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Near Triangularizability Implies Triangularizability

With my best regards, dedicated to A. Jafarian, H. Radjavi, P. Rosenthal, and A.R. Sourour

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Abstract. In this paper we consider collections of compact operators on a real or complex Banach space including linear operators on finite-dimensional vector spaces. We show that such a collection is simultaneously triangularizable if and only if it is arbitrarily close to a simultaneously triangularizable collection of compact operators. As an application of these results we obtain an invariant subspace theorem for certain bounded operators. We further prove that in finite dimensions near reducibility implies reducibility whenever the ground field is $\mathbb{R}$ or $\mathbb{C}$.

1 Introduction

We start by recalling some definitions and standard notations. Throughout this paper, unless otherwise stated, $\mathcal{X}$ stands for a separable real or complex Banach space. As is usual, by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. The terms *subspace* and *operator* or *linear operator* will, respectively, be used to describe a closed subspace of a Banach space $\mathcal{X}$ and a bounded linear operator on $\mathcal{X}$. If F is a field and $\mathcal{V}$ is a finite-dimensional vector space over F, then we use $\mathcal{L}(\mathcal{V})$ to denote the set (in fact the algebra) of linear transformations on $\mathcal{V}$. We use $\mathcal{B}(\mathcal{X})$ to denote the set (in fact the algebra) of bounded operators on $\mathcal{X}$; $\mathcal{B}_0(\mathcal{X})$ is used to denote the set (in fact the ideal) of compact operators on $\mathcal{X}$. We note that if $\mathcal{X}$ is a finite-dimensional real or complex Banach space, then $\mathcal{L}(\mathcal{X}) = \mathcal{B}(\mathcal{X}) = \mathcal{B}_0(\mathcal{X})$, and that every linear subspace of $\mathcal{X}$ is necessarily closed. A subspace $\mathcal{M}$ is *invariant* for a collection $\mathcal{F}$ of bounded operators (resp. linear transformations) if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F}$; $\mathcal{M}$ is *hyperinvariant* for $\mathcal{F}$ if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F} \cup \mathcal{F}'$ where $\mathcal{F}'$ denotes the *commutant* of $\mathcal{F}$. A collection $\mathcal{F}$ of operators (resp. transformations) is called *reducible* if $\mathcal{F} = \{0\}$ or it has a nontrivial invariant subspace. This definition is slightly unconventional, but it simplifies some of the statements in what follows. For a collection $\mathcal{C}$ of vectors, $\langle \mathcal{C} \rangle$ denotes *the (not necessarily closed) linear manifold spanned by* $\mathcal{C}$. A *projection* or an *idempotent* is a bounded operator $P \in \mathcal{B}(\mathcal{X})$ (resp. linear transformation $P \in \mathcal{L}(\mathcal{V})$) satisfying $P^2 = P$. If P is an idempotent,

$$\mathcal{M} := \{x \in \mathcal{X} : Px = x\} = P\mathcal{X}, \ \mathcal{N} := \{x \in \mathcal{X} : Px = 0\} = \ker P,$$

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then P is said to be *the projection on* $\mathcal{M}$ *along* $\mathcal{N}$, and $\mathcal{M}$ and $\mathcal{N}$ are *complementary* subspaces of $\mathcal{X}$, *i.e.*, $\mathcal{M} + \mathcal{N} = \mathcal{X}, \mathcal{M} \cap \mathcal{N} = \{0\}$. In case the space is an arbitrary Hilbert space and $\mathcal{M} \perp \mathcal{N}$, the the projection P is called *the orthogonal projection on* $\mathcal{M}$ *along* $\mathcal{N}$, or simply *the orthogonal projection on* $\mathcal{M}$. Let P and Q be two idempotents, by definition $P \leq Q$ if $PQ = P = QP$, equivalently $P\mathcal{X} \subset Q\mathcal{X}$ (resp. $P\mathcal{V} \subset Q\mathcal{V}$), or $\ker P \supset \ker Q$ (see Lemma 7.5.2 of [9]). Existence of invariant (resp. hyperinvariant) subspaces for a collection $\mathcal{F}$ of transformations or operators can be expressed in terms of idempotents as follows. By Theorem 6.4.5(i) of [9], a subspace $\mathcal{M}$ is invariant (resp. hyperinvariant) for a family $\mathcal{F}$ of transformations or operators iff there exists a projection P on $\mathcal{M}$ such that $TP = PTP$ for all $T \in \mathcal{F}$ (resp. $TP = PTP$ for all $T \in \mathcal{F} \cup \mathcal{F}'$). If the underlying space $\mathcal{X}$ happens to be a finite-dimensional Hilbert space (resp. normed linear space), then the projection P above can be chosen to be orthogonal, equivalently $\|P\| = 1$, (resp. $\|P\| \leq \sqrt{\dim \mathcal{X}}$ in view of Theorem 4.15 of [1]), where $\|\cdot\|$ denotes the operator norm induced by the norm of $\mathcal{X}$.

A collection $\mathcal{F}$ of operators (resp. transformations) is called *simultaneously triangularizable* or simply *triangularizable* if there exists a maximal chain of subspaces of $\mathcal{X}$ (resp. $\mathcal{V}$) each of which is invariant for $\mathcal{F}$. In case the underlying space is finite-dimensional, it is easily seen that triangularizability of a family of linear transformation is equivalent to the existence of a basis for the vector space such that all transformations in the family have upper triangular matrix representation with respect to that basis. In other words, there exists a finite chain of idempotents P_i ($i = 0, \dots, \dim \mathcal{V}$) such that

$$0 = P_0 < P_1 < \cdots < P_{\dim \mathcal{V}} = I,$$

and that $TP_i = P_i T P_i$ for all $T \in \mathcal{F}$ and $i = 1, \dots, \dim \mathcal{V}$. If the underlying space $\mathcal{X}$ happens to be a finite-dimensional real or complex Hilbert space (resp. Banach space), we may assume that $\|P_i\| = 1$ (resp. $\|P_i\| \leq \sqrt{\dim \mathcal{X}}$ in view of Theorem 4.15 of [1]) for all $1 \leq i \leq \dim \mathcal{X}$ where $\|\cdot\|$ denotes the operator norm induced by the norm of $\mathcal{X}$.

Note that a collection of (individually) triangularizable operators is not necessarily triangularizable.

2 Near Triangularizability in Finite Dimensions

We start off with a well-known theorem due to O. Perron (See [9], Theorem 1.6.2).

Theorem 2.1 **(O. Perron)** *If $\mathcal{A}$ is the algebra of $n \times n$ upper triangular matrices on $\mathbb{F}$, relative to a given basis, then for every $\epsilon > 0$ there is an invertible matrix $S_\epsilon = \operatorname{diag}(\eta, \dots, \eta^n)$ where $\eta = \eta(\epsilon)$ depends on ϵ such that*

$$\|S_\epsilon^{-1} A S_\epsilon - D(A)\| < \epsilon \|A\|$$

for all $A \in \mathcal{A}$ and where $D(A)$ is the diagonal matrix with the same entries as A on its main diagonal.

