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THE GENERALIZED WIELANDT SUBGROUP OF A GROUP

To Otto Kegel on his sixtieth birthday

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ABSTRACT. The intersection $IW(G)$ of the normalizers of the infinite subnormal subgroups of a group G is a characteristic subgroup containing the Wielandt subgroup $W(G)$ which we call the generalized Wielandt subgroup. In this paper we show that if G is infinite, then the structure of $IW(G)/W(G)$ is quite restricted, being controlled by a certain characteristic subgroup $S(G)$. If $S(G)$ is finite, then so is $IW(G)/W(G)$, whereas if $S(G)$ is an infinite Prüfer-by-finite group, then $IW(G)/W(G)$ is metabelian. In all other cases, $IW(G) = W(G)$.

1. **Introduction.** If G is a group, the *Wielandt subgroup* of G, denoted by $W(G)$, is defined to be the intersection of the normalizers of all the subnormal subgroups of G. This subgroup has been the subject of a number of papers since its introduction by Wielandt in [13]. Wielandt obtained many of the basic properties of $W(G)$, showing in particular that all non-abelian simple subnormal subgroups of G are contained in $W(G)$, as are all minimal normal subgroups with min $-n$ (see [12, 13.3.2 and 13.3.7]). The Wielandt subgroup has also been discussed in Camina [2] and more recently in Casolo [3], Cossey [4] and Brandl, Franciosi and de Giovanni [1]. Of course, $W(G)$ always contains the centre $Z(G)$, but the example of the infinite dihedral group shows that $W(G)$ can be trivial.

A closely related concept is that of a T-*group*, or group in which normality is a transitive relation. The Wielandt subgroup of a group G is always a T-group, and G is a T-group if and only if $G = W(G)$. The basic properties of T-groups are elucidated in [10]. In [5], de Giovanni and Franciosi introduced the notion of an IT-*group*. This is a group in which every infinite subnormal subgroup is normal. Soluble IT-groups were classified in [5], while Heineken [6] studied IT-groups in general. These two papers have influenced much of the current work, which provides a substantial generalization of them.

In this paper we shall consider the *generalized Wielandt subgroup* $IW(G)$ of a group G; this is defined to be the intersection of the normalizers of all the infinite subnormal subgroups of G. Of course, if G has no infinite subnormal subgroups, then $IW(G) = G$. It is clear that $IW(G)$ is a characteristic subgroup of G containing the Wielandt subgroup. Also $IW(G)$ is an IT-group and G is an IT-group precisely when $G = IW(G)$.

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In Section 2 we show that the structure of a group G with $IW(G) \neq W(G)$ is quite restricted. The crucial result here is that in such a group the normal abelian subgroups are Prüfer-by-finite; this enables us to show that the Baer radical of G is also of this type. Here, and throughout this paper, we shall allow the term *Prüfer-by-finite* to include the possibility that the group is finite.

In Section 3 we obtain our main results concerning the structure of $IW(G)/W(G)$. An important role is played by a characteristic subgroup $S(G)$, which is defined to be the join of all the finite soluble subnormal subgroups of G. In fact $S(G)$ controls the structure of $IW(G)/W(G)$. Our main results can be summarized in the following surprisingly strong form.

THEOREM. *Let G be a group and write $I = IW(G)$, $W = W(G)$, $S = S(G)$. Then the following hold.*

(i) I/W is residually finite.
(ii) If G is infinite, then I/W is a T-group.
(iii) If S is not Prüfer-by-finite, then $I = W$.
(iv) If S is an infinite Prüfer-by-finite group, then I/W is metabelian.
(v) If S is finite, then I/W is finite.

Some of the conclusions of this theorem follow from the interesting fact (Corollary 1) that if I does not normalize a subnormal subgroup H of G, then H has only finitely many conjugates in G. The theorem is complemented by a number of examples. The finite insoluble subnormal subgroups or lack thereof also play a decisive role, as is demonstrated by the following result.

COROLLARY 5. *Suppose that G is a group in which all the finite subnormal subgroups are soluble. Then $IW(G)/W(G)$ is finite.*

We use our results to obtain large classes of groups for which $IW(G) = W(G)$. For example, according to Theorem 7, this is the case if G is an infinite residually finite group.

In Section 4 we show that our results are particularly strong for soluble groups, obtaining necessary and sufficient conditions for a soluble group G to satisfy $IW(G) \neq W(G)$.

Section 5 contains several examples which indicate the limitations of the theory. In particular we show that if $S(G)$ is finite, then $IW(G)/W(G)$ can be an arbitrary finite group. We have also modified an example of Heineken [6] to demonstrate that $IW(G)/W(G)$ need not be torsion.

Our notation is standard and, where not explained, can be found in [11].

2. **The Baer radical.** We begin with a fundamental result which shows that the structure of the normal abelian subgroups of a group G with $IW(G) \neq W(G)$ is very restricted.

LEMMA 1. *Let G be a group with $IW(G) \neq W(G)$. Then every abelian normal subgroup A of G is Prüfer-by-finite.*

PROOF. By hypothesis there is a finite subnormal subgroup H and an element $g \in IW(G)$ such that $H^g \neq H$. We first show that A is torsion. Let $a \in A$ be of infinite order.

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Then $B_0 = \langle a^H \rangle$ is a finitely generated abelian group, so there is a positive integer n such that $B = B_0^n$ is a non-trivial torsion-free subnormal subgroup of G. Since H is finite and subnormal, $[B, H] = 1$ and HB is an infinite subnormal subgroup of G. Therefore $HB = (HB)^g = H^g B$ and $H = H^g$, since H is the torsion subgroup of HB. This contradiction shows that A is torsion.

Next we show that A is a Černikov group. Suppose that $\pi(A)$, the set of primes dividing the orders of elements of A, is infinite, and put $B = A_{\pi(H)'}$. Then B is infinite, and HB is subnormal. Thus $H^g B = HB$, which yields $H^g = H$, a contradiction.

Now assume that A has infinite p-rank for some prime p. Then $A[p] := \{a \in A \mid a^p = 1\}$ is infinite. There is a subgroup B_0 of $A[p]$ such that $|A[p] : B_0| < \infty$ and $H \cap B_0 = 1$. Put $B = \bigcap_{h \in H} B_0^h$; then $|A[p] : B| < \infty$, B is infinite and $H \cap B = 1$. Also HB is subnormal in G since $\langle B^H \rangle$ is. As usual $H^g B^g = HB$ and thus

$$H^g \leq \bigcap HB = H,$$

where the intersection is taken over all H-invariant subgroups B with finite index in $A[p]$ which satisfy $H \cap B = 1$. It follows that A has finite p-rank and hence is a Černikov group.

