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LINEAR METRIC SPACES AND ANALYTIC SETS

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A problem in descriptive set theory, in which the objects of interest are compact convex sets in linear metric spaces, primarily those having extreme points.

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Introduction

Let E be a linear metric space with norm $\|x\|$ and distance $d(x,y)=\|x-y\|$, and suppose that E is separable and complete. The space $c(E)$ of all non-void compact subsets of E is then a Polish space in the Hausdorff metric, and its topology depends only on that of E. The space $\mathscr{K}(E)$ of all compact, convex (non-void) subsets of E is closed in $c(E)$; we are interested primarily in the subset $\mathscr{E}\mathscr{K}(E)\subseteq\mathscr{K}(E)$, consisting of those sets in $\mathscr{K}(E)$ having at least one extreme point. A remarkable example by Roberts [**4**, **5**, **9**] shows that $\mathscr{E}\mathscr{K}(E)\neq\mathscr{K}(E)$ when $E=L^{1/2}(0,1)$ for example.

We push this a step further:

Theorem. *$\mathscr{E}\mathscr{K}(E)$ is always an analytic subset of $\mathscr{K}(E)$, and is a complete analytic subset when $E=l^2\oplus L^{1/2}$.*

The second part of the theorem means this: for each analytic set A in a compact metric space M, there is a continuous mapping ϕ of M into $\mathscr{K}(E)$, such that $A=\phi^{-1}(\mathscr{E}\mathscr{K}(E))$. Previous work on extreme boundaries of convex sets in Banach spaces, and on the class of (non-compact) convex sets admitting extreme points, is presented in [**2**, **3**, **6**, **7**]; applications of Roberts' technique appear in [**1**, **10**].

The basic construction

Let K_0 be a compact, convex set in $L^{1/2}$ containing 0 but no extreme point, and K_1 a compact, convex set in a Banach space, say l^2. (In the space $l^2\oplus L^{1/2}$, the subspace $l^2\oplus(0)$ is identified with l^2, and likewise for $(0)\oplus L^{1/2}$). We define a sort of extension of K_1 by K_0, parametrized by a function $f\geqq 0$ in $C(K_1)$. Let $L(f)$ be the compact subset of K_1+K_0 containing all (x,y) such that

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$$x \in K_1,\ y \in f(x) \cdot K_0$$

and

$$K(f) = \overline{co}(L(f)) \subseteq K_1 + \|f\|_\infty \cdot K_0$$

Lemma 1. *$K(f)$ has an extreme point if and only if f has a zero in $ex\, K_1$.*

This depends on an elementary result, whose proof is omitted.

Lemma 0. *Let C be a compact convex set in E, F a closed subset of C, and z and extreme point of $\overline{co}(F)$. Then there is an element w of F, such that $z \in \overline{co}(F \cap V)$ for every neighbourhood V of w.*

We apply this with $C = K_1 + \|f\|_\infty \cdot K_0$, and $F = L(f)$. Let $z = (x, y)$ be an extreme point of $K(f)$. The element w of $L(f)$ must have the same first co-ordinate as z (since l^2 is locally convex), so we write $w = (x, y_1)$, $y_1 \in f(x) \cdot K_0$. To see that $y \in f(x) \cdot K_0$, let $\delta > 0$ and let V be a neighbourhood of (x, y_1) such that $f(\xi) < f(x) + \delta$ whenever $(\xi, \eta) \in V$. Then $\overline{co}(V \cap L(f)) \subseteq K_1 + (f(x) + \delta) \cdot K_0$; since K_0 is compact, and $\delta > 0$ was arbitrary, we conclude that $y \in f(x) \cdot K_0$.

Continuing with the consideration of $z \in ex\, K(f)$, we see that it is necessary that $f(x) = 0$, whence $y = 0$. But then it is necessary as well as that $x \in ex K_1$, i.e. $f^{-1}(0)$ meet $ex\, K_1$.

Conversely, suppose that $x \in ex\, K_1$ and $f(x) = 0$, and $2(x, 0) = (x_1, y_1) + (x_2, y_2)$, with $(x_i, y_i) \in K(f)$. Clearly $x_1 = x_2 = x$, and we proceed to prove that $y_1 = y_2 = 0$. There is a formula $(x, y_1) = \lim \int (x, y)\, d\mu_n(x, y)$, where each μ_n is an atomic probability measure in $L(f)$, with finite support. Writing $c_n = \int f(x)\, d\mu_n$, and $c = \liminf c_n$, we see that $y_1 \in c \cdot K_0$. Let μ be any w^*-limit of the sequence (μ_n), so that $x = \int x\, d\mu(x, y)$—since l^2 is a Banach space. This is possible only if μ is concentrated on $(x) \oplus L^{1/2}$, and thus $\lim c_n = 0$. We have proved that $y_1 = 0$, i.e. $(x, 0)$ is an extreme point, thereby proving Lemma 1.

A small observation is necessary: $K(f)$ depends continuously on the parameter $f \in C^+(K_1)$. Indeed, when $\|f - g\|_\infty < \delta$, then $f \leqq g + \delta$, so $K(f) \subseteq K(g + \delta) \subseteq K(g) + \delta \cdot K_0$. This implies continuous dependence of $K(f)$ on f.