Perron's Theorem immediately implies the following:

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Corollary 2.2 *Let $\mathcal{F} = \{A_\alpha : \alpha \in \Lambda\}$ be a norm bounded triangularizable family of linear transformations on $\mathbb{F}$ (i.e., on $\mathbb{R}$ or $\mathbb{C}$). Then there is a diagonalizable, thus commutative, family $\{D_\alpha : \alpha \in \Lambda\}$ (relative to the triangularizing basis for $\mathcal{F}$) such that for every $\epsilon > 0$ there is an invertible transformation T_ϵ satisfying*

$$\|T_\epsilon^{-1} A_\alpha T_\epsilon - D_\alpha\| < \epsilon,$$

for all $\alpha \in \Lambda$

Proof Triangularize $\mathcal{F}$ by a similarity T. Set

$$D_\alpha = D(T^{-1}A_\alpha T);\ T_\epsilon = TS_{\frac{\epsilon}{M+1}}$$

where $M = \sup\{\|A_\alpha\| : \alpha \in \Lambda\}$. ∎

Motivated by Theorem 1.6.4 of [9] and its proof, due to A. A. Jafarian, H. Radjavi, P. Rosenthal, and A. R. Sourour, we were able to prove the following generalization.

Theorem 2.3 *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$ with the following property: for each finite subfamily $\{A_1, \ldots, A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exists a triangularizable family $\{T_1, \ldots, T_m\}$, and an invertible transformation $S = S_\epsilon$ satisfying*

$$\|T_j\| \le K,\ \|S^{-1}A_jS - T_j\| < \epsilon,$$

for every $1 \le j \le m$ where $\|\cdot\|$ denotes any given norm on $\mathcal{B}(\mathcal{V})$. Then $\mathcal{F}$ is triangularizable.

Proof First, we prove the assertion on $\mathbb{C}$. Note that if $\mathcal{F}$ is a singleton, then we have nothing to prove, so we may assume that $|\mathcal{F}| > 1$. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$. In view of Theorem 1.3.2 and Lemma 2.1.15 of [9], it suffices to show that the trace of $(BC - CB)^n$ is 0 for all $n \in \mathbb{N}$ and $B, C \in \mathcal{A}$. Given $B, C \in \mathcal{A}$, there are $A_i \in \mathcal{F}$, $(1 \le i \le m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, \ldots, A_m),\ C = q(A_1, \ldots, A_m).$$

Since all norms on $\mathcal{B}(\mathcal{V})$ are equivalent (for $\mathcal{B}(\mathcal{V})$ is a finite-dimensional vector space, see [2], Theorem 3.3.1), without loss of generality, we may assume that $\|\cdot\| = \|\cdot\|_1$ with respect to a fixed basis of the space. So in particular, for every $T \in \mathcal{B}(\mathcal{V})$, we have

$$|\mathrm{tr}(T)| \le \|T\|,$$

where "tr" means the trace linear functional. Let $K > 0$ be the appropriate constant for $\{A_1, \ldots, A_m\}$. Define: $h_n : \mathcal{B}(\mathcal{V})^m \longrightarrow \mathcal{B}(\mathcal{V})$ by

$$h_n(X_1, \ldots, X_m) = (p(X_1, \ldots, X_m)q(X_1, \ldots, X_m) - q(X_1, \ldots, X_m)p(X_1, \ldots, X_m))^n.$$

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We observe that h_n, $n \in \mathbb{N}$, is a noncommutative polynomial in m linear transformations. It is easily seen that every such h_n is a uniformly continuous function of its arguments on any bounded set in $\langle \mathcal{B}(\mathcal{V})^m, \|\cdot\|_\infty \rangle$ where $\|(X_1, \dots, X_m)\|_\infty = \max\{\|X_1\|, \dots, \|X_m\|\}$. In particular, for every $\eta > 0$, there is a positive δ with $0 < \delta < 1$ such that

$$\|h_n(X_1, \dots, X_m) - h_n(Y_1, \dots, Y_m)\| < \eta,$$

whenever $\|X_j - Y_j\| < \delta$, $\|X_j\| \le K + 1$, $\|Y_j\| \le K + 1$ for all $1 \le j \le m$. Now, for a given $\eta > 0$, find the corresponding δ with $0 < \delta < 1$. By hypothesis, for this δ, there exists a triangularizable family $\{T_1, \dots, T_m\}$, and an invertible transformation $S = S_\delta$ satisfying

$$\|T_j\| \le K, \ \|S^{-1} A_j S - T_j\| < \delta,$$

for every $1 \le j \le m$. Clearly,

$$\|S^{-1} A_j S - T_j\| < \delta, \|S^{-1} A_j S\| \le K + 1, \|T_j\| \le K + 1,$$

for every $1 \le j \le m$. It follows from uniform continuity of h_n that

$$\|h_n(S^{-1} A_1 S, \dots, S^{-1} A_m S) - h_n(T_1, \dots, T_m)\| < \eta.$$

We note that $\operatorname{tr}(h_n(T_1, \dots, T_m)) = 0$, for $\{T_1, \dots, T_m\}$ is triangularizable. So we can write

$$\begin{aligned} |\operatorname{tr}(h_n(A_1, \dots, A_m))| &= |\operatorname{tr}(S^{-1} h_n(A_1, \dots, A_m) S)| \\ &= |\operatorname{tr}(h_n(S^{-1} A_1 S, \dots, S^{-1} A_m S))| \\ &= |\operatorname{tr}(h_n(S^{-1} A_1 S, \dots, S^{-1} A_m S)) - \operatorname{tr}(h_n(T_1, \dots, T_m))| \\ &= |\operatorname{tr}(h_n(S^{-1} A_1 S, \dots, S^{-1} A_m S) - h_n(T_1, \dots, T_m))|. \end{aligned}$$

Now, since $|\operatorname{tr}(T)| \le \|T\|$ for every $T \in \mathcal{B}(\mathcal{V})$, we can write

$$|\operatorname{tr}(h_n(A_1, \dots, A_m))| \le \|h_n(S^{-1} A_1 S, \dots, S^{-1} A_m S) - h_n(T_1, \dots, T_m)\| < \eta.$$

Thus $|\operatorname{tr}(h_n(A_1, \dots, A_m))| < \eta$. Since $\eta > 0$ was arbitrary, it follows that $\operatorname{tr}(h_n(A_1, \dots, A_m)) = 0$. That is, $\operatorname{tr}((BC - CB)^n) = 0$ for all $n \in \mathbb{N}$. Thus $\mathcal{A}$, and therefore $\mathcal{F}$, is triangularizable by Theorem 1.3.2 of [9].

