

NONLINEAR HEMIVARIATIONAL INEQUALITIES AT RESONANCE

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In this paper we consider nonlinear hemivariational inequalities involving the p -Laplacian at resonance. We prove the existence of a nontrivial solution. Our approach is variational based on the critical point theory for nonsmooth, locally Lipschitz functionals due to Chang.

1. INTRODUCTION

Let $Z \subset \mathbb{R}^N$ be a bounded domain with a C^1 -boundary Γ . In this paper we study the following nonlinear hemivariational inequality at resonance:

$$(HVI) \quad \begin{cases} -\operatorname{div}(\|\nabla x(z)\|^{p-2} \nabla x(z)) - \lambda_1 |x(z)|^{p-2} x(z) \in \partial j(z, x(z)) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{almost everywhere on } Z \\ x|_{\Gamma} = 0, \end{cases}$$

where $2 \leq p < +\infty$. Here λ_1 is the first eigenvalue of the p -Laplacian $-\Delta_p x \stackrel{\text{df}}{=} -\operatorname{div}(\|\nabla x\|^{p-2} \nabla x)$ with Dirichlet boundary condition (that is, of the operator $(-\Delta_p, W_0^{1,p}(Z))$), $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a functional measurable in $z \in Z$ which is locally Lipschitz in $\zeta \in \mathbb{R}$ and $\partial j(z, \zeta)$ denotes the subdifferential of $j(z, \cdot)$ in the sense of Clarke [7] (generalised subdifferential). Our starting point is the recent paper by Goeleven-Motreanu-Panagiotopoulos [9], where the authors considered eigenvalue problems for semilinear hemivariational inequalities. In that paper the authors proved the existence of multiple solutions for the eigenvalue problems they examined, and also mentioned as an open problem for future research the resonant case. Our work here can also be viewed as the continuation of the recent work by the authors (see Gasiński-Papageorgiou [8]) where the work of Goeleven-Motreanu-Panagiotopoulos [9] was extended to eigenvalue problems for nonlinear hemivariational inequalities involving the p -Laplacian. In this paper we assume that the generalised subdifferential of $j(z, \zeta)$ has nonzero limits as $\zeta \rightarrow \pm\infty$. Assuming that $j(\cdot, 0) \in L^\infty(Z)$, by the Lebourg mean value theorem (see

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Clarke [7, Theorem 2.3.7, p.41]) we infer that $j(z, \zeta)$ (the generalised potential) goes to infinity as $\zeta \rightarrow \pm\infty$. This is the case studied by Ahmad-Lazer-Paul [1] and Rabinowitz [14], where $p = 2$ (semilinear case) and $j(z, \cdot) \in C^1$. The case of finite limits for $j(z, \zeta)$ as $\zeta \rightarrow \pm\infty$ was examined by Landesman-Lazer [10] (the pioneering work in this area), Thews [17], Ward [18] and Benci-Bartolo-Fortunato [3], where $p = 2$ and $j(z, \cdot) \in C^1$. We should mention that Benci-Bartolo-Fortunato called their problem “strongly resonant”.

Hemivariational inequalities arise in physical problems when we deal with nonconvex nonsmooth energy functionals. Such functions appear quite often in mechanics and engineering if one wants to consider more realistic mechanical laws of a nonmonotone, multivalued nature. For concrete applications of hemivariational inequalities to problems in mechanics we refer to the book of Panagiotopoulos [13], Nanievich-Panagiotopoulos [12] and references therein.

Our approach is variational based on the critical point theory for nonsmooth Lipschitz functionals as this was developed by Chang [6]. For the convenience of the reader in the next section we recall the basic notions and facts from that theory, which we shall need in the sequel.