Finally we need to show that the divisible subgroup R of A is of type p^∞. Suppose that R_p and R_q are non-trivial primary components of R with $p \neq q$. Then $H^g \leq HR_p \cap HR_q = H(HR_p \cap R_q)$. Now

$$\begin{aligned} HR_p \cap R_q &\leq \big(H \cap (R_p \times R_q)\big) R_p \cap R_q \\ &= \big((H \cap R_p) \times (H \cap R_q)\big) R_p \cap R_q \\ &= H \cap R_q, \end{aligned}$$

which yields $H^g = H$. Thus $\pi(R) = \{p\}$ or else $R = 1$.

Suppose that $H \cap R$ is contained in a proper divisible subgroup P_1 of R; then $R = P_1 \times P_2$ where $P_2 \neq 1$. Since $[H, R] = 1$, the subgroups HP_1 and HP_2 are subnormal in G. Therefore

$$H^g \leq HP_1 \cap HP_2 = H(HP_1 \cap P_2) = H\big((H \cap R)P_1 \cap P_2\big) = H$$

since $H \cap R \leq P_1$. Hence the minimal divisible subgroup containing $H \cap R$ is R itself; it is now easy to deduce that $R[p] \leq H$, so that we can pass to $G/R[p]$. By repeating the argument if necessary, we eventually obtain a contradiction and the result follows.

The preceding result is crucial throughout this paper. We use it immediately to obtain the structure of the Baer radical of a group G with $IW(G) \neq W(G)$.

THEOREM 1. *Let G be a group such that $IW(G) \neq W(G)$. Then the Baer radical $B(G)$ is Prüfer-by-finite and hence is nilpotent.*

First we prove:

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LEMMA 2. *Let G be a Baer group which satisfies the minimal condition on subnormal abelian subgroups. Then G is a nilpotent Černikov group.*

PROOF. In the first place note that G is a torsion group and $\pi(G)$ is finite. Next we show that Fit(G), the Fitting subgroup of G, is nilpotent.

Here we can assume that G is a p-group. Consider a chain of subgroups $N_1 \leq N_2 \leq \cdots$ where N_i is a normal nilpotent subgroup of G. Assume that $U = \bigcup_i N_i$ is not nilpotent and put $Z_i = \{x \in Z(N_i) \mid x^p = 1\}$. Then $\langle Z_1, Z_2, \ldots \rangle$ is an elementary abelian normal subgroup of G, so it is finite. Hence for sufficiently large i we have $Z_{i+1} \leq \langle Z_1, Z_2, \ldots, Z_i \rangle \leq N_i$ and $Z_{i+1} \leq Z_i$. Thus, if i is large enough, $Z_i = Z_{i+1} = \cdots$ and $Z_i \neq 1$, which shows that $Z(U) \neq 1$.

Next observe that $U/Z(U)$ inherits the finiteness property of G. For let $A/Z(U)$ be a subnormal abelian subgroup of $U/Z(U)$. Then A is a subnormal nilpotent subgroup of G. Let B be a maximal abelian normal subgroup of A. Then B satisfies the minimal condition (min) by hypothesis, and $C_A(B) = B$ by 5.2.3 of [12]. Since A is torsion, $A/C_A(B)$ has min by [11, Theorem 3.29] and hence A has min. Then $U/Z(U)$ is not nilpotent, and the preceding argument gives

$$1 < Z_1(U) < Z_2(U) < \cdots .$$

Next $Z_\omega(U)$ is a normal hypercentral torsion group whose subnormal abelian subgroups satisfy min. By 1.G.2 of [7] the group $Z_\omega(U)$ is Černikov. Also the divisible part of $Z_\omega(U)$ is contained in $Z_1(U)$ by 12.2.9 of [12]. Thus $Z_\omega(U)/Z_1(U)$ is finite, a contradiction which shows that U is nilpotent, as must be Fit(G).

Now let A be a subnormal abelian subgroup of G. We show that $A \leq \mathrm{Fit}(G)$. Let $A = A_n \lhd A_{n-1} \lhd \cdots \lhd A_1 \lhd A_0 = G$. Note that A_1 inherits the properties of G. By induction on $n > 1$ we obtain $A \leq \mathrm{Fit}(A_1)$. However $\mathrm{Fit}(A_1)$ is nilpotent by the first part of the proof, so $\mathrm{Fit}(A_1) \leq \mathrm{Fit}(G)$, and the required conclusion follows. Therefore $G = \mathrm{Fit}(G)$ is nilpotent and so G is Černikov.

PROOF OF THEOREM 1. By hypothesis there exists a finite subnormal subgroup H of G and $g \in IW(G)$ such that $H \neq H^g$. Let $B = B(G)$ and assume that B is not Prüfer-by-finite. Since there is a positive integer i such that $[B, {}_iH] \leq H$, there is a largest r for which $B_1 = [B, {}_rH]$ is not Prüfer-by-finite. Then $[B_1, H]$ is Prüfer-by-finite, whence $[B_1, H, H]$ is finite since H centralizes a subnormal Prüfer subgroup.

Let $d = |H|$. Then

$$\begin{aligned}[B_1, H]^d = [B, {}_{r+1}H]^d &\leq [B, {}_rH, H^d][B, {}_{r+2}H] \\ &= [B, {}_{r+2}H],\end{aligned}$$

which is finite. Since $[B_1, H]$ is Prüfer-by-finite, it must be finite.

If B_1 is Černikov, then Lemma 1 implies that the divisible part of B_1 is of rank at most 1, a contradiction. Thus Lemma 2 shows that B_1 does not satisfy the minimal condition on subnormal abelian subgroups. If $h \in H$, then $[B_1, h]$ is finite, and so $|B_1 : C_{B_1}(h)|$ is finite.

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Therefore $B_2 := C_{B_1}(H)$ has finite index in B_1, and thus B_2 does not satisfy the minimal condition on subnormal abelian subgroups. Now B_2 is a torsion subgroup. For if $c \in B_2$ has infinite order, then $\langle c^H \rangle$ is a finitely generated nilpotent group, and it is subnormal in G (being contained in B). Hence there exists a non-trivial free abelian subgroup V such that $V \leq \langle c^H \rangle$ and $[V, H] = 1$. Then $H^g \leq \bigcap_p HV^p = H$, a contradiction. It is also easy to see that $\pi(B_2)$ is finite. Consequently B_2 has for some prime p an infinite elementary abelian p-subgroup A, and $[H, A] = 1$. If $A_1 \leq A$ and $|A : A_1| < \infty$, then HA_1 is an infinite subnormal subgroup of G. Thus $H^g \leq \bigcap_{A_1} HA_1 = H$, a final contradiction.

