Canad. Math. Bull. Vol. 37 (4), 1994 pp. 461–467

TRACE, SYMMETRY AND ORTHOGONALITY

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ABSTRACT. Does there exist a circulant conference matrix of order > 2? When is there a symmetric Hadamard matrix with constant diagonal? How many pairwise disjoint, amicable weighing matrices of order n can there be? These are questions concerning which the trace function gives a great deal of insight. We offer easy proofs of the known solutions to the first two, the first being new, and develop new results regarding the latter question. It is shown that there are 2^t disjoint amicable weighing matrices of order $2^t p$, where p is odd, and that this is an upper bound for $t \leq 1$. An even stronger bound is obtained for certain cases.

1. **Introduction and preliminaries.** Weighing matrices A, B are said to be *amicable* if $AB^t = BA^t$ (and *antiamicable* if $AB^t + BA^t = 0$). Clearly A and B are amicable (antiamicable) if and only if A^t and B^t are as well. We denote by $A \cap B$ the Hadamard, or entry-wise product of matrices A, B having the same size. These are *disjoint* if $A \cap B = 0$.

One of the most powerful means by which Hadamard matrices are constructed is with *orthogonal designs* (see [7]). An orthogonal design $OD(n; s_1, \ldots, s_k)$ corresponds to k pairwise disjoint, antiamicable weighing matrices of order n, with weights $s_1, \ldots, s_k$. Similar constructions use disjoint, amicable weighing matrices (see [3, Theorem 4.2], [5]). Even without the disjointness property, amicable weighing matrices are of considerable interest (see pp. 217–219 of [7]). For one, they provide an interesting variation on some algebraic questions regarding orthogonal designs, perhaps having implications in the theory of quadratic forms.

At the problem session for the 1991 Ontario Combinatorics Workshop, I gave the following problems, which I had considered hard:

1. Does there exist a symmetric weighing matrix of odd order with zero diagonal? Conjecture: No.
2. Do there exist disjoint, amicable weighing matrices of odd order? Conjecture: No.

Of course, the second question generalizes the first. I was surprised and pleased when David Gregory offered a proof of the former, as follows.

THEOREM 1 (D. GREGORY). *There does not exist a symmetric weighing matrix of odd order, having zero diagonal.*

PROOF. Let $W = (n, w)$, n odd, $\lambda = \sqrt{w}$, $W \cap I = 0$, $W = W^t$. Then $W^2 = wI$, and so W has all real eigenvalues $\lambda_i = \pm\lambda$, $i = 1, \ldots, n$. Now $\mathrm{tr}(W) = 0 = \sum_{i=1}^{n} \lambda_i = k\lambda$. Since n is odd, k is odd and therefore nonzero—contradiction! ∎

Supported by an NSERC Postdoctoral Fellowship.

Received by the editors May 13, 1992; revised January 4, 1993 and July 13, 1994.
AMS subject classification: Primary: 05B20; secondary: 15A15, 05C50.

The same trick gives an easy proof of the classical result [7] that there is no skew weighing matrix of odd order. I was immediately able to generalize the method to prove the second conjecture, as follows.

THEOREM 2. *There does not exist a pair of disjoint, amicable weighing matrices of odd order.*

PROOF. Let $A = W(n,a)$, $B = W(n,b)$, n odd, $A \cap B = 0$, $AB^t = BA^t$. Then let $W = AB^t$. W is then symmetric, with all eigenvalues $\pm\lambda$, $\lambda^2 = ab$, and since A and B are disjoint, W has zero diagonal. A contradiction is obtained as above. ∎

The simplicity of the method used here motivated a closer look into the possibility of exploiting the trace function as a tool for studying orthogonal matrices satisfying symmetry conditions. The method is not entirely new, for it has been used in the related setting of symmetric designs (see, for example, [1]).

2. Trace and symmetric weighing matrices.

We state here the relationship between symmetric weighing matrices and their trace.

LEMMA 3. *Let $W = W(n,w)$ be symmetric. Then either*

1. *w is not a perfect square, and* $\mathrm{tr}(W) = 0$, *or*
2. *$w = k^2$, and* $\mathrm{tr}(W) = tk$, *where* $|t| \le \lfloor \frac{n}{k} \rfloor$ *and* $t \equiv n \bmod 2$.