To settle the assertion on $\mathbb{R}$, fixing a basis for the space, it suffices to prove the matrix version of the assertion. That being noted, first we claim that every $A \in \mathcal{F}$ is triangularizable over $\mathbb{R}$. To see this, using the hypothesis and if necessary by passing to a subsequence, it is easily seen that for every $A \in \mathcal{F}$ there exists a subsequence $(S_i)_{i \in \mathbb{N}}$ of invertible matrices and a triangularizable matrix T in $M_N(\mathbb{R})$ ($N = \dim \mathcal{V}$) such that $\lim_i S_i^{-1} A S_i = T$. Thus, $\lim_i S_i^{-1} A^n S_i = T^n$ for $n \in \mathbb{N}$, and hence

$$\operatorname{tr}(A^n) = \lim_i \operatorname{tr}(S_i^{-1} A^n S_i) = \operatorname{tr}(T^n),$$

for all $n \in \mathbb{N}$. Now, by Theorem 2.1.16 of [9], the two matrices A and T share the same characteristic polynomial. Therefore, triangularizability of A follows from that of T, for the characteristic polynomial for T splits over $\mathbb{R}$, settling the claim. Now, to establish the assertion on $\mathbb{R}$, note that the complex counterpart of the assertion we just proved, implies that the family $\mathcal{F}$ is triangularizable over $\mathbb{C}$. Now, since $\mathcal{F}$ consists of individually triangularizable matrices over $\mathbb{R}$, the triangularizability of $\mathcal{F}$ over $\mathbb{R}$ follows from Lemma 2.2.8 of [13], as desired. ■

Remarks 1. Using the same argument one can prove that the Near Triangularizability Theorem above holds for families of linear transformations on a finite-dimensional vector space over any topologically closed subfield of an algebraically closed complete field F with a nontrivial absolute value. (See Chapter XII of [7] for a nice exposition of fields with absolute values).

2. In view of Theorem 7.6.1 of [9] and Lemma 2.5.8(iv) of [13], using an argument almost identical to that of the preceding theorem, one can prove an analogue of the theorem for $\mathcal{C}_p$ operators on complex Hilbert spaces (see [13], Theorem 3.3.3).

In the preceding theorem, if the invertible transformation $S = S_\epsilon$ can always be chosen to be bounded by a function in K, or equivalently can be chosen to be the identity transformation, then we can drop K and prove the following result.

Theorem 2.4 (**Near Triangularizability Theorem**) *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$ with the following property: For each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$ and for every $\epsilon > 0$ there exists a triangularizable family $\{T_1, \dots, T_m\}$ satisfying $\|A_j - T_j\| < \epsilon$, for every $1 \le j \le m$ where $\|\cdot\|$ denotes any given norm on $\mathcal{B}(\mathcal{V})$. Then $\mathcal{F}$ is triangularizable.*

Proof Fixing a basis for the space, it suffices to prove the matrix version of the assertion. Set $n = \dim \mathcal{V}$. Since $M_n(\mathbb{F})$ is n^2-dimensional, it suffices to show that each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$ is triangularizable. In doing so, it is easily seen from the hypothesis that there exist triangularizable families $\{T_{k1}, \dots, T_{km}\}$ $(k \in \mathbb{N})$ such that $A_j = \lim_k T_{kj}$ for all $1 \le j \le m$. Now, since for each $k \in \mathbb{N}$ the family $\{T_{k1}, \dots, T_{km}\}$ is triangularizable, it follows from the definition that there exists a finite chain of orthogonal projections

$$0 = P_{k0} < P_{k1} < \cdots < P_{kn} = I,$$

such that $T_{ki}P_{kj} = P_{kj}T_{ki}P_{kj}$ for all $k \in \mathbb{N}$, $1 \le i \le m$, $1 \le j \le n$. In view of a well-known theorem of Riesz about compactness of the unit ball of finite-dimensional normed spaces, if necessary, by passing to convergent subsequences, we may assume that there exists a finite chain of orthogonal projections

$$0 = P_0 < P_1 < \cdots < P_n = I,$$

such that $P_j = \lim_k P_{kj}$. This together with $A_j = \lim_k T_{kj}$ and $T_{ki}P_{kj} = P_{kj}T_{ki}P_{kj}$ for all $k \in \mathbb{N}$, $1 \le i \le m$, $1 \le j \le n$, obviously, shows that $\{A_1, \dots, A_m\}$ is triangularizable relative to the chain P_i's above, completing the proof. ■

Recall that given a transformation T, a collection $\mathcal{F}$ of linear transformations on a real or complex vector space $\mathcal{V}$, and a norm $\|\cdot\|$ on $\mathcal{B}(\mathcal{V})$, by definition $\operatorname{dist}(\mathcal{F}, T) = \inf\{\|A - T\| : A \in \mathcal{F}\}$. The following result is a quick consequence of the preceding theorem.

Corollary 2.5 *Let $\mathcal{F}_i, \mathcal{F}$ $(i \in \mathbb{N})$ be nonempty families of linear transformations on a finite-dimensional real or complex vector space $\mathcal{V}$. If each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is triangularizable and* $\lim_n \operatorname{dist}(\mathcal{F}_n, A) = 0$ *for all $A \in \mathcal{F}$, then $\mathcal{F}$ is triangularizable.*

Proof Theorem 2.4. ■

We need the following lemma.

Lemma 2.6 *Let A be a linear transformation on a real or complex finite-dimensional vector space. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of simultaneously diagonalizable linear transformations and $(S_n)_{n\in\mathbb{N}}$ a sequence of invertible linear transformations such that* $\lim_n \|S_n^{-1}AS_n - D_n\| = 0$, *then $(D_n)_{n\in\mathbb{N}}$ is bounded.*

Proof Diagonalize $(D_n)_{n\in\mathbb{N}}$ by a similarity T. Set $N_n = T^{-1}D_nT$; $S'_n = S_nT$. We note that $(N_n)_{n\in\mathbb{N}}$ is a sequence of diagonal, hence normal, linear transformations. We can write

$$\begin{aligned}\lim_n \|S_n^{'-1}AS'_n - N_n\| &= \|T^{-1}S_n^{-1}AS_nT - T^{-1}D_nT\| \\ &\leq \|T^{-1}\|\|T\| \lim_n \|S_n^{-1}AS_n - D_n\| = 0\end{aligned}$$

So $\lim_n \|S_n^{'-1}AS'_n - N_n\| = 0$. Now it follows from Lemma 1.6.5 of [9] that $(N_n)_{n\in\mathbb{N}}$ is bounded. This implies that $(D_n)_{n\in\mathbb{N}}$ is bounded. ■

Corollary 2.7 *Let $\mathcal{F}$ be a collection of linear transformations on a real or complex finite-dimensional vector space $\mathcal{V}$. Then the following assertions are equivalent:*