Recall that a group G is called *subsoluble* if it has an ascending subnormal series whose factors are abelian. Every group G has a *subsoluble radical*, which contains all subnormal subsoluble subgroups; this radical is also the limit of the upper Baer series of G. (For these facts see [9] or [11].) We shall be particularly concerned with $S(G)$, the subgroup generated by all the finite soluble subnormal subgroups of G. This subgroup, which is always a locally soluble torsion group, has a decisive influence on the structure of $IW(G)/W(G)$. For groups G with $IW(G) \neq W(G)$ we have the following result concerning the structure of the subsoluble radical.

THEOREM 2. *Let G be a group such that $IW(G) \neq W(G)$. Then the subsoluble radical $H = H(G)$ is soluble and has a torsion subgroup $T(H)$ with $H/T(H)$ torsion-free abelian and $T(H)$ Prüfer-by-finite. If P is the divisible part of $T(H)$, then $C_{T(H)}(P) = C_H(P) = S(G)$.*

PROOF. Let $B = B(G)$ be the Baer radical of G. Then $B = B(H)$ and $C_H(B) \leq B$ by [9]. According to Theorem 1 the subgroup B is Prüfer-by-finite and nilpotent. Now $\mathrm{Aut}(B)$ is finite-by-torsion free abelian, as is H/B since $B/C_H(B)$ is finite. The indicated structure of H now follows.

Since P is the divisible part of $T(H)$, we have $P \leq B$ and either $P = 1$ or P is a p^∞-group by Lemma 1. Also $|B : P|$ is finite and hence $C_H(P) = C_{T(H)}(P)$. Put $C = C_H(P)$ and note that $S = S(G) \leq C$. Since C/P is finite, there is a finite soluble subgroup F of C such that $C = PF$. Then F is normal in C and hence is subnormal in G. Therefore $F \leq S$ and so $C = S$.

3. **The structure of** $IW(G)/W(G)$.

For any group G let $V(G)$ denote the subgroup generated by all the finite subnormal subgroups of G. Thus $S(G) \leq V(G)$. Then $V(G)$ is locally finite by 1.3.3 of [8]. The subgroup $V(G)$ plays a particularly important role in Heineken's paper [6], whereas in the present work it is $S(G)$ that plays the dominant role. Theorem 2 shows that if $S(G)$ is not Prüfer-by-finite, then $IW(G) = W(G)$. Thus the ensuing discussion is concentrated on two cases, where $S(G)$ is finite and where $S(G)$ is a finite extension of a p^∞-group.

We denote the completely reducible radical of a group G by $\mathrm{cr}(G)$. This is the join of all the non-abelian simple subnormal subgroups of G (see [12, p. 85]). The following result may be well known.

LEMMA 3. *Let $G = \langle H, K\rangle$ be a group with H and K subnormal in G. Then* $\mathrm{cr}(G) = \langle \mathrm{cr}(H), \mathrm{cr}(K)\rangle$.

PROOF. It is clear that $\langle \mathrm{cr}(H), \mathrm{cr}(K)\rangle \leq \mathrm{cr}(G)$. Let T be a simple non-abelian subnormal subgroup of G. Assume that T is not contained in H or K. Then $T \cap H = 1 = T \cap K$ and so $T \leq Z(G)$ by 13.3.1 of [12], a contradiction. Therefore T is contained in H or K, so that $T \leq \langle \mathrm{cr}(H), \mathrm{cr}(K)\rangle$ and the result follows.

The next result is critical for most of our major theorems.

THEOREM 3. *Let G be any group, let $S = S(G)$ and let $d > 0$. Then $IW(G)$ normalizes almost all subnormal subgroups H of G such that $|H \cap S| \leq d$.*

PROOF. Let $I = IW(G)$. Assume that the result is false and $H_1, H_2, \ldots$ are infinitely many finite subnormal subgroups such that $H_i^I \neq H_i$ and $|H_i \cap S| \leq d$. Consider the groups H_iS/S and suppose that these contain only finitely many subnormal non-abelian simple subgroups. Put $J_i = \langle H_1, H_2, \ldots, H_i\rangle$; then by Lemma 3,

$$\mathrm{cr}(J_iS/S) = \langle \mathrm{cr}(H_jS/S) \mid 1 \leq j \leq i\rangle.$$

Let R/S be generated by all the finite non-abelian simple subnormal subgroups of G/S which are contained in the subgroups H_iS/S. By Theorem 1.3.3 of [8] R/S is a finite subnormal subgroup of G/S. Note also that R/S is completely reducible. Hence $\mathrm{cr}(J_iS/S) \leq R/S$, so that the order of $\mathrm{cr}(J_iS/S)$ is bounded. Now J_iS/S is a finite semisimple group with centreless completely reducible radical of order $\leq |R/S|$. By 3.3.18 of [12], we have $|J_iS/S| \leq \left|\mathrm{Aut}\big(\mathrm{cr}(J_iS/S)\big)\right|$ and so $|J_iS/S| \leq |R/S|!$. Hence JS/S is finite where $J = \langle H_1, H_2, \ldots\rangle$. Since $|H_i \cap S| \leq d$, it follows that $|H_i|$ is bounded by some integer e. But J is Černikov, so it has only finitely many subnormal subgroups of given order. Thus J is finite, which is impossible.

These considerations show that we can proceed to a subsequence of the H_i's and assume that each H_iS/S contains a subnormal non-abelian simple subgroup T_i/S where the T_i's are all distinct.

Note that $T_i' \leq H_i[S, H_i]S'$. Thus, since H_i is subnormal in G, there is a positive integer k such that $T_i^{(k)} \leq H_i[S, {}_kH_i]S' \leq H_iS'$. Therefore, since S is soluble by Theorem 2, there is a positive integer k_i for which $T_i^{(k_i)} \leq H_i$. Put $V_i = T_i^{(k_i)}$ and note that V_i is subnormal in G.

Next we observe that there are only finitely many subgroups $V_i \cap S$. Proceeding to a subsequence again, we can assume that $U = V_i \cap S \leq H_i$ is independent of i. Then $T_i = V_iS$ since T_i/S is simple. Hence $T_i/S = V_iS/S \cong V_i/U$, a non-abelian simple group.

Let L be the subgroup of G generated by all the V_j. By 13.3.1 of [12] L/U is the normal product of the non-abelian simple groups V_i/U. In addition $(V_i/U) \cap \prod_{j \neq i}(V_j/U) = 1$ since all the V_j/U are distinct, so that $L/U = \mathrm{Dr}_j\, V_j/U$. The subgroup S normalizes T_i and hence V_i, so $D_\ell = \langle V_j \mid j \geq \ell\rangle$ is normal in $D_\ell S$. Further, $D_\ell S/S = \langle T_j/S \mid j \geq \ell\rangle$ is subnormal in G/S, so D_ℓ is subnormal in G.