PROOF. Since we have $W^2 = wI$, W has minimal polynomial $x^2 - w$. It follows that the eigenvalues of W are all $\pm k$, where $k^2 = w$. Now $\mathrm{tr}(W) = \sum_{(\lambda \text{ an eigenvalue of } W)} \lambda = (a-b)k$, where $a + b = n$. Letting $a - b = t$, we have $t = n - 2b \equiv n \bmod 2$. Finally, $|\mathrm{tr}(W)| \le n$, and this gives the bound on t. ∎

Symmetric Hadamard matrices with constant diagonal are sometimes called *graphical*. The following was obtained by Wallis [14], [15] in the setting of graph theory and design theory. (See also [9]).

THEOREM 4. *If there is a symmetric Hadamard matrix of order n, with constant diagonal, then n is a square.*

PROOF. The trace of such a matrix is $\pm n \ne 0$. So from Lemma 3, n is a square. ∎

Lemma 3 also tells us something about symmetric Hadamard matrices of non-square orders.

THEOREM 5. *A symmetric Hadamard matrix of non-square order has trace* 0.

We may use this lemma, more generally, to study the existence of symmetric weighing matrices with constant diagonal. Now it is not known whether there exists a symmetric weighing matrix of odd order, other than the identity matrix, which has constant diagonal, although Theorem 1 dispenses with the case of zero diagonal. In contrast, we may generalize and slightly modify Example 3 of [4] to obtain symmetric $W(2p^2, p^2)$ with constant diagonal 1, where p is any odd prime power, and consequently a number of other classes of orders $2q$, q odd. Examples abound even more in orders divisible by 4.

Let us consider the case $W = W(n, w)$, $W = W^t$, $\mathrm{tr}(W) = n$. By Lemma 3, we have $w = k^2$, $n = kt$, $t \equiv n \bmod 2$, from which we obtain the following.

THEOREM 6. *Let $W = W(n, w)$ be a symmetric weighing matrix with all diagonal entries* 1, *where $n = 2^q p$, p odd. Then $w = k^2$. Moreover, if $q = 0$ then $k \mid p$, and otherwise, $k \mid 2^{q-1}p$.*

EXAMPLE 1. We can eliminate symmetric $W(n, w)$ with constant diagonal 1, for $n = 2p$, p odd and $w > 1$, in the cases $n = 2, 6, 10, 14, 22, 26$ as well as $W(18, 4)$, $W(18, 16)$, $W(30, 4)$, $W(30, 16)$. As mentioned above, we can construct such a $W(18, 9)$. The matrix

$$\begin{pmatrix} J & -A & -A & -A \\ A & J & -A & A \\ A & A & J & -A \\ A & -A & A & J \end{pmatrix},$$

with A the 3×3 circulant matrix with first row $(0\ 1\ -)$ and J the 3×3 matrix of 1's, is an example with $n = 12$, $w = 9$, and so direct sum gives us an example with $n = 30$, $w = 9$. The first unresolved case for $n = 2p$, p odd, is $W(30, 25)$, and for n odd, $W(15, 9)$.

3. **Circulant conference matrices.** A *conference matrix* is a $W(n, n-1)$ with zero diagonal. Clearly, n must be even. A matrix is *k-regular* if all its row and column sums equal k. Clearly, every circulant matrix is k-regular for some k.

LEMMA 7. *A k-regular conference matrix of order n is symmetric, and we must have $n = k^2 + 1$, k odd.*

PROOF. By assumption, we have $JW = JW^t = kJ$. Thus, $JWW^t = k^2 J = (n-1)J$. So $n = k^2 + 1$, and k is odd. Now let the first two rows of W be $\begin{pmatrix} 0 & x & * & \cdots & * \\ y & 0 & * & \cdots & * \end{pmatrix}$, where a of the unspecified columns are of the form $\binom{1}{1}$, b are of the form $\binom{1}{-1}$, c of the form $\binom{-1}{1}$, and d of the form $\binom{-1}{-1}$. Then from the regularity of W and orthogonality of its rows, we have the relations

$$\begin{aligned} a + b + c + d &= k^2 - 1, \\ x + a + b - c - d &= k, \\ y + a - b + c - d &= k, \\ a - b - c + d &= 0. \end{aligned}$$

Adding the four equations, we get $x + y + 4a = k^2 + 2k - 1$. Since k is odd, we have $x + y \equiv 1 + 2 - 1 \equiv 2 \bmod 4$. Therefore, $x = y$. Similarly arranging the ith and jth rows, we show that the (i, j) and (j, i) entries of W are equal. Therefore, W is symmetric. ∎

EXAMPLE 2. There is a symmetric circulant conference matrix of order 2, namely

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

https://doi.org/10.4153/CMB-1994-067-1 Published online by Cambridge University Press

The following result shows that this is an exceptional case. The usual proof requires at least a page or two of counting arguments, and consideration of several cases (see, for example, [7, pp. 201–205]).