2. PRELIMINARIES

Let X be a Banach space and X^* its topological dual. A function $f : X \rightarrow \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood U of x and a constant $k > 0$ depending on U such that $|f(z) - f(y)| \leq k \|z - y\|$ for all $z, y \in U$. It is well known from convex analysis that a proper, convex and lower semicontinuous function $g : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its domain $\text{dom } g = \{x \in X : g(x) < +\infty\}$. In analogy with the directional derivative of a convex function, we define the generalised directional derivative of a locally Lipschitz function f at $x \in X$ in the direction $h \in X$, by

$$f^0(x; h) = \limsup_{\substack{x' \rightarrow x \\ \lambda \searrow 0}} \frac{f(x + x' + \lambda h) - f(x + x')}{\lambda}.$$

It is easy to check that $X \ni h \mapsto f^0(x; h) \in \mathbb{R}$ is sublinear and continuous and $|f^0(x; h)| \leq k \|h\|$. So by the Hahn-Banach theorem $f^0(x; \cdot)$ is the support function of the nonempty, convex and w^* -compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^0(x, h) \text{ for all } h \in X\}$$

known as the “generalised subdifferential” of f at x . Note that for every $x^* \in \partial f(x)$ we have $\|x^*\| \leq k$. Also if $f, g : X \rightarrow \mathbb{R}$ are locally Lipschitz functions, then $\partial(f + g) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbb{R}$. Moreover, if $f : X \rightarrow \mathbb{R}$ is convex then it is well-known that f is locally Lipschitz and the subdifferential of f in the

sense of convex analysis coincides with the generalised subdifferential introduced above. Moreover, if f is strictly differentiable at x (in particular if f is continuously Gateaux differentiable at x), then $\partial f(x) = \{f'(x)\}$.

Let $f : X \rightarrow \mathbb{R}$ be a locally Lipschitz function on a Banach space X . A point $x \in X$ is said to be a “critical point” of f , if $0 \in \partial f(x)$. It is easy to see that, if $x \in X$ is a local minimum of f , then x is a critical point. We say that f satisfies the “Palais-Smale condition” ((PS)-condition) if any sequence $\{x_n\}_{n \geq 1} \subseteq X$ along which $\{f(x_n)\}_{n \geq 1}$ is bounded and $m(x_n) \stackrel{df}{=} \min\{\|x^*\| : x^* \in \partial f(x_n)\} \rightarrow 0$ as $n \rightarrow +\infty$, has a strongly convergent subsequence. If $f \in C^1(X)$, then since $\partial f(x_n) = \{f'(x_n)\}$, we see that the above definition of the Palais-Smale condition coincides with the classical one (see Rabinowitz [15]).

The first theorem is due to Chang [6] and extends to a nonsmooth setting the well known “mountain pass theorem” due to Ambrosetti-Rabinowitz [2].

THEOREM 1. *If X is a reflexive Banach space, $R : X \rightarrow \mathbb{R}$ is locally Lipschitz functional which satisfies the (PS)-condition and for some $\rho > 0$ and $y \in X$ with $\|y\| > \rho$ we have*

$$\max\{R(0), R(y)\} < \inf_{\|x\|=\rho} \{R(x)\},$$

then R has a nontrivial critical point $x \in X$ such that the critical value $c = R(x)$ is characterised by the following minimax principle

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} \{R(\gamma(\tau))\},$$

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = y\}$.

In (HVI) appears the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. This is the least real number λ for which the problem

$$(EP) \quad \begin{cases} -\operatorname{div}(\|\nabla x(z)\|^{p-2} \nabla x(z)) = \lambda |x(z)|^{p-2} x(z) \text{ almost everywhere on } Z \\ x|_{\Gamma} = 0 \end{cases}$$

has a nontrivial solution. This first eigenvalue λ_1 is positive, isolated and simple (that is, the associated eigenfunctions are constant multiples of each other). Furthermore we have a variational characterisation of λ_1 via the Rayleigh quotient, that is,

$$\lambda_1 = \min \left\{ \frac{\|\nabla x\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \neq 0 \right\}.$$

This minimum is realised at the normalised eigenfunction u_1 . Note that if u_1 minimises the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change sign on Z . In fact we can show that $u_1(z) \neq 0$ almost everywhere on Z and so we can assume that $u_1 > 0$ almost everywhere on Z . For details we refer to Lindqvist [11].

3. EXISTENCE OF SOLUTIONS

We start by introducing our hypotheses on the function $j(z, \zeta)$.