Now choose j and fix it, and put $H = H_j$. Then HS/S is a finite subnormal subgroup of G/S, so there is a positive integer ℓ_0 such that

$$(HS/S) \cap \Big(\prod_{i \geq \ell_0} T_i\Big)/S = 1.$$

Hence $H \cap D_{\ell_0} \leq H \cap (\prod_{i \geq \ell_0} T_i) \leq S$, which yields $H \cap D_{\ell_0} \leq \langle V_i \mid i \geq \ell_0 \rangle \cap S$. Now the T_i/S generate $\mathrm{Dr}_i(T_i/S)$ and $V_i \leq T_i$. It therefore follows that $H \cap D_{\ell_0} \leq \langle V_i \cap S \mid i \geq \ell_0 \rangle = U$. Since $(HS/S) \cap (T_i/S) = 1$ if $i \geq \ell_0$, we see that H normalizes T_i and hence V_i. Also $U = V_k \cap S$ for all k, so H also normalizes U. Therefore $U \lhd \langle H_1, H_2, \ldots \rangle$.

Put $M = \langle H_1, H_2, \ldots \rangle$ and let $g \in I$ be such that $H^g \neq H$. We know that V_i/U is a subnormal non-abelian simple subgroup of M/U. Thus, by 13.3.1 of [12], it centralizes H/U if $i \geq \ell_0$. For $\ell \geq \ell_0$ the subgroup D_ℓ normalizes H, from which it follows that HD_ℓ is subnormal in G. Therefore

$$H^g \leq \bigcap_{\ell \geq \ell_0} HD_\ell = H\Big(\bigcap_{\ell \geq \ell_0} D_\ell\Big) = HU = H.$$

This contradiction completes the proof.

COROLLARY 1. *Let G be a group. If H is a finite subnormal subgroup of G which is not normalized by $IW(G)$, then H has only finitely many conjugates in G. Thus $\langle H^G \rangle$ is finite.*

PROOF. We apply Theorem 3 with $d = |H|$. Since $IW(G)$ does not normalize H, it cannot normalize any conjugate of H. Furthermore, if $g \in G$, then $|H^g \cap S(G)| = |H \cap S(G)|$.

An immediate consequence of this is

COROLLARY 2. *Let G be a group and let $I = IW(G)$. If H is a subnormal subgroup of G, then $|I : N_I(H)|$ is finite*

The next corollary is a useful special case of Theorem 3.

COROLLARY 3. *Let G be a group such that $S(G)$ is finite. Then $IW(G)$ normalizes all but a finite number of subnormal subgroups of G.*

COROLLARY 4. *For each group G there is a normal subgroup R such that $IW(G) \cap R = W(G)$ and G/R is residually finite.*

PROOF. Let $R = \bigcap_H N_G(H)$ where H is subnormal in G and is not normalized by $IW(G)$. Then R has the desired properties.

In particular we have from Corollaries 2 and 3

THEOREM 4. *Let G be any group. Then*
(i) $IW(G)/W(G)$ is residually finite;
(ii) if $S(G)$ is finite, then $IW(G)/W(G)$ is finite.

We know of no example of a group G with $IW(G) \neq W(G)$ containing a finite subnormal subgroup H with infinitely many conjugates.

If G is infinite, we can point to an additional property of $IW(G)/W(G)$.

THEOREM 5. *If G is an infinite group, then $IW(G)/W(G)$ is a T-group.*

PROOF. Clearly we may assume $I = IW(G) \neq W(G) = W$, and that W is finite. Then $S(G)$ is finite. Let $W \leq H \leq I$, with H subnormal in G. Let $H_1, \ldots, H_n$ be the finitely many subnormal subgroups of G that are not normalized by I, and set $U = \mathrm{core}_G\big(\bigcap_{i=1}^n N_G(H_i)\big)$. Then Corollary 1 shows that G/U is finite, so U must be infinite. Therefore HU is normalized by I and

$$H^I \leq HU \cap I = H(U \cap I) \leq HW = H$$

since $U \cap I \leq W \leq H$. Hence $H \lhd I$, as required.

We remark that Theorem 5 is not valid for finite groups (see Example 4 in Section 5). The reader should compare Theorems 4 and 5 with Proposition 3a of [6]. In Example 1 of [6] Heineken shows that in general $IW(G)/W(G)$ can be infinite, so Theorem 4 is in some sense best possible. We provide a similar example in Section 5 of this paper which shows that $IW(G)/W(G)$ can have elements of infinite order.

In the case where $V(G)$ is Černikov we can improve upon Theorem 5 as follows.

THEOREM 6. *Let G be a group for which $V(G)$ is a Černikov group. Then*
(i) $IW(G)/W(G)$ is finite;
(ii) G possesses a finite normal subgroup N such that $IW(G)/N$ is a T-group.

PROOF. Put $I = IW(G)$ and $W = W(G)$. Clearly we may assume that $I \neq W$. First suppose that $V = V(G)$ is finite. Then $S(G)$ is finite and there are only finitely many finite subnormal subgroups not normalized by I by Corollary 3. The join U of these subgroups is a finite normal subgroup of G. Then $I/U \cap I$ is a T-group. (We can also use Proposition 1 of [6] to see this.) Thus we may assume that V is infinite.

By Lemma 1 the finite residual P of V is a p^∞-group for some prime p. Note that $P \leq V \leq C_G(P)$. There is a finite characteristic subgroup E of V such that $V = PE$; put $J = V \cap I$. Then $J = P(J \cap E)$. Now $I/J \cap E$ is an extension of a p^∞-group by a group that is isomorphic with IV/V; moreover the latter has no non-trivial finite subnormal subgroups. Hence $I/J \cap E$ is a T-group and (ii) is proved.

To establish (i) let $I_1 = C_I(E)$ and note that I/I_1 is finite. Denote by p^d the highest power of p dividing the order of E, and let $P[p^d] = \{x \in P \mid x^{p^d} = 1\}$. Next put $I_2 = C_{I_1}(P[p^d])$ and observe that I/I_2 is finite. Hence it is sufficient to show that $I_2 \leq W$.

Let $y \in I_2$, let e be a p-element of E, and let $a \in P$. Then $a^y = a^\alpha$ where α is a p-adic integer which satisfies $\alpha \equiv 1 \pmod{p^d}$. Hence $(ae)^y = a^y e^y = a^y e = a^\alpha e^\alpha$ since $e \in E$ and $y \in I_2$. Thus y induces a power automorphism in $V = PE$ since it centralizes the p'-elements of E.