(i) *The collection $\mathcal{F}$ is triangularizable.*
(ii) *There is a basis $\mathcal{B}$ for the space such that for each finite subset $\{A_1, \ldots, A_m\}$ of $\mathcal{F}$, there exists a diagonalizable, hence commutative, set $\{D_1, \ldots, D_m\}$, relative to $\mathcal{B}$, of linear transformations such that for every $\epsilon > 0$ there is an invertible transformation $S = S_\epsilon$ satisfying $\|S^{-1}A_jS - D_j\| < \epsilon$, for all $1 \leq j \leq m$.*
(iii) *There is a basis $\mathcal{B}$ for the space such that for each finite subset $\{A_1, \ldots, A_m\}$ of $\mathcal{F}$ and every $\epsilon > 0$, there exists a diagonalizable, hence commutative, set of linear transformations $\{D_1, \ldots, D_m\}$, relative to $\mathcal{B}$, and an invertible linear transformation $S = S_\epsilon$ such that $\|S^{-1}A_jS - D_j\| < \epsilon$, for all $1 \leq j \leq m$.*
(iv) *For each finite subfamily $\{A_1, \ldots, A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exists a triangularizable family $\{T_1, \ldots, T_m\}$, and an invertible transformation $S = S_\epsilon$ satisfying $\|T_j\| \leq K$, $\|S^{-1}A_jS - T_j\| < \epsilon$, for every $1 \leq j \leq m$.*

(v) *There exist triangularizable families* $\mathcal{F}_n$ $(n \in \mathbb{N})$ *of linear transformations on* $\mathcal{V}$ *such that* $\lim_n \operatorname{dist}(\mathcal{F}_n, f) = 0$ *for all* $f \in \mathcal{F}$.

Proof Obviously Corollary 2.2 shows that (i) implies (ii). That (ii) implies (iii) is obvious. That (iii) implies (iv) follows from Lemma 2.6. Taking $\epsilon = 1/n$ in (iii), we get a diagonalizable set $\{D_{n1}, \dots, D_{nm}\}$ of linear transformations, relative to $\mathcal{B}$, and an invertible linear transformation S_n such that $\|S_n^{-1}A_jS_n - D_{nj}\| < 1/n$ for all $1 \le j \le m$. Hence $\lim_n \|S_n^{-1}A_jS_n - D_{nj}\| = 0$ for all $1 \le j \le m$. So Lemma 2.6 implies that $\{\|D_{nj}\|\}_{n\in\mathbb{N}}$ is bounded for all $1 \le j \le m$. Thus there exists $0 < K \in \mathbb{R}$ such that $\|D_{nj}\| \le K$ for all $i \in \mathbb{N}$, $1 \le j \le m$. Now it is obvious that (iii) implies (iv). That (iv) implies (i) is nothing but Theorem 2.3. Finally, (i) obviously implies (v). That (v) implies (i) is a quick consequence of Corollary 2.5. ∎

In order to prove the Near Triangularizability Theorem for real Banach spaces, it turns out that we basically need a criterion for triangularizability of an algebra of linear transformations on real vector spaces. It is known that an algebra $\mathcal{A}$ of linear transformations on a finite-dimensional vector space over $\mathbb{C}$ is triangularizable if and only if $AB - BA$ is nilpotent for all $A, B \in \mathcal{A}$ (see Theorem 1.3.2 of [9]). It will turn out that the criterion for triangularizability of an algebra of linear transformations over reals is rather surprising in the sense that individual triangularizability of the members of the algebra implies triangularizability of the algebra. As a matter of fact, it turns out that $\mathbb{R}$ does not play an important role in the proof of our criterion. The aforementioned triangularization criterion holds for algebras of linear transformations on finite-dimensional vector spaces over any field over which there exists an irreducible polynomial of degree 2.

For a given field F and $k \in \mathbb{N}$ with $k > 1$, we say that F is *k-closed* if every polynomial of degree k over F is reducible over F. It is plain that a field F is algebraically closed if and only if F is k-closed for all $k \in \mathbb{N}$ with $k > 1$.

Recall that if $\mathcal{V}$ is a finite-dimensional vector space over a field F, then a linear transformation T on $\mathcal{V}$ is triangularizable if and only if the characteristic polynomial for T is a product of linear polynomials over F (to prove this, use the Triangularization Lemma, Lemma 1.1.4 of [9], or see Theorem 6.4.5 of [4]), or equivalently $\bar{\sigma}(T) \subset F$ where $\bar{\sigma}(T)$ denotes the spectrum of T in the algebraic closure of F. Also recall that Burnside's Theorem asserts that the only irreducible algebra in $\mathcal{L}(\mathcal{V})$ is $\mathcal{L}(\mathcal{V})$ provided that the ground field is algebraically closed.

Motivated by the proof of Burnside's Theorem due to I. Halperin and P. Rosenthal (see Theorem 1.2.2 of [9], or [5]), we restate Burnside's Theorem as follows.

Theorem 2.8 *Let* $\mathcal{V}$ *be a finite-dimensional vector space over a field F of dimension greater than* 1. *If there exists an irreducible algebra* $\mathcal{A}$ *of linear transformations with* $\bar{\sigma}(\mathcal{A}) \subseteq F$, *then* $\mathcal{A} = \mathcal{L}(\mathcal{V})$, *and therefore F is k-closed for each* $k = 2, \dots, n$. *In particular, Burnside's Theorem holds in* $\mathcal{L}(\mathcal{V})$.

Proof In view of individual triangularizability of the members of the algebra $\mathcal{A}$, the proof is identical to that of Burnside's Theorem due to I. Halperin and P. Rosenthal (see Theorem 1.2.2 of [9], or [5]). ∎

Theorem 2.8 implies the following. Later on, as mentioned before, we will use part (ii) of the following theorem to prove the Near Triangularizability Theorem for compact operators on real Banach spaces.

Theorem 2.9 (i) *Let F be a field that is not 2-closed, and $\mathcal{V}$ an n-dimensional vector space over F with $n > 1$. Let $\mathcal{A}$ be an algebra in $\mathcal{L}(\mathcal{V})$. Then $\mathcal{A}$ is triangularizable if and only if every $A \in \mathcal{A}$ is triangularizable. Conversely, let a field F be given. If every algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{V})$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$ is triangularizable, then F is not 2-closed.*

(ii) *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\mathcal{A}$ be an algebra of linear transformations in $\mathcal{B}(\mathcal{V})$. Then $\mathcal{A}$ is triangularizable if and only if every element of $\mathcal{A}$ is triangularizable.*

Proof (i) "$\Rightarrow$" Obvious.

"$\Leftarrow$" The proof is just a quick consequence of Theorem 2.8 together with the Triangularization Lemma (see Lemma 1.1.4 of [9]).

For the converse, fixing a basis for $\mathcal{V}$, we need to prove the matrix version of the assertion. Again we use contradiction. Suppose that F is 2-closed. This hypothesis easily implies that the nontriangularizable algebra $\mathcal{A} := \operatorname{diag}(M_2(F), 0_{n-2})$ has the property that $\bar{\sigma}(\mathcal{A}) \subseteq F$. On the other hand, $\mathcal{A}$ must be triangularizable by hypothesis, for $\bar{\sigma}(\mathcal{A}) \subseteq F$, a contradiction.