$H(j)$ $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a functional such that:

- (i) for all $\zeta \in \mathbb{R} : Z \ni z \mapsto j(z, \zeta) \in \mathbb{R}$ is measurable;
- (ii) for almost all $z \in Z : \mathbb{R} \ni \zeta \mapsto j(z, \zeta) \in \mathbb{R}$ is locally Lipschitz;
- (iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $v \in \partial j(z, \zeta)$ we have $|v| \leq a(z)$ with some $a \in L^\infty(Z)$;
- (iv) $j(\cdot, 0) \in L^\infty(Z)$ and $\int_Z j(z, 0) dz \geq 0$;
- (v) for almost all $z \in Z$ and all $v(z, \zeta) \in \partial j(z, \zeta) : v(\cdot, \zeta)$ is measurable, we have $v(z, \zeta) \rightarrow v_-(z)$ as $\zeta \rightarrow -\infty$ and $v(z, \zeta) \rightarrow v_+(z)$ as $\zeta \rightarrow +\infty$, where v_- and v_+ are measurable and $v_-(z) \leq 0 \leq v_+(z)$ almost everywhere on Z with strict inequalities on a set of positive Lebesgue measurable;
- (vi) there exists $\mu > \lambda_1$ such that

$$\limsup_{\zeta \rightarrow 0} \frac{pj(z, \zeta)}{|\zeta|^p} < -\mu$$

uniformly for almost all $z \in Z$.

We introduce two functionals $K : W_0^{1,p}(Z) \mapsto \mathbb{R}_+$ and $L : W_0^{1,p}(Z) \mapsto \mathbb{R}$ defined by

$$K(x) \stackrel{df}{=} \frac{1}{p} \|\nabla x\|_p^p$$

$$L(x) \stackrel{df}{=} \frac{\lambda_1}{p} \|x\|_p^p + \int_Z j(z, x(z)) dz.$$

Clearly $K \in C^1(W_0^{1,p}(Z))$ and is convex (thus K is locally Lipschitz). Using [7, Theorem 2.7.5, p.83], we see that L is locally Lipschitz too. Set $R = K - L$. Then $R : W_0^{1,p}(Z) \mapsto \mathbb{R}$ is locally Lipschitz.

LEMMA 2. *If hypotheses $H(j)$ hold, then R satisfies the (PS)-condition.*

PROOF: Let $\{x_n\}_{n \geq 1} \subset W_0^{1,p}(Z)$ be a sequence such that $\{R(x_n)\}_{n \geq 1}$ is bounded and $m(x_n) \rightarrow 0$ as $n \rightarrow +\infty$. Then there exists $M_1 > 0$ such that for all $n \geq 1$ we have $|R(x_n)| \leq M_1$, so

$$(1) \quad -M_1 \leq \frac{1}{p} \|\nabla x_n\|_p^p - \frac{\lambda_1}{p} \|x_n\|_p^p - \int_Z j(z, x_n(z)) dz \leq M_1.$$

Suppose that the sequence $\{x_n\}_{n \geq 1} \subset W_0^{1,p}(Z)$ was unbounded. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$. Set $y_n =$

$(x_n / \|x_n\|)$ for $n \geq 1$. By passing to a further subsequence, if necessary, we may assume that

$$\begin{aligned} y_n &\rightharpoonup y \text{ weakly in } W_0^{1,p}(Z), \\ y_n &\rightarrow y \text{ in } L^p(Z), \\ y_n(z) &\rightarrow y(z) \text{ almost everywhere on } Z \text{ as } n \rightarrow +\infty, \end{aligned}$$

and $|y_n(z)| \leq k(z)$ almost everywhere on Z with $k \in L^p(Z)$ (see [4, Theorem IV.9, p.58]).

Dividing (1) by $\|x_n\|^p$, we obtain

$$(2) \quad -\frac{M_1}{\|x_n\|^p} \leq \frac{1}{p} \|\nabla y_n\|_p^p - \frac{\lambda_1}{p} \|y_n\|_p^p - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^p} dz \leq \frac{M_1}{\|x_n\|^p}.$$

Let us establish the asymptotic behaviour of the integral $\int_Z (j(z, x_n(z)) / \|x_n\|^p) dz$. By the Lebourg mean value theorem (see Clarke [7, Theorem 2.3.7, p.47], for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and for some $v \in \partial j(z, \bar{\zeta})$, $\bar{\zeta} = \alpha\zeta$, $0 < \alpha < 1$, we have, using hypothesis $H(j)$ (iii), that