Finally, let H be a finite subnormal subgroup of G. Then $H \leq V$, so that $H^y = H$. Therefore $I_2 \leq W$.

An immediate consequence of Theorems 2 and 6 is

COROLLARY 5. *Let G be a group in which $S(G) = V(G)$. Then $IW(G)/W(G)$ is finite.*

In particular, if G is a radical group, $IW(G)/W(G)$ will be finite. However, as our examples show, the presence of insoluble subnormal subgroups can greatly affect the structure of $IW(G)/W(G)$.

For residually finite groups we have the following consequence of Corollary 3.

THEOREM 7. *If G is an infinite residually finite group, then $IW(G) = W(G)$.*

PROOF. Let $I = IW(G)$ and suppose that $I \neq W(G)$. Then $S(G)$ is finite and there are only finitely many subnormal subgroups not normalized by I; let J be the join of these subgroups. Then J is finite, so there is a normal subgroup N of finite index in G such that $N \cap J = 1$. If H is a subnormal subgroup not normalized by I, then $H^I \leq HN \cap J = H$ since HN is infinite and $H \leq J$. The result follows at once.

Finally in this section, we obtain the analogue of Proposition 3(c) of Heineken's paper [6], and we extend Theorem 4 in the case when $S(G)$ is infinite. It is convenient to split off the following general lemma from the main argument. In what follows $\mathbb{Z}_p^*$ is the multiplicative group of p-adic integers and $\mathrm{Paut}(G)$ is the group of power automorphisms of a group G.

LEMMA 4. *Let A be an abelian p-group with divisible part D of type p^∞ and suppose $A = F \times D$ for some finite group F. Let X denote the group of automorphisms of A inducing power automorphisms on A/D. Then the following hold.*

(i) X is a metabelian residually finite $\{p, p-1\}$-group.

(ii) $X = \big(\mathrm{Paut}(A)\big)Y$ and $[\mathrm{Paut}(A), Y] = 1$ where Y is the subgroup of all elements of X acting trivially on A/D.

(iii) If $p > 2$ or F has exponent at least 4, then $T(X) = T\big(\mathrm{Paut}(A)\big) \times T(Y)$.

(iv) If $p = 2$ and F is elementary abelian, then $X = Y$.

PROOF. Since D is a characteristic subgroup of A, it follows that

$$X \cong \left\{ \begin{bmatrix} \alpha & \theta \\ 0 & \beta \end{bmatrix} \;\middle|\; \alpha \in \mathrm{Paut}(F), \theta \in \mathrm{Hom}(F, D), \beta \in \mathbb{Z}_p^* \right\},$$

$$\mathrm{Paut}(A) \cong \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \;\middle|\; \alpha \in \mathrm{Paut}(F), \beta \in \mathbb{Z}_p^*, \alpha \equiv \beta \pmod{\exp F} \right\} \quad \text{and}$$

$$Y \cong \left\{ \begin{bmatrix} 1 & \theta \\ 0 & \beta \end{bmatrix} \;\middle|\; \theta \in \mathrm{Hom}(F, D), \beta \in \mathbb{Z}_p^* \right\},$$

where $\exp F$ is the exponent of F. It is easily seen that $X = \big(\mathrm{Paut}(A)\big)Y$ and that $[\mathrm{Paut}(A), Y] = 1$. Furthermore $Y \cong \mathrm{Hom}(F, D) \rtimes \mathbb{Z}_p^*$, with the natural action of $\mathbb{Z}_p^*$, and hence X is a metabelian residually finite $\{p, p-1\}$-group. Since F is finite, the torsion subgroup $T(X)$ is isomorphic with the set of matrices of the form $\begin{bmatrix} \alpha & \theta \\ 0 & \beta \end{bmatrix}$, where β has finite order in $\mathbb{Z}_p^*$. Thus if p is odd or if $p = 2$ and $\exp F \geq 4$, then $T(X) = T\big(\mathrm{Paut}(A)\big) \times T(Y)$, by a result of Baer (see [11, Lemma 3.28]). If F has exponent 2, then clearly $X = Y$.

We proceed now to the generalization of Heineken's result.

THEOREM 8. *Let G be a group such that S(G) is a finite extension of a p^∞-subgroup P. Write $I = IW(G)$ and $W = W(G)$. Then the following hold.*

(i) I/W is a metabelian residually finite $\{p, p-1\}$-group.

(ii) I/W is abelian if either $I/C_I(P)$ is infinite or $p = 2$.

(iii) If $p > 2$, then I/W is a semidirect product of an abelian group by a cyclic group of power automorphisms of order dividing $p-1$.

PROOF. Let H be a finite subnormal subgroup of G. Then $[H, P] = 1$ and $(HP)^I = HP$. Hence $(H')^I \leq (HP)' = H'$. Consider the abelian factor

$$HP/H'(H \cap P) = H/H'(H \cap P) \times H'P/H'(H \cap P).$$

The second direct factor here, which we denote by $\bar{P}$, is isomorphic to P, and I acts on it by conjugation. Furthermore, the p'-component of $HP/H'(H \cap P)$ is characteristic, so it is I-invariant. Hence

$$N_I(H) = N_I\Big(\big(H/H'(H \cap P)\big)_p\Big). \tag{1}$$

Now consider the abelian p-factor $\big(HP/H'(H \cap P)\big)_p$, which we denote by F_H. Then

$$F_H = \big(H/H'(H \cap P)\big)_p \times \bar{P}.$$

Each element of I normalizes every subgroup of F_H containing $\bar{P}$, and hence $I/C_I(F_H)$ is isomorphic with a subgroup of the group X of all automorphisms of F_H that induce power automorphisms in $F_H/\bar{P}$. Since $[H, P] = 1$, we have $P \leq C_I(F_H)$. Hence $I_H = I/C_I(F_H)$ is a T-group. Since I_H is isomorphic to a subgroup of X, the preceding lemma implies that I_H is a metabelian residually finite T-group. If $J_H/C_I(F_H) = I_H \cap \mathrm{Paut}(F_H)$, then I/J_H is a finite $\{p, p-1\}$-group. Now W induces power automorphisms in the abelian group F_H, so $W \leq J_H$. Moreover J_H normalizes $\big(H/H'(H \cap P)\big)_p$ and hence

$$J_H \leq N_I(H).$$

Therefore I/W is a metabelian residually finite $\{p, p-1\}$-group, and (i) follows.