(ii) This is a quick consequence of (i) because $\mathbb{R}$ is not 2-closed. ∎

Remark In [10], we extend Theorems 2.8 and 2.9 to F-algebras of triangularizable linear transformations in $\mathcal{L}(\mathcal{V})$ with spectra in F where $\mathcal{V}$ is a finite-dimensional vector space over a field K, and F is a subfield of K that is not 2-closed.

We now prove a near reducibility theorem. The idea of the proof below is similar to that of Theorem 2.4. However, we include the proof for the sake of completeness.

Theorem 2.10 (**Near Reducibility Theorem**) *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$ with the following property: For each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$ and for every $\epsilon > 0$ there exists a reducible family $\{T_1, \dots, T_m\}$ satisfying*

$$\|A_j - T_j\| < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is reducible.

Proof Fixing a basis for the space, it suffices to prove the matrix version of the assertion. Set $n = \dim \mathcal{V}$. Since $M_n(\mathbb{F})$ is n^2-dimensional, it suffices to show that each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$ is reducible. In doing so, it is easily seen from the hypothesis that there exist reducible families $\{T_{k1}, \dots, T_{km}\}$ ($k \in \mathbb{N}$) of matrices such that $A_j = \lim_k T_{kj}$ for all $1 \leq j \leq m$. Now from reducibility of $\{T_{k1}, \dots, T_{km}\}$, we see that there exist orthogonal projections P_k's such that $T_{ki}P_k = P_kT_{ki}P_k$ for all $k \in \mathbb{N}$ and $1 \leq i \leq m$. In view of a well-known theorem of Riesz about compactness of the unit ball of finite-dimensional normed spaces, if necessary, by passing to a convergent subsequence, we may assume that there exists an orthogonal projection P such that $P = \lim_k P_k$. This together with $A_j = \lim_k T_{kj}$ and $T_{ki}P_k = P_kT_{ki}P_k$ for all $k \in \mathbb{N}$, $1 \leq i \leq m$, obviously, shows that $\{A_1, \dots, A_m\}$ is reducible, completing the proof. ■

Remarks 1. Using the argument of Theorem 3.2.12 of [13], one can prove that the near reducibility theorem above holds for families of linear transformations on a finite-dimensional vector space over an algebraically closed complete field F with a nontrivial absolute value.

2. Having proved the above near reducibility theorem, one can prove an analogue of Corollary 2.5 for reducible collections of transformations on a real or complex vector space. More precisely, one can prove: *Let $\mathcal{V}$ be a finite-dimensional real or complex vector space, $\mathcal{F}_i, \mathcal{F}$ $(i \in \mathbb{N})$ nonempty families of linear transformations on $\mathcal{V}$. If each family $\mathcal{F}_n (n \in \mathbb{N})$ is reducible and $\lim_n \operatorname{dist}(\mathcal{F}_n, A) = 0$ for all $A \in \mathcal{F}$, then $\mathcal{F}$ is reducible.*

The following useful lemma, taken from [13], is needed for the proof of the counterpart of Theorem 2.9 for algebras of compact operators on real Banach spaces. It is worth mentioning that we are quoting the lemma below and some of its consequences for reader's convenience.

Lemma 2.11 *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$, and T a nonzero linear operator in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = \overline{T\mathcal{X}}$ is the closure of the range of T.*

Proof See [11] or Lemma 2.5.1 of [13]. ■

Let $\mathcal{X}$ be a complex (resp. real) Banach space, and S a subset of $\mathbb{C}$ (resp. $\mathbb{R}$). By an *S-semigroup* $\mathcal{S}$ of $\mathcal{B}(\mathcal{X})$, we mean a multiplicative semigroup of bounded operators that is closed under scalar multiplication by the elements of S. The following lemma is the counterpart of Lemma 7.4.5 of [9] which is due to Radjavi. We should point out that the proof below is essentially an imitation of Radjavi's proof.

Lemma 2.12 *Let $\mathcal{X}$ be a real Banach space and let $\mathcal{S}$ be a uniformly closed $\mathbb{R}^+$-semigroup of compact triangularizable operators on $\mathcal{X}$ where $\mathbb{R}^+$ denotes the set of positive real numbers. If $\mathcal{S}$ contains an operator that is not quasinilpotent, then $\mathcal{S}$ contains a nonzero finite-rank operator that is either idempotent or nilpotent.*

Proof The idea of proof is identical to that of Lemma 7.4.5 of [9]. First note that by multiplying by an appropriate sequence of positive reals, we can assume that there is a $K \in \mathcal{S}$ of spectral radius 1. Since $\sigma(K) \in \mathbb{R}$, it follows that K has either one or two eigenvalues of absolute value 1, namely either 1 or -1. If necessary, by repeated application of Corollary 6.4.13 of [9], one can conclude that there are complementary invariant subspaces $\mathcal{N}$ and $\mathcal{R}$ of K such that $\mathcal{N}$ is finite-dimensional, $\varnothing \neq \sigma(K|_{\mathcal{N}}) \subseteq \{-1, 1\}$, and $\rho(K|_{\mathcal{R}}) < 1$. From this point on, the proof is identical to that of Lemma 7.4.5 of [9] which we omit for the sake of brevity. ■

As pointed out in [11], Turovskii's Theorem [9, Theorem 8.1.11] and Lomonosov's Lemma [9, Lemma 7.3.1] hold on arbitrary real Banach spaces as well. The proofs are

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identical to those of their counterparts over complex Banach spaces. The following lemma is needed to prove the counterpart of Theorem 2.9 (ii) for algebras of compact operators on real Banach spaces.

Lemma 2.13 *Let $\mathcal{X}$ be a real Banach space of dimension greater than* 1*, and let $\mathcal{A}$ be a subalgebra of triangularizable compact operators. Then $\mathcal{A}$ is reducible.*

Proof In view of Lemma 5 on page 1091 of [3], we may, without loss of generality, assume that the algebra $\mathcal{A}$ is uniformly closed. Use contradiction. If $\mathcal{A}$ is a Volterra algebra, *i.e.*, an algebra of quasinilpotent operators, then $\mathcal{A}$ is triangularizable, hence reducible, by Lomonosov's Lemma (Lemma 7.3.1 of [9]) which is a contradiction. So suppose that $\mathcal{A}$ contains an operator that is not quasinilpotent. It follows from the preceding lemma that $\mathcal{A}$ then contains a nonzero finite-rank operator F that is either idempotent or nilpotent. Since $\mathcal{A}$ is assumed to be irreducible, without loss of generality, we may assume that rank(F) > 1. Let $\mathcal{R}$ denote the range of F, by Lemma 2.11 the algebra $F\mathcal{A}|_{\mathcal{R}}$, on the finite-dimensional space $\mathcal{R}$ over $\mathbb{R}$ of dimension greater than 1, is irreducible. On the other hand, by Theorem 2.9(ii) the algebra $F\mathcal{A}|_{\mathcal{R}}$ is triangularizable, hence reducible, for $\sigma(A) \subset \mathbb{R}$ for all $A \in \mathcal{A}$. This contradiction proves the assertion. ■

The following is the infinite-dimensional counterpart of Theorem 2.9(ii).