$$|j(z, \zeta) - j(z, 0)| = |\langle v, \bar{\zeta} \rangle| \leq a(z)|\zeta|,$$

and using $H(j)$ (iv) we obtain

$$|j(z, \zeta)| \leq |j(z, 0)| + a(z)|\zeta| \leq a_1 + a_2|\zeta|,$$

for some $a_1, a_2 > 0$. So we can write that

$$\begin{aligned} \left| \int_Z \frac{j(z, x_n(z))}{\|x_n\|^p} dz \right| &\leq \int_Z \frac{|j(z, x_n(z))|}{\|x_n\|^p} dz \leq \int_Z \frac{a_1 + a_2|x_n(z)|}{\|x_n\|^p} dz \\ &\leq \frac{a_3}{\|x_n\|^p} + \frac{a_4}{\|x_n\|^{p-1}}, \end{aligned}$$

with $a_3 = a_1|Z|$ and $a_4 = a_2 \int_Z |k(z)| dz$. So

$$\int_Z \frac{j(z, x_n(z))}{\|x_n\|^p} dz \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Thus if we pass to the limit as $n \rightarrow +\infty$ in (2), we obtain

$$\frac{1}{p} \liminf_{n \rightarrow +\infty} \|\nabla y_n\|_p^p = \frac{\lambda_1}{p} \|y\|_p^p.$$

But since $\nabla y_n \rightharpoonup \nabla y$ weakly in $L^p(Z, \mathbb{R}^N)$ as $n \rightarrow +\infty$ (recall that $y_n \rightarrow y$ weakly in $W_0^{1,p}(Z)$), from the weak lower semicontinuity of the norm functional, we have that

$$\frac{1}{p} \|\nabla y\|_p^p \leq \liminf_{n \rightarrow +\infty} \frac{1}{p} \|\nabla y_n\|_p^p,$$

and so

$$\frac{1}{p} \|\nabla y\|_p^p \leq \frac{\lambda_1}{p} \|y\|_p^p.$$

On the other hand from the variational characterisation of λ_1 (Rayleigh quotient, see section 2), we have

$$\frac{\lambda_1}{p} \|y\|_p^p \leq \frac{1}{p} \|\nabla y\|_p^p.$$

So finally we deduce that

$$\|\nabla y\|_p^p = \lambda_1 \|y\|_p^p.$$

Also, by passing to the limit in (2), we obtain

$$\limsup_{n \rightarrow +\infty} \frac{1}{p} \|\nabla y_n\|_p^p \leq \frac{\lambda_1}{p} \|y\|_p^p = \frac{1}{p} \|\nabla y\|_p^p \leq \liminf_{n \rightarrow +\infty} \frac{1}{p} \|\nabla y_n\|_p^p,$$

so

$$\|\nabla y_n\|_p \rightarrow \|\nabla y\|_p \quad \text{as } n \rightarrow +\infty.$$

Since $\nabla y_n \rightharpoonup \nabla y$ weakly in $L^p(Z, \mathbb{R}^N)$ as $n \rightarrow +\infty$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex, we infer that $\nabla y_n \rightarrow \nabla y$ in $L^p(Z, \mathbb{R}^N)$ as $n \rightarrow +\infty$. Hence $y_n \rightarrow y$ in $W_0^{1,p}(Z)$ as $n \rightarrow +\infty$. But $\|y_n\| = 1$ for $n \geq 1$, so $\|y\| = 1$, thus $y \neq 0$. Therefore from the equality $\|\nabla y\|_p^p = \lambda_1 \|y\|_p^p$, it follows that $y = \pm u_1$.

Without loss of generality we may assume that $y = +u_1$ (the analysis is similar if $y = -u_1$). So from section 2 we know that $y(z) = u_1(z) > 0$ almost everywhere on Z . Now let $x_n^* \in \partial R(x_n)$ for $n \geq 1$, be such that $m(x_n) = \|x_n^*\|$. The existence of such an element follows from the fact that $\partial R(x_n)$ is a nonempty weakly compact subset of $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ (see section 2) and the weak lower semicontinuity of the norm functional. Consider the operator $A : W_0^{1,p}(Z) \mapsto W^{-1,q}(Z)$ defined by

$$\langle Ax, u \rangle = \int_Z \|\nabla x(z)\|^{p-2} (\nabla x(z), \nabla u(z))_{\mathbb{R}^N} dz \quad \forall x, u \in W_0^{1,p}(Z).$$

Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. It is easy to check that A is monotone, hemicontinuous, hence maximal monotone. So it is also generalised pseudomonotone (see Browder-Hess [5]). Then we have

$$x_n^* = Ax_n - \lambda_1 \|x_n\|^{p-2} x_n - v_n,$$

where $v_n \in \partial\psi(x_n)$, $n \geq 1$, with $\psi : W_0^{1,p}(Z) \rightarrow \mathbb{R}$ being defined by $\psi(x) = \int_Z j(z, x(z)) dz$. We know that, if $v \in \partial\psi(x)$, then $v(z) \in \partial j(z, x(z))$ almost everywhere on Z (see Clarke [7]). Note that $\psi = \widehat{\psi}|_{W_0^{1,p}(Z)}$ where $\widehat{\psi} : L^p(Z) \rightarrow \mathbb{R}$ is defined by $\widehat{\psi}(x) = \int_Z j(z, x(z)) dz$. So invoking [6, Theorem 2.2], we have that $\partial\psi(x) \subseteq \widehat{\partial\psi}(x) \subseteq L^q(Z)$. Hence $v_n \in L^q(Z)$ for all $n \geq 1$.

From the choice of the sequence $\{x_n\}_{n \geq 1}$ we have that $|R(x_n)| \leq M_1$ and

$$(3) \quad |\langle x_n^*, u \rangle| \leq \varepsilon_n \|u\| \quad \forall u \in W_0^{1,p}(Z) \quad \text{with } \varepsilon_n \searrow 0.$$

From (1), we have

$$(4) \quad -pM_1 \leq \|\nabla x_n\|_p^p - \lambda_1 \|x_n\|_p^p - p \int_Z j(z, x_n(z)) \, dz \leq pM_1,$$

and taking $u = x_n$ in (3), we have

$$(5) \quad -\varepsilon_n \|x_n\| \leq -\langle Ax_n, x_n \rangle + \lambda_1 \|x_n\|_p^p + \int_Z v_n(z)x_n(z) \, dz \leq \varepsilon_n \|x_n\|.$$

We add (4) and (5). Because $\langle Ax_n, x_n \rangle = \|\nabla x_n\|_p^p$, we have

$$-pM_1 - \varepsilon_n \|x_n\| \leq \int_Z (v_n(z)x_n(z) - pj(z, x_n(z))) \, dz \leq pM_1 + \varepsilon_n \|x_n\|.$$

Dividing by $\|x_n\|$ we obtain

$$(6) \quad -\frac{pM_1}{\|x_n\|} - \varepsilon_n \leq \int_Z \left(v_n(z)y_n(z) - \frac{pj(z, x_n(z))}{\|x_n\|} \right) \, dz \leq \frac{pM_1}{\|x_n\|} + \varepsilon_n.$$

We know that $y_n(z) \rightarrow u_1(z) > 0$ almost everywhere on Z as $n \rightarrow +\infty$ and so $x_n(z) \rightarrow +\infty$ almost everywhere on Z as $n \rightarrow +\infty$. Then because of hypothesis $H(j)(v)$ and since $v_n(z) \in \partial j(z, x_n(z))$ almost everywhere on Z , we have that $v_n(z) \rightarrow v_+(z)$ almost everywhere on Z as $n \rightarrow +\infty$. Hence by the Lebesgue dominated convergence theorem, we have that $\int_Z v_n(z)y_n(z) \, dz \rightarrow \int_Z v_+(z)u_1(z) \, dz$ as $n \rightarrow +\infty$. Next let $N \subseteq Z$ be the Lebesgue-null set outside of which we have $x_n(z) \rightarrow +\infty$ as $n \rightarrow +\infty$ and hypothesis $H(j)(v)$ holds. Fix $z \in Z \setminus N$. For a given $0 < \varepsilon < 1$, from the Lebourg mean value theorem, we have

