARK 2. There are some generalisations of uniform convexity which may be linked with the coefficient introduced in this note. We mention for instance the concept of a uniformly noncreasy Banach space introduced by Prus [**9**] and the concept of a 2-uniformly rotund Banach space, introduced by Sullivan [**12**]. Recall that a Banach space X is said to be 2-uniformly rotund if for each $\varepsilon > 0$ there is $\delta > 0$ such that if x, y, z are points in the unit sphere of X with $\|x+y+z\| > 3-\delta$ then

$$A(x_1,x_2,x_3) = \sup\left\{\begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z)\end{vmatrix} : f,g \in X^*,\ \|f\| \leqslant 1,\ \|g\| \leqslant 1\right\} < \varepsilon .$$

It is known that $A(x_1,x_2,x_3) \geqslant D(x_1,x_2,x_3)$, where

$$D(x_1,x_2,x_3) = \|x_2 - x_3\| \cdot \operatorname{dist}\big(x_1,[x_2,x_3]\big)$$

and $[x_2,x_3]$ is the affine span of x_2, x_3.

Bernal and Sullivan proved in [**2**] that a Banach space X has normal structure (and hence the w-fpp) whenever X satisfies the following property: there exist $\delta > 0$, $0 < \varepsilon < 1$ and a positive integer m such that if $x_1,\dots,x_m$ are points in the unit sphere of X with $\big\|(x_1+\dots+x_m)/m\big\| > 1-\delta$ then $D(x_1,\dots,x_m) < \varepsilon$.

The next theorem shows, via theorem 1, that the w-fpp still holds if in the above property, with $m = 3$, we replace the condition "$0 < \varepsilon < 1$"by "$0 < \varepsilon < 2$ and X has the WORTH property".

THEOREM 2. *Suppose that there exist $\delta > 0$ and $0 < \varepsilon < 2$ such that for all points x_1, x_2, x_3 in the unit sphere of X with $\big\|(x_1+x_2+x_3)/3\big\| > 1-\delta$ we have $D(x_1,x_2,x_3) < \varepsilon$.*

Then $\widetilde{\varepsilon}_0(X) < 2$.

PROOF: Suppose that $\widetilde{\varepsilon}_0(X) = 2$. Then there exist sequences (x_n^1), (x_n^2) and (x_n^3) in B_X such that

$$\lim_n \|x_n^1 + x_n^2 + x_n^3\| = 3 \quad \text{and} \quad \lim_n \|x_n^i - x_n^j\| = 2 \quad \text{for all } i \neq j . \tag{6}$$

Since this implies that $\|x_n^i\| \to 1$, $i = 1,2,3$, we may assume without loss of generality that (x_n^i), $i = 1,2,3$ are sequences in the unit sphere of X.

On the other hand, since

$$\begin{aligned} D\big(x_n^1,x_n^2,x_n^3\big) &= \|x_n^2 - x_n^3\| \cdot \operatorname{dist}\big(x_n^1,[x_n^2,x_n^3]\big) \\ &\geqslant \|x_n^2 - x_n^3\| \left(\min\{\|x_n^1 - x_n^2\|, \|x_n^1 - x_n^3\|\} - \frac{1}{2}\|x_n^2 - x_n^3\|\right), \end{aligned}$$

and $\|x_n^i - x_n^j\| \to 2$ for all $i \neq j$, then $\liminf_n D\big(x_n^1,x_n^2,x_n^3\big) \geqslant 2$. This together with (6) gives a contradiction. □

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