Theorem 2.14 *Let $\mathcal{X}$ be a real Banach space of dimension greater than* 1*, and let $\mathcal{A}$ be a subalgebra of compact operators. Then $\mathcal{A}$ is triangularizable if and only if every element of $\mathcal{A}$ is triangularizable.*

Proof Necessity is obvious. Sufficiency is established by the Triangularization Lemma [9, Lemma 7.1.11] and Theorem 2.13. ■

Remark In [11], the preceding theorem is extended to R-algebras of compact operators on a real or complex Banach space with spectra in R where R is a subring of $\mathbb{R}$.

3 Near Triangularizability in Infinite Dimensions

In this section we prove the infinite-dimensional version of near triangularizability. Here is the Near Triangularizability Theorem for arbitrary collections of compact operators on a real or complex Banach space.

Theorem 3.1 *Let $\mathcal{F}$ be a family of compact operators on a real or complex Banach space with the following property: For each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$ and for every $\epsilon > 0$ there exists a triangularizable family $\{T_1, \dots, T_m\}$ of compact operators satisfying $\|A_j - T_j\| < \epsilon$, for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.*

Proof We first prove the assertion for the case when the underlying space is a real Banach space. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$. In view of Theorem 2.14, it

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suffices to show that every element of $\mathcal{A}$ is triangularizable. Given $A \in \mathcal{A}$, there are $A_i \in \mathcal{F}, (1 \le i \le m, m \in \mathbb{N})$, and noncommutative polynomial p such that

$$A = p(A_1, \dots, A_m).$$

Let $K = \max\{\|A_i\| : 1 \le i \le m\}$. As we mentioned before, it is easily seen that every such p is a uniformly continuous function of its arguments on any bounded set in $\langle (\mathcal{B}(\mathcal{X}))^m, \|\cdot\|_\infty \rangle$ where $\|(X_1, \dots, X_m)\|_\infty = \max\{\|X_1\|, \dots, \|X_m\|\}$, and where $\|\cdot\|$ is the operator norm on $\mathcal{B}(\mathcal{X})$. In particular, for every $n > 0$, there is a δ_n with $0 < \delta_n < 1$ such that

$$\|p(X_1, \dots, X_m) - p(Y_1, \dots, Y_m)\| < \frac{1}{n},$$

whenever

$$\|X_j - Y_j\| < \delta_n, \ \|X_j\| \le K + 1, \ \|Y_j\| \le K + 1$$

for all $1 \le j \le m$.

Now, by the hypothesis for this δ_n, there is a triangularizable family $\{T_{n1}, \dots, T_{nm}\}$ of linear transformations satisfying

$$\|A_j - T_{nj}\| < \delta_n,$$

for every $1 \le j \le m$. Clearly,

$$\|A_j\| \le K + 1, \ \|T_{nj}\| \le K + 1, \ \|A_j - T_{nj}\| < \delta_n,$$

for every $1 \le j \le m$. Thus it follows from $(*)$ that

$$\|p(A_1, \dots, A_m) - p(T_{n1}, \dots, T_{nm})\| < \frac{1}{n}.$$

Thus $\|p(A_1, \dots, A_m) - T_n\| < \frac{1}{n}$, where $T_n = p(T_{n1}, \dots, T_{nm})$. Obviously, T_n is a triangularizable operator, for $\{T_{n1}, \dots, T_{nm}\}$ is a triangularizable family of linear operators. In particular, $\sigma(T_n) \subseteq \mathbb{R}$, $n \in \mathbb{N}$. So we have $p(A_1, \dots, A_m) = \lim_n T_n$ and $\sigma(T_n) \subseteq \mathbb{R}$, $n \in \mathbb{N}$; hence it follows from Lemma 5 on page 1091 of [3] that $\sigma(p(A_1, \dots, A_m)) \subseteq \mathbb{R}$ and that $A = p(A_1, \dots, A_m)$ is triangularizable, as desired.

We now prove the assertion for the case when the underlying space is a complex Banach space. As before, we note that if $\mathcal{F}$ is a singleton, then we have nothing to prove, for every compact operator is triangularizable. So we may assume that $|\mathcal{F}| > 1$. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$. In view of Theorem 7.6.1 of [9], it suffices to prove that each commutator $BC - CB$ is quasinilpotent for all B and C in $\mathcal{A}$.

To do so, we will show that $\rho(BC - CB) = 0$ where "ρ" denotes the spectral radius. As before, given $B, C \in \mathcal{A}$, there are $A_i \in \mathcal{F}, (1 \le i \le m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, \dots, A_m), \ C = q(A_1, \dots, A_m).$$

As before, set $K = \max\{\|A_i\| : 1 \le i \le m\}$. Define

$$h(x_1, \dots, x_m) = p(x_1, \dots, x_m)q(x_1, \dots, x_m) - q(x_1, \dots, x_m)p(x_1, \dots, x_m).$$

Since the spectral radius is continuous at $h(A_1, \dots, A_m)$, for $h(A_1, \dots, A_m)$ is a compact operator, it follows that for a given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $|\rho(h(A_1, \dots, A_m)) - \rho(A)| < \epsilon$ whenever $\|h(A_1, \dots, A_m) - A\| < \delta$.

Now, for this $\delta = \delta(\epsilon) > 0$, there is an η with $0 < \eta < 1$ such that

$$\|h(X_1, \dots, X_m) - h(Y_1, \dots, Y_m)\| < \delta,$$

whenever

$$\|X_j - Y_j\| < \eta, \ \|X_j\| \le K + 1, \ \|Y_j\| \le K + 1$$

for all $1 \le j \le m$. By the hypothesis, for this $0 < \eta < 1$ there is a triangularizable family $\{T_1, \dots, T_m\}$ of compact operators satisfying $\|A_j - T_j\| < \eta$ for every $1 \le j \le m$. Clearly,

$$\|A_j\| \le K + 1, \ \|T_j\| \le K + 1, \ \|A_j - T_j\| < \eta,$$

for every $1 \le j \le m$. Thus it follows from $(*)$ that

$$\|h(A_1, \dots, A_m) - h(T_1, \dots, T_m)\| < \delta.$$

So it follows from the continuity of spectral radius at $h(A_1, \dots, A_m)$ that

$$|\rho(h(A_1, \dots, A_m)) - \rho(h(T_1, \dots, T_m))| < \epsilon.$$

But $\rho(h(T_1, \dots, T_m)) = 0$, for $\{T_1, \dots, T_m\}$ is triangularizable. Thus we conclude that

$$|\rho(h(A_1, \dots, A_m))| < \epsilon,$$

for all $\epsilon > 0$. Hence, $\rho(BC - CB) = \rho(h(A_1, \dots, A_m)) = 0$ for each commutator $BC - CB$, $B, C \in \mathcal{A}$. Therefore, $BC - CB$ is quasinilpotent for all $B, C \in \mathcal{A}$, completing the proof. ■

Remark Having proved the above near triangularizability theorem, one can prove an analogue of Corollary 2.5 for collections of compact operators on a real or complex Banach space where nearness is measured by the operator norm.