$$j(z, x_n(z)) = j(z, \varepsilon x_n(z)) + w_n(z)(1 - \varepsilon)x_n(z),$$

with $w_n(z) \in \partial j(z, u_n(z))$, where $u_n(z) = (1 - \alpha_n)x_n(z) + \alpha_n \varepsilon x_n(z)$, with $0 < \alpha_n < 1$, for $n \geq 1$. Hence $u_n(z) = x_n(z) - \alpha_n(1 - \varepsilon)x_n(z) \geq x_n(z) - (1 - \varepsilon)x_n(z) = \varepsilon x_n(z)$. Since $x_n(z) \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\varepsilon > 0$, we have that $u_n(z) \rightarrow +\infty$ as $n \rightarrow +\infty$ and so $w_n(z) \rightarrow v_+(z)$ as $n \rightarrow +\infty$ (see hypothesis $H(j)(v)$). Now let $n_0 = n_0(\varepsilon, z) \geq 1$ be such that for all $n \geq n_0$ we have $x_n(z) > 0$ and $|w_n(z) - v_+(z)| \leq \varepsilon$. We have, for $n \geq 1$, that

$$\frac{pj(z, x_n(z))}{x_n(z)} = \frac{pj(z, \varepsilon x_n(z))}{x_n(z)} + \frac{pw_n(z)(1 - \varepsilon)x_n(z)}{x_n(z)}.$$

Recall that $|j(z, \varepsilon x_n(z))| \leq a_1 + a_2 \varepsilon |x_n(z)|$, while $-\varepsilon + v_+(z) \leq w_n(z) \leq \varepsilon + v_+(z)$. So for $n \geq n_0$ we can write that

$$\begin{aligned} & \frac{-pa_1 - pa_2\varepsilon x_n(z)}{x_n(z)} + \frac{p(-\varepsilon + v_+(z))(1 - \varepsilon)x_n(z)}{x_n(z)} \\ & \leq \frac{pj(z, x_n(z))}{x_n(z)} \leq \frac{pa_1 + pa_2\varepsilon x_n(z)}{x_n(z)} + \frac{p(\varepsilon + v_+(z))(1 - \varepsilon)x_n(z)}{x_n(z)}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, from the above inequalities which are valid for $n \geq n_0$, we infer that

$$\frac{pj(z, x_n(z))}{x_n(z)} \rightarrow pv_+(z) \quad \text{as } n \rightarrow +\infty.$$

Therefore it follows that

$$\begin{aligned} \int_Z \frac{pj(z, x_n(z))}{\|x_n\|} dz &= \int_Z \frac{pj(z, x_n(z))}{x_n(z)} \cdot \frac{x_n(z)}{\|x_n\|} dz \\ &= \int_Z \frac{pj(z, x_n(z))}{x_n(z)} y_n(z) dz \rightarrow p \int_Z v_+(z) u_1(z) dz \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

So if we pass to the limit as $n \rightarrow +\infty$ in (6), we obtain

$$(1 - p) \int_Z v_+(z) u_1(z) dz = 0$$

so

$$\int_Z v_+(z) u_1(z) dz = 0.$$

But recall that $u_1(z) > 0$ almost everywhere on Z and $v_+(z) \geq 0$ almost everywhere on Z with strict inequality on a set of positive Lebesgue measure (see hypothesis $H(j)(v)$). Thus $\int_Z v_+(z) u_1(z) dz > 0$, a contradiction. This implies that $\{x_n\}_{n \geq 1} \subset W_0^{1,p}(Z)$ is bounded. So by passing to a subsequence if necessary we may assume that $x_n \rightharpoonup x$ weakly in $W_0^{1,p}(Z)$, $x_n \rightarrow x$ in $L^p(Z)$, $x_n(z) \rightarrow x(z)$ almost everywhere on Z and $|x_n(z)| \leq k_1(z)$ almost everywhere on Z with $k_1 \in L^p(Z)$.

Putting $u = x_n - x$ in (3) we obtain

$$\begin{aligned} (7) \quad -\varepsilon_n \|x_n - x\| &\leq \langle Ax_n, x_n - x \rangle - \lambda_1 \int_Z |x_n(z)|^{p-2} x_n(z) (x_n - x)(z) dz \\ &\quad - \int_Z v_n(z) (x_n - x)(z) dz \leq \varepsilon_n \|x_n - x\|. \end{aligned}$$

Note that

$$\begin{aligned} \lambda_1 \int_Z |x_n(z)|^{p-2} x_n(z) (x_n - x)(z) dz &\rightarrow 0 \quad \text{as } n \rightarrow +\infty, \text{ and} \\ \int_Z v_n(z) (x_n - x)(z) dz &\rightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