If $I/C_I(P)$ is infinite, then I_H contains an element of infinite order. Since I_H is residually finite, it cannot be a soluble T-group of type 2; also the elements of finite order in I_H form a subgroup by Lemma 4 and hence I_H is not a T-group of type 1 (for these facts see [10]). It follows that I_H is abelian and (1) implies that $I' \leq N_I(H)$. Hence I/W is abelian in this case.

If, on the other hand, $I/C_I(P)$ is finite, then I_H is finite, so $I_H \leq T(X)$, and $J_H/C_I(F_H) = I_H \cap T\big(\mathrm{Paut}(F_H)\big)$. Let Y be the subgroup of elements of X acting trivially on $F_H/\bar{P}$. If $p = 2$ and $\exp(F_H/\bar{P}) \geq 4$, then Lemma 4(iii) shows that I/J_H is isomorphic to a subgroup of $T(Y)$. Since I/J_H is a T-group, it is Dedekind. However $T(Y)$ has no subgroups isomorphic to the quaternion group Q_8, so I/J_H is abelian. If $F_H/\bar{P}$ has exponent 2, then again I/J_H is abelian. Since this holds for all H and $\bigcap_H J_H = W$, we conclude that I/W is abelian.

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Observe that I/W is an extension of an abelian residually finite p-group by an abelian group with exponent dividing $p-1$. If $p>2$, such a group cannot have a subgroup isomorphic to Q_8 and hence the Fitting subgroup F/W of I/W is abelian. It is easy to see that F/W is residually finite-p. Then I/W induces a torsion group of power automorphisms in F/W, so I/F is a cyclic group of order m dividing $p-1$. Suppose that $I/W=\langle xW, F/W\rangle$ with $x^m \in F$. If $x^m \notin W$, then xW induces a power automorphism of order relatively prime to p in the residually finite p-group F/W, and it fixes a non-trivial element. Such an automorphism must act trivially on the whole of F/W, contradicting the fact that F/W is the Fitting subgroup of I/W. Hence xW has order m and (iii) follows.

We know of no example of a group G where $IW(G)/W(G)$ is an infinite non-abelian group. If such an example were to exist, then $IW(G)/W(G)$ would be either a soluble T-group of type 1 or an infinite non-abelian soluble torsion T-group. T-groups of type 2 are ruled out since these are never residually finite.

4. **Soluble groups in which** $IW(G)\neq W(G)$.

In Theorem 2 we saw that the subsoluble radical of a group G with $IW(G)\neq W(G)$ is soluble. This, together with the results of de Giovanni and Franciosi [5], suggests that in the presence of solubility our results can be strengthened further.

First we obtain a result concerning nilpotent groups.

THEOREM 9. *Let G be an infinite nilpotent group. Then $IW(G)\neq W(G)$ if and only if (i) $G=P\times Q$ where P is a Prüfer-by-finite p-group and Q is a finite p'-group, and (ii) if R is the Prüfer subgroup of G, then $W(P/R)\neq Z(P)/R$.*

PROOF. Assume that $IW(G)\neq W(G)$. By Theorem 1 the group G is Prüfer-by-finite. Thus $G=P\times Q$ where P is an infinite Prüfer-by-finite p-group and Q is a finite p'-group. Let R be the Prüfer subgroup of G. If H is an infinite subgroup of G, then $R\leq H$, so that

$$IW(G)/R=W(G/R)=W(P/R)\times W(Q)R/R.$$

Also $W(G)=W(P)\times W(Q)$. We claim that $W(P)=Z(P)$. Indeed let $g\in W(P)$ and $x\in P$, and choose $y\in R$ of the same order as x. Then g induces a power automorphism in the abelian group $\langle x,y\rangle$ which is of the form $u\longmapsto u^k$ with k an integer. Now $y^g=y$ and so $y^k=y$, whence $x^k=x$ since $|x|=|y|$. Thus $x^g=x$ and $g\in Z(P)$. It follows that

$$IW(G)/R=W(P/R)\times W(Q)R/R\neq W(G)/R$$

and $W(G)/R=Z(P)/R\times W(Q)R/R$. Therefore $W(P/R)\neq Z(P)/R$.

Conversely, if G has the given structure, it is immediate that $IW(G)\neq W(G)$.

We mention here that Theorem 1.4 of [5] is a consequence of Theorem 9. Next we aim to characterize the soluble groups G such that $IW(G)\neq W(G)$.

THEOREM 10. *Let G be an infinite soluble group. Then $IW(G) \neq W(G)$ if and only if G has a normal Prüfer subgroup P such that $C_G(P)/P$ is finite and $W(G/P) \neq W(G)/P$.*

PROOF. Assume that $IW(G) \neq W(G)$ and let B be the Baer radical of G. By Theorems 1 and 2 the subgroup B is a finite extension of a Prüfer group P with $P \lhd G$ and $C_G(P)/P$ is finite. Also note that $W(G)/P < IW(G)/P \leq W(G/P)$.

Conversely, assume that G has a normal Prüfer subgroup P with the properties stated. Observe that the set of elements of finite order forms a subgroup T, and T/P is finite. Suppose that Y is an infinite subnormal subgroup of G. If Y is torsion, then YP/P is finite, so that $P \leq Y$. Now let $x \in Y$ be of infinite order. Then $x \notin C_G(P)$ and thus $[P,x] = P$. Thus again $P \leq Y$. Let $aP \in W(G/P) \setminus \big(W(G)/P\big)$. Then $Y^a = Y$, whence $a \in IW(G) \setminus W(G)$ and the result follows.

As a consequence we obtain rather large classes of infinite soluble groups in which the generalized Wielandt subgroup and the Wielandt subgroup coincide.

COROLLARY 6. *Let G be a finitely generated infinite soluble group. Then $IW(G) = W(G)$.*

PROOF. Assume that $IW(G) \neq W(G)$. By Theorem 10 the group G has a normal Prüfer subgroup P such that G/P is finite-by-abelian and hence finitely presented. This implies that P is the normal closure of a finite subset, which is of course impossible.

COROLLARY 7. *Let G be an infinite soluble group without Prüfer subgroups. Then $IW(G) = W(G)$.*

This follows immediately from Theorem 10.

We know of no example of a locally nilpotent group G with $IW(G) \neq W(G)$ which is not soluble. If such a group were to exist, there would have to be a locally finite p-group with this property for some prime p. Also we know of no finitely generated infinite groups G such that $IW(G) \neq W(G)$.

The following result summarizes what can be said concerning the structure of a soluble group G with $IW(G) \neq W(G)$.