THEOREM 8 (MULLIN AND STANTON, [13]). *There is no circulant conference matrix of order $n > 2$.*

PROOF (NEW). If W is a circulant conference matrix of order n, then from Lemma 7, $n = k^2 + 1$, and without loss of generality, W has first row $(0, a_1, \ldots, a_m, 1, a_m, \ldots, a_1)$, where $m = \frac{n}{2} - 1$. So also, $U = W(n, n-1)$, the circulant matrix with first row $(1, a_m, \ldots, a_1, 0, a_1, \ldots, a_m)$, is symmetric and has trace n. Therefore, by Lemma 3, $n = tk$—which is clearly impossible unless $k = 1$, $n = 2$. ∎

In contrast to the circulant case, it is possible for regular conference matrices of other orders to exist.

EXAMPLE 3. The following is a 3-regular $W(10, 9)$:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - \\ 1 & 0 & 1 & 1 & 1 & - & - & - & 1 & 1 \\ 1 & 1 & 0 & 1 & - & 1 & - & 1 & - & 1 \\ 1 & 1 & 1 & 0 & - & - & 1 & 1 & 1 & - \\ 1 & 1 & - & - & 0 & 1 & 1 & - & 1 & 1 \\ 1 & - & 1 & - & 1 & 0 & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & 1 & 0 & 1 & 1 & - \\ - & - & 1 & 1 & - & 1 & 1 & 0 & 1 & 1 \\ - & 1 & - & 1 & 1 & - & 1 & 1 & 0 & 1 \\ - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 0 \end{pmatrix}.$$

4. **Pairwise disjoint, amicable weighing matrices.** A result of Radon gives an upper bound on the number of pairwise disjoint antiamicable weighing matrices of order n [7], [10]. Can we find a similar upper bound for pairwise disjoint amicable weighing matrices?

EXAMPLE 4. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are two pairwise disjoint amicable weighing matrices of order 2.

Now if $\{X_i\}_{i=1,\ldots,k}$ and $\{Y_j\}_{j=1,\ldots,l}$ are sets of disjoint amicable weighing matrices of orders m and n respectively, it is clear that $\{X_i \otimes Y_j\}_{i=1,\ldots,k, j=1,\ldots,l}$ is a set of kl disjoint amicable weighing matrices of order mn. Thus we obtain 2^t disjoint amicable weighing matrices of order 2^t, $t \geq 0$ (these actually form the regular representation of the elementary abelian group of order 2^t). Now let $n = 2^t p$, where p is odd. Letting $\{X_i\}$ be a set of 2^t disjoint amicable weighing matrices of order 2^t, and $Y_1 = I_p$, we obtain 2^t disjoint amicable weighing matrices of order n. Observe that this number exceeds Radon's function when $t > 3$, so in the case of amicable matrices we must expect a weaker bound than is known for antiamicable matrices.

THEOREM 9. *There are 2^t disjoint amicable weighing matrices of order $2^t p$, where p is odd. Further, we may choose all the weights to be* 1.

Could it be that this is also an upper bound? It certainly is when $p = 1$. Also, Theorem 2 demonstrates this is the case when $t = 0$. We now extend this to the case $t = 1$.

THEOREM 10. *If A, B, C are disjoint amicable weighing matrices of order n, then* $4 \mid n$.

PROOF. Let a, b, c be the weights of A, B, C respectively, and set $X_1 = \frac{1}{\sqrt{ab}}AB^t$, $X_2 = \frac{1}{\sqrt{bc}}BC^t$, $X_3 = \frac{1}{\sqrt{ca}}CA^t$. It is easy to verify that $G = \{I, X_1, X_2, X_3\}$ is a group of commuting matrices all squaring to the identity, and X_1, X_2, X_3 all have zero diagonal. So G is a representation of EA(4) with character $(n, 0, 0, 0)$. Since every character of a group may be uniquely expressed as an integer combination of its irreducible characters, it follows that G is equivalent to an $\frac{n}{4}$-fold direct sum of the regular representation with itself (the regular representation has character $(4, 0, 0, 0)$, and is the direct sum of the irreducible representations of EA(4)). Hence, $4 \mid n$. ∎

The use of trace in this result is hidden in the fact that characters of matrix representations are defined using trace; both result and proof, therefore, are extensions of Theorem 2. Unfortunately, this method does not extend in the obvious way to the case $t = 2$, as the following example shows.