So from (7), we have

$$\limsup_{n \rightarrow +\infty} \langle Ax_n, x_n - x \rangle \leq 0.$$

But we already know that A is generalised pseudomonotone. So we have

$$\langle Ax_n, x_n \rangle \rightarrow \langle Ax, x \rangle \quad \text{as } n \rightarrow +\infty,$$

so

$$\|\nabla x_n\|_p^p \rightarrow \|\nabla x\|_p^p \quad \text{as } n \rightarrow +\infty.$$

Also $\nabla x_n \rightarrow \nabla x$ weakly in $L^p(Z, \mathbb{R}^N)$ as $n \rightarrow +\infty$. Since $L^p(Z, \mathbb{R}^N)$ is uniformly convex, we conclude that $\nabla x_n \rightarrow \nabla x$ in $L^p(Z, \mathbb{R}^N)$ as $n \rightarrow +\infty$ and so $x_n \rightarrow x$ in $W_0^{1,p}(Z)$ as $n \rightarrow +\infty$. Thus R satisfies the (PS)-condition. \square

LEMMA 3. *If hypotheses $H(j)$ hold, then there exist $\beta_1, \beta_2 > 0$ such that for all $x \in W_0^{1,p}(Z)$, we have*

$$R(x) \geq \beta_1 \|x\|^p - \beta_2 \|x\|^\vartheta$$

with $p < \vartheta \leq p^* = Np/(N - p)$.

PROOF: Let $\varepsilon > 0$ be such that $\lambda_1 + \varepsilon < \mu$. From hypothesis $H(j)(vi)$ we can find $\delta > 0$ such that for almost all $z \in Z$ and all ζ such that $|\zeta| \leq \delta$ we have

$$j(z, \zeta) \leq \frac{1}{p}(-\mu + \varepsilon)|\zeta|^p.$$

On the other hand from the proof of Lemma 2 we know that for almost all $z \in Z$ and all ζ such that $|\zeta| > \delta$ we have

$$|j(z, \zeta)| \leq a_1 + a_2|\zeta|,$$

with some $a_1, a_2 > 0$. Thus for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$ we have

$$j(z, \zeta) \leq \frac{1}{p}(-\mu + \varepsilon)|\zeta|^p + \gamma|\zeta|^\vartheta,$$

with $\gamma = (a_1 + a_2\delta)\delta^{-\vartheta} + (\mu - \varepsilon)\delta^{p-\vartheta}/p$ and $p < \vartheta \leq p^* = Np/(N - p)$. Using this we obtain that

$$\begin{aligned} R(x) &= \frac{1}{p} \|\nabla x\|_p^p - \frac{\lambda_1}{p} \|x\|_p^p - \int_Z j(z, x(z)) \, dz \\ &\geq \frac{1}{p} \|\nabla x\|_p^p - \frac{\lambda}{p} \|x\|_p^p + \frac{1}{p}(\mu - \varepsilon) \|x\|_p^p - \gamma \|x\|_\vartheta^\vartheta \\ &= \frac{1}{p} \|\nabla x\|_p^p - \frac{1}{p}(\lambda_1 - \mu + \varepsilon) \|x\|_p^p - \gamma \|x\|_\vartheta^\vartheta. \end{aligned}$$

From the choice of ε , we have $-(1/p)(\lambda_1 - \mu + \varepsilon) \|x\|_p^p > 0$, so

$$R(x) \geq \frac{1}{p} \|\nabla x\|_p^p - \gamma \|x\|_\vartheta^\vartheta.$$

Because $\vartheta < p^* = Np/(N - p)$, from the Sobolev embedding theorem we have that $W_0^{1,p}(Z)$ is embedded continuously in $L^\vartheta(Z)$. So using the Poincare inequality, it follows that

$$R(x) \geq \beta_1 \|x\|^p - \beta_2 \|x\|^\vartheta,$$

for some $\beta_1, \beta_2 > 0$ and all $x \in W_0^{1,p}(Z)$. □

Using Lemmas 2 and 3 we have the following existence theorem for problem (HVI).