4 A Reducibility Result

In this section we use the Near Triangularizability Theorem to prove a rather surprising reducibility result. Let $\mathcal{X}$ be a real or complex Banach space, and $A_n, A \in \mathcal{B}(\mathcal{X})$. By $\text{s-lim}_n A_n = A$ we mean A is the limit of A_n's in the strong operator topology on $\mathcal{B}(\mathcal{X})$, *i.e.*, $\lim_n \|A_n x - Ax\| = 0$ for all $x \in \mathcal{X}$. To present our reducibility result, we need the following two results.

Lemma 4.1 *Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K_n, K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ $(n \in \mathbb{N})$. If* $\text{s-lim}_n A_n = A$ *and* $\lim_n K_n = K$, *then* $\lim_n A_n K_n = AK$.

Proof We give the proof in three stages.

(i) *If* $\text{s-lim}_n A_n = A$, *then* $\lim_n A_n F = AF$ *for all* $F \in \mathcal{B}_{00}(\mathcal{X})$.

Since F is a finite-rank operator, it follows that we can write

$$F = \sum_{i=1}^{m} \phi_i \otimes x_i,$$

where $m \in \mathbb{N}$, $\phi_i \in \mathcal{X}^*$, $x_i \in \mathcal{X}$ $(1 \leq i \leq m)$, and $\phi_i \otimes x_i$ is the rank-one operator defined on $\mathcal{X}$ by $\phi_i \otimes x_i(x) = \phi_i(x) x_i$. It is easily seen that $AF = \sum_{i=1}^{m} \phi_i \otimes Ax_i$. Therefore, we can write

$$\begin{aligned}\|A_n F - AF\| = \|(A_n - A)F\| &= \sup_{\|y\|=1} \|(A_n - A)F(y)\| \\ &= \sup_{\|y\|=1} \Big\| \sum_{i=1}^{m} \phi_i(y)(A_n x_i - Ax_i) \Big\|.\end{aligned}$$

On the other hand, since $\text{s-lim}_n A_n = A$, it follows that $\lim_n A_n x_i = Ax_i$ for each $i = 1, \ldots, m$. Hence, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|A_n x_i - Ax_i\| < \frac{\epsilon}{2mM},$$

for all $n \geq N$ and $1 \leq i \leq m$ where $M = \max_{1 \leq i \leq m} \|\phi_i\|$. So for all $n \geq N$ we can write

$$\|A_n F - AF\| \leq \sum_{i=1}^{m} \|\phi_i\| . \|(A_n x_i - Ax_i)\| \leq \sum_{i=1}^{m} M \frac{\epsilon}{2mM} = \frac{\epsilon}{2} < \epsilon.$$

That is, $\lim_n A_n F = AF$.

(ii) *If* $\text{s-lim}_n A_n = A$ *and* $\lim_n F_n = K$ *where* $F_n \in \mathcal{B}_{00}(\mathcal{X})$ $(n \in \mathbb{N})$, *then* $\lim_n A_n K = AK$.

Since $\text{s-lim}_n A_n = A$, it follows from the Principle of Uniform Boundedness (Theorem III.14.1 of [2]) that there exists $M > 0$ such that $\|A\|, \|A_n\| \leq M$ for each $n \in \mathbb{N}$. We have $\lim_n F_n = K$. Therefore, for a given $\epsilon > 0$, there exists $N_1 > 0$ such that

$$\|F_n - K\| < \frac{\epsilon}{3(M+1)},$$

for all $n \geq N_1$. We can write

$$\begin{aligned}\|A_n K - AK\| &\leq \|A_n K - A_n F_{N_1}\| + \|A_n F_{N_1} - AF_{N_1}\| + \|AF_{N_1} - AK\| \\ &\leq \|A_n\| \cdot \|K - F_{N_1}\| + \|A_n F_{N_1} - AF_{N_1}\| + \|A\| \cdot \|F_{N_1} - K\|\end{aligned}$$

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On the other hand, (i) implies that $\lim_n \|A_n F_{N_1} - AF_{N_1}\| = 0$. Hence there exists $N_2 > 0$ such that

$$\|A_n F_{N_1} - AF_{N_1}\| < \frac{\epsilon}{3},$$

for all $n \geq N_2$. Now for all $n \geq \max(N_1, N_2)$ we can write

$$\|A_n K - AK\| < M\frac{\epsilon}{3(M+1)} + \frac{\epsilon}{3} + M\frac{\epsilon}{3(M+1)} < \epsilon.$$

That is, $\lim_n \|A_n K - AK\| = 0$. In other words, $\lim_n (A_n K - AK) = 0$ which is what we wanted.

(iii) *We now prove the general statement.*

Again, in view of the Principle of Uniform Boundedness, it is easily seen that there exists $M > 0$ such that $\|A\|, \|A_n\|, \|K\|, \|K_n\| \leq M$ for each $n \in \mathbb{N}$. Since $\lim_n K_n = K$, we conclude that for a given $\epsilon > 0$ there exists $N_1 > 0$ such that

$$\|K_n - K\| < \frac{\epsilon}{2(M+1)},$$

for all $n \geq N_1$. The fact that $K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ along with (ii) implies that

$$\lim_n \|A_n K - AK\| = 0.$$

Thus, there exists $N_2 > 0$ such that $\|A_n K - AK\| < \frac{\epsilon}{2}$ for all $n \geq N_2$. Now for all $n \geq \max(N_1, N_2)$ we can write

$$\begin{aligned}\|A_n K_n - AK\| &\leq \|A_n K_n - A_n K\| + \|A_n K - AK\| \\ &\leq \|A_n\| \cdot \|K_n - K\| + \|A_n K - AK\| < M\frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

That is, $\lim_n \|A_n K_n - AK\| = 0$. In other words, $\lim_n A_n K_n = AK$ which is what we wanted. ■

The following result is needed for the proof of the main theorem of this section.

Theorem 4.2 *Let $\mathcal{X}$ be a real or complex Banach space of dimension greater than one, $\mathcal{S}$ a semigroup of operators in $\mathcal{B}(\mathcal{X})$, and $B \in \mathcal{B}(\mathcal{X})$ a bounded operator with* $\operatorname{rank}(B) \geq 2$. *If $\mathcal{S}B$ is triangularizable, then $\mathcal{S}$ has a nontrivial invariant subspace.*

Proof Let $\mathcal{A}$ denote the algebra generated by the semigroup $\mathcal{S}$. We note that $\mathcal{A} = \langle \mathcal{S} \rangle$. That being noted, it suffices to prove the assertion for an algebra $\mathcal{A}$ of operators in $\mathcal{B}(\mathcal{X})$. Let $\mathcal{C}$ be a maximal chain of subspaces each of which is invariant for $\mathcal{A}B$. Let $\mathcal{X}_- := \overline{\cup_{\mathcal{X} \neq \mathcal{Y} \in \mathcal{C}} \mathcal{Y}}$. We now distinguish two cases.