EXAMPLE 5. Let $A_i = \text{diag}(d_i)$, $i = 1, \ldots, 5$, where

$$\begin{aligned}
d_1 &= (1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1),\\
d_2 &= (1\,1\,1\,1\,1\,1\,-\,-\,-\,-\,-\,-),\\
d_3 &= (1\,1\,1\,-\,-\,-\,1\,1\,1\,-\,-\,-),\\
d_4 &= (1\,1\,-\,1\,-\,-\,1\,-\,-\,1\,1\,-),\\
d_5 &= (1\,1\,-\,-\,1\,-\,-\,1\,-\,1\,-\,1).
\end{aligned}$$

The pairwise products of these five orthogonal, symmetric, commuting matrices of order $12 = 2^2 \cdot 3$ all have trace 0. Since our use of character theory, as in Theorem 10, will not detect that these matrices are not disjoint, it cannot prohibit the existence of five disjoint, amicable weighing matrices of weight 1. Diagonal matrices can be used analogously to exceed our conjectured bound for any order divisible by 4.

Here is a result similar to Lemma 3, involving (not neccesarily disjoint) amicable matrices.

LEMMA 11. *Let $W_1, \ldots, W_{2p}$ be pairwise amicable weighing matrices of order n, having weights $w_1, \ldots, w_{2p}$. Let $M = W_1 {W_2}^t W_3 {W_4}^t \cdots W_{2p-1} {W_{2p}}^t$ and $m = w_1 w_2 \cdots w_p$. Then either*

1. *m is not a perfect square, and* $\text{tr}(M) = 0$, *or*
2. *$m = k^2$, and* $\text{tr}(M) = tk$, $t \equiv n \bmod 2$.

https://doi.org/10.4153/CMB-1994-067-1 Published online by Cambridge University Press

PROOF. As in Lemma 3, except that we must first show that M is symmetric. For $p = 1$, this is clear. For $p = 2$, taking A, B, C, D to be the given matrices, we have $M^t = (AB^tCD^t)^t = DC^tBA^t = CD^tAB^t = CA^tDB^t = AC^tBD^t = AB^tCD^t = M$. Similar calculations give the general case. ∎

We may now give a stronger upper bound for the number of pairwise disjoint amicable weighing matrices in some special cases.

THEOREM 12. *If there are q pairwise disjoint amicable weighing matrices of order n, such that no product of $2p$ of their weights is square, $1 < p \leq \lfloor \frac{q}{2} \rfloor$, then $2^{q-1} \mid n$.*

PROOF. Let the weighing matrices be $W_1, \ldots, W_q$, with weights $w_1, \ldots, w_q$. It is not hard to see that the matrices $\frac{1}{\sqrt{w_i w_j}} W_i W_j{}^t, i \neq j$, generate a multiplicative group isomorphic to $\mathrm{EA}(2^{q-1})$. Let $M \neq I$ be an element of this group. If M is one of the generators, then $\mathrm{tr}(M) = 0$ since the weighing matrices are disjoint. Otherwise, $\mathrm{tr}(M) = 0$ by Lemma 11. Therefore, this representation of $\mathrm{EA}(2^{q-1})$ has character $(n, 0, \ldots, 0)$, and the result follows as in Theorem 10. ∎

This result eliminates many possibilities that would not be covered by a general bound on the number of weighing matrices in such a set. For example, it excludes most 4-tuples (of weights) of order 12, such as $(1,1,1,2)$ and $(1,2,3,5)$, although Theorem 9 shows that some 4-tuples do exist. A few 4-tuples, such as $(1,1,3,3)$ remain unresolved by the results of this paper. Incidentally, Theorem 12 also eliminates all but four 5-tuples of order 12, and $(1,1,1,1,4)$ is the only 5-tuple it leaves unresolved in order 8.

We may drop the condition that the weighing matrices are pairwise disjoint by including the case $p = 1$ in the statement of the theorem.

ACKNOWLEDGEMENT. I am grateful to D. Gregory, H. Kharaghani (my postdoctoral supervisor) and J. Seberry for helpful discussions. It has been brought to my attention that similar ideas were pursued in [9], which I was not aware of. Some other related references are [2], [6], [8], [11], [12].

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