THEOREM 4. *If hypothesis H(j) hold, then the problem (HVI) has a nontrivial solution.*

PROOF: From Lemma 3 we know that there exist $\beta_1, \beta_2 > 0$ such that for all $x \in W_0^{1,p}(Z)$ we have

$$R(x) \geq \beta_1 \|x\|^p - \beta_2 \|x\|^\vartheta = \beta_1 \|x\|^p \left(1 - \frac{\beta_2}{\beta_1} \|x\|^{\vartheta-p} \right).$$

Since $p < \vartheta$, we can find $\rho > 0$ small enough such that

$$\inf_{\|x\|=\rho} \{R(x)\} > 0.$$

Because $\int_Z j(z, 0) dz \geq 0$, we have that $R(0) \leq 0$. Also for all $\xi > 0$, we have

$$R(\xi u_1) = \frac{\xi^p}{p} \|\nabla u_1\|_p^p - \frac{\lambda_1 \xi^p}{p} \|u_1\|_p^p - \int_Z j(z, \xi u_1(z)) dz = - \int_Z j(z, \xi u_1(z)) dz,$$

since $\|\nabla u_1\|_p^p = \lambda_1 \|u_1\|_p^p$ (Rayleigh quotient). From the proof of Lemma 2 we know that

$$\frac{j(z, \xi u_1(z))}{\xi u_1(z)} \rightarrow v_+(z) \text{ almost everywhere on } Z, \text{ as } \xi \rightarrow +\infty.$$

So, recalling that $u_1(z) > 0$ almost everywhere on Z , we have

$$j(z, \xi u_1(z)) \rightarrow +\infty \text{ almost everywhere on } Z, \text{ as } \xi \rightarrow +\infty.$$

So for $\xi > 0$ large enough we shall have that $R(\xi u_1) \leq 0$. This permits the use of Theorem 1 which gives us $x \in W_0^{1,p}(Z)$ such that $R(x) > 0 \geq R(0)$ (hence $x \neq 0$) and $0 \in \partial R(x)$. From this last inclusion we obtain

$$0 = Ax - \lambda_1 |x|^{p-2} x - v,$$

with $v \in \partial\psi(x) \subseteq L^q(Z)$. Hence

$$Ax = \lambda_1 |x|^{p-2} x + v,$$

and

$$\langle Ax, \varphi \rangle = \lambda_1 \left(|x|^{p-2} x, \varphi \right)_{pq} + (v, \varphi)_{pq} \quad \forall \varphi \in C_0^\infty(Z).$$

Here by $(\cdot, \cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$. So, for all $\varphi \in C_0^\infty(Z)$, we have

$$\int_Z \|\nabla x(z)\|^{p-2} (\nabla x(z), \nabla \varphi(z))_{\mathbb{R}^N} dz = \int_Z (\lambda_1 |x(z)|^{p-2} x(z) + v(z)) \varphi(z) dz.$$

From the definition of the distributional derivative we have

$$\begin{cases} -\operatorname{div}(\|\nabla x(z)\|^{p-2} \nabla x(z)) = \lambda_1 x(z) |x(z)|^{p-2} x(z) + v(z) & \text{almost everywhere on } Z \\ x|_\Gamma = 0, \end{cases}$$

so

$$\begin{cases} -\operatorname{div}(\|\nabla x(z)\|^{p-2} \nabla x(z)) - \lambda_1 x(z) |x(z)|^{p-2} x(z) \in \partial j(z, x(z)) & \text{almost everywhere on } Z \\ x|_\Gamma = 0, \end{cases}$$

(recall that $v \in \partial\psi(x)$ implies $v(z) \in \partial j(z, x(z))$ almost everywhere on Z). Therefore $x \in W_0^{1,p}(Z)$ is a nontrivial solution of (HVI). \square

REMARK 5. In a companion paper we shall deal with the existence of multiple solutions for semilinear (that is, $p = 2$) hemivariational inequalities at resonance. Our formulation here incorporates problems with discontinuities. In this case $j(z, \zeta) = \int_0^\zeta h(z, r) dr$ with $h : Z \times \mathbb{R} \rightarrow \mathbb{R}$ a Borel measurable function. In this case we know (see Chang [6]) that $\partial j(z, \zeta) \subseteq [h_1(z, \zeta), h_2(z, \zeta)]$ with $h_1(z, \zeta) = \liminf_{\bar{\zeta} \rightarrow \zeta} h(z, \bar{\zeta})$ and $h_2(z, \zeta) = \limsup_{\bar{\zeta} \rightarrow \zeta} h(z, \bar{\zeta})$. Such problems were considered by Chang [6] for semilinear equations ($p = 2$).

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