PROPOSITION 1. *Let G be an infinite soluble group such that $I \neq W(G)$ where $I = IW(G)$. Let P be the normal Prüfer subgroup of G in Theorem 2. Then the following hold.*

(i) G'' is finite.
(ii) $G/W(G)$ has finite exponent.
(iii) If I' is infinite, then $P \leq I$ and either $P = I'$ or I is a torsion group.
(iv) If I' is finite, then I is torsion and I/P is finite.
(v) If G is not torsion, then I/W is abelian.

PROOF. By Theorem 10 the factor $C_G(P)/P$ is finite. Put $C = C_G(P)$ and note that $G' \leq C$. Let E be a finite normal subgroup of C such that $C = PE$. Then $G'' \leq C' = E'$, which is finite. Thus (i) follows.

Next G/P is finite-by-abelian, and so it is centre-by-finite exponent. Also all infinite subnormal subgroups of G contain P; therefore $IW(G)/P = W(G/P) \geq Z(G/P)$, and (ii) follows from Corollary 5.

Now assume that I' is infinite. As noted above, $I'P/P \leq C/P$ is finite, so that $P \leq I'$. Because of Theorem 4.3.1 and Corollary 2, (p. 26) of [10], I/P is abelian or torsion. Therefore either I is torsion or $I' = P$, which gives (iii).

Suppose that I' is finite. Then $I/Z(I)$ is torsion, whence so is $I/C_I(P)$. Part (iv) now follows by Theorems 10 and 2.

Finally, assume that G is not torsion. By part (ii), G/I has finite exponent, so I is not torsion. By (iii) and (iv) $I' = P$, and thus $I/W(G)$ is abelian because $P \leq W(G)$.

The example in 1.8 of [5] shows that $IW(G)/W(G)$ can be non-abelian in the case where G is soluble. For the convenience of the reader we give brief details here. Let $A = \langle a \rangle \times P$ where a has order p, $P \cong C_{p^\infty}$ and p is an odd prime. Define automorphisms s, t of A by $s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$ where $\ell \in \mathbb{Z}_p^*$ has order $p - 1$. Then $t^{-1}st = s^\ell$. Put $X = \langle s, t \rangle$ and let $G = A \rtimes X$ be the natural semidirect product. Note that infinite subnormal subgroups contain P and that G/P is a T-group; thus $IW(G) = G$. Also $W(G) = P$, so $IW(G)/W(G) \cong G/P$, which is not abelian.

Theorems 1.10 and 1.11 of [5] are consequences of Theorem 2, Corollary 5 and Proposition 1. The derived length of an infinite soluble IT-group is at most 3 by Theorem 1.6 of [5]. However, the derived length of G is not bounded when G is a soluble group with $IW(G) \neq W(G)$. For example, let P be a p^∞-group with a an element of order p in P; also let $Q = \langle x \rangle$ be a group of order p, and let α be the automorphism of $F = P \times Q$ defined by $x^\alpha = xa$ and $c^\alpha = c$ for all $c \in P$. Put $H = \langle \alpha \rangle \ltimes F$ and note that $W(H) = P$ and $IW(H) = H$. Finally, let $G = H \times L$ where L is a finite soluble p'-group of arbitrary derived length. Then $IW(G) = H \times W(L)$, but $W(G) = W(H) \times W(L)$. Notice that the derived subgroup of G is finite.

5. **Examples.** Our first group is a variation of Example 1 of Heineken [6]. Here G is an IT-group, so $G = IW(G)$; also $S(G) = Z(G)$ is a Prüfer group and $G/W(G)$ is an abelian group with elements of infinite order. Furthermore G is not T-by-finite or finite-by-T.

EXAMPLE 1. Let p and q be distinct primes. For $i = 1, 2, \ldots$ let X_i, Y_i be finite groups which satisfy the following requirements:

(a) $Z(X_i) \leq X_i' = X_i''$, $Z(Y_i) \leq Y_i' = Y_i''$;

(b) $Z(X_i) = \langle u_i \rangle$ and $Z(Y_i) = \langle v_i \rangle$ are cyclic of order p^i;

(c) $X_i'/Z(X_i)$ and $Y_i'/Z(Y_i)$ are non-abelian simple groups;

(d) $(X_i)_{ab} = \langle x_i X_i' \rangle$ and $(Y_i)_{ab} = \langle y_i Y_i' \rangle$ are cyclic of order p^{2i};

(e) $C_{X_i}\big(X_i'/Z(X_i)\big) = Z(X_i)$ and $C_{Y_i}\big(Y_i'/Z(Y_i)\big) = Z(Y_i)$.

For example, we can take $X_i = Y_i$ to be a suitable factor of $\mathrm{GL}(p^{2i}, q^{m_i})$ where $q^{m_i} \equiv 1 \pmod{p^{2i}}$.

Next define

$$W_i = \langle s_i, t_i \mid s_i^{p^{2i}} = t_i^{p^{2i}} = 1, [s_i, t_i] \in Z(W_i) \rangle,$$

so that $d_i = [s_i, t_i]$ has order p^{2i}, and form the group $X_i \times Y_i \times W_i$. Now put

$$D_i = \langle X_i', Y_i', x_i s_i, y_i t_i \rangle / \langle u_i v_i^{-1}, u_i d_i^{-p^i} \rangle,$$

and observe that D_i contains $[x_i s_i, y_i t_i] = d_i$ (with some notational abuse here). Clearly $Z(D_i) = \langle d_i \rangle$.

Next define

$$H = D_1 \times D_2 \times \cdots / \langle d_1^{-1} d_2^{p^2}, d_2^{-1} d_3^{p^2}, \ldots, d_i^{-1} d_{i+1}^{p^2}, \ldots \rangle$$

Conjugation by $x_i y_i t_i$ leaves D_i invariant. Let α be the automorphism of H which acts via conjugation by $x_i y_i t_i$ on D_i. Then α fixes $y_i t_i$ and $(x_i s_i)^\alpha = x_i s_i d_i$. Clearly α has infinite order. Finally put $G = \langle \alpha \rangle \ltimes H$. We shall establish the following facts about the group G:

(i) $\langle \alpha \rangle \cap W(G) = 1$;
(ii) $Z(H) = P$ is a Prüfer p-group and G/P is a T-group;
(iii) every infinite subnormal subgroup of G contains P, so that $IW(G) = G$;
(iv) $W(G) = \langle X_i', Y_i', (x_i s_i)^{p^i}, (y_i t_i)^{p^i} \mid i \geq 1 \rangle$ and $G/W(G) \simeq \mathrm{Dr}_{i=1,2,\ldots}(C_{p^i} \times C_{p^i}) \times \mathbb{Z}$;
(v) $S(G) = P$ and $V(G) = H$;
(vi) G is neither T-by-finite nor finite-by-T.