(a) $\mathcal{X}_- = \mathcal{X}$.

Obviously, there exists a $\mathcal{Y} \in \mathcal{C}$ with $\mathcal{Y} \neq \mathcal{X}$, and a $y_0 \in \mathcal{Y}$ such that $By_0 \neq 0$. Define $\mathcal{M} := \overline{\mathcal{A}By_0}$. If $\mathcal{M} = \{0\}$, then $\langle By_0 \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$.

If $\mathcal{M} \neq \{0\}$, then we would have $0 \neq \mathcal{M} = \overline{\mathcal{A}By_0} \subseteq \mathcal{Y} \neq \mathcal{X}$. So $\mathcal{M}$ would then be a nontrivial invariant subspace for $\mathcal{A}$.

(b) $\mathcal{X}_- \neq \mathcal{X}$.

Since $\mathcal{C}$ is maximal, it follows that $\mathcal{X}_- \in \mathcal{C}$ is a closed subspace of $\mathcal{X}$ of codimension one, *i.e.*, $\dim \frac{\mathcal{X}}{\mathcal{X}_-} = 1$. Since $\operatorname{rank}(B) \geq 2$, we see that there exists $x_0 \in \mathcal{X}_-$ such that $Bx_0 \neq 0$. Again define $\mathcal{M} := \overline{\mathcal{A}Bx_0}$. If $\mathcal{M} = \{0\}$, then $\langle Bx_0 \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$. If $\mathcal{M} \neq \{0\}$, then we would have $0 \neq \mathcal{M} = \overline{\mathcal{A}Bx_0} \subseteq \mathcal{X}_- \neq \mathcal{X}$. So $\mathcal{M}$ would then be a nontrivial invariant subspace for $\mathcal{A}$. ■

Remarks 1. In the preceding theorem, if the triangularizing chain, say $\mathcal{C}$, for $\mathcal{S}B$ happens to have the property that $\mathcal{X}_- = \mathcal{X}$, *e.g.*, any continuous chain, then, by case (a) of the proof above, the assertion holds under the weaker hypothesis that B is nonzero.

2. By adjusting case (b) of the proof above, it is easily seen that the preceding theorem holds on finite-dimensional vector spaces over general fields. It is worth noting that in the preceding theorem the hypothesis that $\operatorname{rank}(B) \geq 2$ cannot be weakened. To see this, let F be a field and $n > 1$. Note that $M_n(F)$ is irreducible whereas $M_n(F)E$ is triangularizable where E is any rank-one matrix in $M_n(F)$. Also note that $\mathcal{B}(\mathcal{X})$ is irreducible but $\mathcal{B}(\mathcal{X})T$ is triangularizable where T is any rank-one operator on $\mathcal{X}$. This, in view of the preceding remark, shows that if T is a rank-one operator, then any triangularizing chain, say $\mathcal{C}$, for $\mathcal{B}(\mathcal{X})T$ is not continuous at $\mathcal{X} \in \mathcal{C}$, *i.e.*, $\mathcal{X}_- \neq \mathcal{X}$.

3. It is not difficult to see that in the preceding theorem if the operator B happens to be 1–1, then reducibility of $\mathcal{S}B$ implies that of $\mathcal{S}$. (See the remarks following Theorem 2.3 of [12].)

Here is the main theorem of this section.

Theorem 4.3 *Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K_n, K \in \mathcal{B}_{00}(\mathcal{X})$ $(n \in \mathbb{N})$ with* $\operatorname{rank}(K) \geq 2$. *If* $\operatorname{s-lim}_n A_n = A$, $\lim_n K_n = K$, *and* $\{A_n, K_n\}$ *is triangularizable for each $n \in \mathbb{N}$, then A has a nontrivial invariant subspace.*

Proof Let $\mathcal{S}$ denote the semigroup generated by A. In light of Theorem 4.2, it suffices to show that $\mathcal{S}K$ is triangularizable. That is, we need to show that the collection $\{A^iK\}_{i=1}^{\infty}$ is triangularizable. In view of Theorem 3.1, it suffices to show that $\{A^iK\}_{i=1}^{\infty}$ satisfies the hypotheses of Theorem 3.1. Suppose that a finite subfamily $\{A^{n_1}K, \dots, A^{n_m}K\}$ is given. Since $\operatorname{s-lim}_i A_i = A$, it easily follows that $\operatorname{s-lim}_i A_i^{n_j} = A^{n_j}$ for each $j = 1, \dots, m$. Now, since $\lim_i K_i = K$, we see from Lemma 4.1 that $\lim_i A_i^{n_j} K_i = A^{n_j}K$ for each $j = 1, \dots, m$. Since $\{A_i, K_i\}$ is triangularizable, it follows that so is $\{A_i^{n_1}K_i, \dots, A_i^{n_m}K_i\}$ for each $i \in \mathbb{N}$. Since $\lim_i A_i^{n_j}K_i = A^{n_j}K$ for each $j = 1, \dots, m$, we conclude that for i large enough $\|A_i^{n_j}K_i - A^{n_j}K\| < \epsilon$, for each $j = 1, \dots, m$. Therefore, the collection $\{A^iK\}_{i=1}^{\infty}$ is triangularizable, finishing the proof. ■

Corollary 4.4 (i) *Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K \in \mathcal{B}_{00}(\mathcal{X})$ $(n \in \mathbb{N})$ with* $\operatorname{rank}(K) \geq 2$. *If* $\operatorname{s-lim}_n A_n = A$, *and* $\{A_n, K\}$ *is triangularizable for each $n \in \mathbb{N}$, then A has a nontrivial invariant subspace.*

(ii) *Let $\mathcal{H}$ be a real or complex Hilbert space, $(\alpha_i)_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and $A_n, A \in \mathcal{B}(\mathcal{H})$. If* s-lim$_n A_n = A$*, and for each $n \in \mathbb{N}$ there exists a permutation π_n on $\mathbb{N}$ such that $(\alpha_{\pi_n(i)})_{i\in\mathbb{N}}$ is a triangularizing basis for A_n, then A has a nontrivial invariant subspace.*

Proof (i) This is a special case of Theorem 4.3 when $K_n = K$ for all $n \in \mathbb{N}$.

(ii) Let K be the compact (and in fact normal) operator defined by $\operatorname{diag}(1/j)_{j=1}^{\infty}$ relative to the orthonormal basis $(\alpha_i)_{i\in\mathbb{N}}$ for the space $\mathcal{H}$. Note that $K \in \overline{\mathcal{B}_{00}(\mathcal{H})}$, for $\overline{\mathcal{B}_{00}(\mathcal{H})} = \mathcal{B}_0(\mathcal{H})$. The hypothesis implies that $\{A_n, K\}$ is triangularizable for each $n \in \mathbb{N}$. So (i) applies. ■

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