PROOF. (i) Let $H_i = \langle X_i', x_i s_i \rangle$, qua subgroup of H. Then $H_i \lhd \langle X_i', x_i s_i, d_1, d_2, \ldots \rangle \lhd H \lhd G$, so H_i is subnormal in G. Suppose that $1 \neq \alpha^m \in W(G)$, $m > 0$. Then, since $(x_i s_i)^{\alpha^m} = x_i s_i d_i^m$, we have $x_i s_i d_i^m \in H_i$, say $x_i s_i d_i^m = (x_i s_i)^r \cdot x'$ where $x' \in X_i'$. Thus $r \equiv 1 \pmod{p^{2i}}$ and hence $d_i^m \in X_i'$. But $\langle d_i \rangle \cap X_i' = \langle d_i^{p^i} \rangle$, so that $p^i \mid m$ for all i, a contradiction.

(ii) Clearly $P := Z(H) = \langle d_1, d_2, \ldots \rangle$ is a Prüfer p-group. Put $Q = G/P$; this is an extension of a completely reducible group R by an abelian group. Let L be a subnormal subgroup of Q. Then $L \cap R$ is a direct factor of R by 3.3.12 of [8]. From the structure of Q we see that $L \cap R$ is normal in Q. Working modulo $L \cap R$, we can assume $L \cap R = 1$. But then L is the soluble radical of LR and $LR \lhd Q$; hence $L \lhd Q$ and Q is a T-group.

(iii) Let L be an infinite subnormal subgroup of G. If $P \not\leq L$, then $P \cap L$ is finite and LP/P is infinite. Assume that $L \not\leq H$. Then there is a positive integer m and an element $g \in G$ such that $\alpha^m g \in L$. Thus there is positive integer k such that L contains $[X_i', {}_k \alpha^m g]$ for all i. Note that $[X_i', {}_k \alpha^m g] = [X_i', {}_k x_i^m]$ and $x_i^m \notin Z(X_i)$ for large enough i. Also, since $X_i'/Z(X_i)$ is simple and $X_i' = X_i''$, we have $[X_i', x_i^m] = X_i'$. Thus $Z(X_i) \leq X_i' \leq L$ for all large i. Therefore $P = Z(H) \leq L$, a contradiction.

We have shown that $L \leq H$. For each i either $X_i'P/P \leq LP/P$ or L centralizes $X_i'P/P$. If the latter happens for almost all i, then L is contained in the subgroup generated by P and a finite number of the X_i', which is impossible. Therefore $X_i'P \leq LP$ for infinitely many i. Then $X_i' = X_i'' \leq L$, so $Z(X_i) \leq L$ for infinitely many i, that is, $P \leq L$. By (ii) G is an IT-group and $IW(G) = G$.

(iv) Let M be a finite subnormal subgroup of G. Then either $X_i' \leq M$ or X_i' centralizes M. It follows that

$$W(G) = \langle X_i', Y_i', (x_i s_i)^{p^i}, (y_i t_i)^{p^i} \mid i \geq 1 \rangle$$

and

$$G/W(G) \cong \mathrm{Dr}_{i=1,2,\ldots}(C_{p^i} \times C_{p^i}) \times \mathbb{Z}$$

is an abelian group with elements of infinite order.

(v) That $S(G) = P$ and $V(G) = H$ follows easily from the construction of G.

(vi) Suppose that B is a normal T-subgroup of G with finite index. Then there is an integer $m > 0$ such that $D_i^m \leq B$ for all i. However D_i^m is not a T-group if i is large. Finally, suppose that A is a finite normal subgroup of G. Then there is an integer n such that $D_n A/A$ is not a T-group, and therefore G/A is not a T-group.

Our next example shows that $IW(G)/W(G)$ can be non-abelian even when $S(G)$ is finite.

EXAMPLE 2. There is an infinite group G such that $S(G)$ is finite and $IW(G)/W(G)$ is non-abelian.

Let $p > 2$ be prime. Let X_i be a finite perfect group such that $X_i/Z(X_i)$ is a non-abelian simple group and $Z(X_i) = \langle u_i \rangle$ has order p for $i = 1, 2, \ldots$. Let $\langle a, b \rangle$ be an elementary abelian group of order p^2 and define automorphisms s and t of $\langle a, b \rangle$ by $a^s = a^t = a$ and $b^s = ab$, $b^t = b^{-1}$. Thus $\langle s, t \rangle$ is a dihedral group of order $2p$. Let $S = \langle s, t \rangle \ltimes \langle a, b \rangle$ be the semidirect product. Then S has order $2p^3$ and $Z(S) = \langle a \rangle$. Now define G to be the central product

$$\big(S \times \operatorname*{Dr}_{i=1,2,\ldots} X_i\big)/\langle u_i^{-1} u_1, u_1^{-1} a \mid i = 2, 3, \ldots \rangle.$$

Then $S(G) = S$ and $Z(G) = \langle a \rangle$. An infinite subnormal subgroup must contain a and $G/Z(G)$ is a T-group; hence G is an IT-group. The non-normal subnormal subgroups are all contained in S and $W(G) = \langle X_1, X_2, \ldots \rangle$, so $G/W(G)$ is non-abelian.

The details of the remaining examples are left to the reader. In the third example $V(G) = S(G)$ is finite and $W(G)$ is a perfect T-group of index 4 in the IT-group G.

EXAMPLE 3. Let G be the central product of the dihedral group of order 8 with $\mathrm{SL}(2, F)$ where F is an infinite field of characteristic not 2. Then $IW(G) = G$, $W(G) = \mathrm{SL}(2, F)$, $V(G) = S(G)$ is of order 8 and $G/W(G)$ is of order 4.

The next example shows that every finite group occurs as the quotient $IW(G)/W(G)$.

EXAMPLE 4. Let H be an arbitrary finite group and put $G = A_5 \operatorname{wr} H$, the standard wreath product. Then $IW(G) = G$, and $W(G) = B$, the base group of G, so that $IW(G)/W(G) \simeq H$.

Our final example is of a group G such that $W(G) \neq IW(G) \neq G$ and $IW(G)$ is a T-group.

EXAMPLE 5. Let $A = \langle x, y \mid x^8 = 1, y^2 = x^4, x^y = x^{-1} \rangle$ be a generalized quaternion group of order 16, and let P be a 2^∞-group. Here A is to act on P with x centralizing P and y acting by inversion. Then $G = A \ltimes P$ has $IW(G) = \langle P, x^2 \rangle$ and $W(G) = \langle P, x^4 \rangle$.

In conclusion we remark that most of our results are still true under the weaker hypothesis that the group possesses an automorphism which fixes all infinite subnormal subgroups but not all finite ones.

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