An elementary step

Let F_m be the subset of $E \times \mathscr{K}(E)$ defined as follows:

$$(x, K) \in F_m \quad \text{if} \quad x + y, x - y \in K$$

for some y, such that $\|y\| \geqq m^{-1}$. (Here $m = 1, 2, 3, \ldots$). To see that F_m is closed in $E \times \mathscr{K}(E)$, suppose $(x_n, K_n) \in F_m$, and $\lim x_n = x$, $\lim K_n = K$. Then $x_n \pm y_n \in K_n$, for some y_n with $\|y_n\| \geqq m^{-1}$. Now $\bigcup_1^\infty K_n$ has compact closure in E, whence $(x_n + y_n)$ has a

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limit-point, and so (y_n) has a limit-point y. Thus $\|y\| \geqq m^{-1}$ and $x+y, x-y$ belong to $K:(x,K)\in F_m$.

The union $\bigcup_1^\infty F_m \equiv F$ is exactly the set of pairs (x,K) such that $x\in K\backslash ex\,K$. Projecting the G_δ-set $E\times\mathscr{K}(E)\backslash F$ on the second factor $\mathscr{K}(E)$, we obtain an analytic set, namely $\mathscr{E}\mathscr{K}(E)$.

A special set in l^2 [8]

Let C and C_0 be the subsets of l^2 containing sequences $(a_k)_1^\infty$ such that $\sum ka_k^2\leqq 1$ (respectively, $\sum ka_k^2=1$). Then C is compact and convex, and $C_0=ex\,C$. The remaining part of the proof is based on the next observation (zero-set representation):

(ZSR). Let A be an analytic set in a compact metric space M. There is a continuous function $f(t,x)\geqq 0$ on $M\times C$ such that $t\in A\Leftrightarrow f(t,x)=0$ for some $x\in C_0$.

The *conclusion* of the main argument is presented next, and then finally the proof of ZSR. We use C in place of K_1. The partial function $x\mapsto f(t,x)$ is denoted f_t, and then M is mapped into $\mathscr{K}(l^2\oplus L^{1/2})$ by the formula $K[t]=K(f_t)$. Then $K[t]$ has an extreme point $\Leftrightarrow f_t$ has a zero in $C_0\Leftrightarrow t\in A$. We have seen that $K(f_t)$ depends continuously on f_t, i.e. $K[t]$ varies continuously with t. This proves the main result.

Proof of ZSR. A certain detail makes this appear complicated. The mapping of a function $g\in C([0,1])$ to its zero set $g^{-1}(0)$ is merely upper semicontinuous; but this can be detoured by enlarging the domain $[0,1]$. The set A is the image $\psi(N)$ of the set J of irrationals in $I=[0,1]$ by a continuous function ψ defined over J. Let Γ be the graph of ψ—thus $\Gamma\subseteq J\times A$—and let d be a metric for $I\times M$, $0\leqq d\leqq 1$. Then each t in M is mapped to a closed set $F(t)$ in $I\times M\times I$: the set of all 3-tuples $(s,t,d((s,t),\Gamma))$. This is continuous and $t\in A\Leftrightarrow F(t)$ meets $J\times M\times(0)$, a G_δ-set in $I\times M\times I$. (The device just introduced is the detour mentioned above.)

Next we define a map h from the cube $Q=[0,1]^N$ to C such that $h^{-1}(C_0)=(0,1]^N$. The triangular function $\delta(u)=\max(1-|u|,0)$; for each $\lambda\geqq 1$ there is a scalar $c(\lambda)>0$ such that $c(\lambda)\delta(n-\lambda)$ belongs to C_0 (as a function of n), and we call this $g(\lambda)$, defining $g(+\infty)=0$. Finally $h(s_1,s_2,s_3,\ldots)=\sum_1^\infty 2^{-k/2}g(s_1^{-1}\cdots s_k^{-1}+3k)$. It is a simple matter to adapt this to an arbitrary metric space Q and any G_δ subset V, in particular $Q=I\times M\times I$ and $V=J\times M\times(0)$. Hence $H(t)\equiv h(F(t))$ is a closed subset of C, $t\in A\Leftrightarrow H(t)$ meets C_0; now we conclude by defining $f(t,x)=d(x,H(t))$.

REFERENCES

1. J. Bourgain and H. P. Rosenthal, Martingales valued in subspaces of L^1, *Israel J. Math.* **37** (1980), 54–75.

2. R. Haydon, A new proof that every Polish space is the extreme boundary of a simplex, *Bull. London Math. Soc.* **7** (1975), 97–100.

3. J. E. Jayne and C. A. Rogers, The extremal structure of convex sets, *J. Funct. Anal.* **26** (1977), 251–288.

https://doi.org/10.1017/S001309150000612X Published online by Cambridge University Press

4. N. J. Kalton and N. T. Peck, A re-examination of the Roberts example of a compact convex set without extreme points, *Math. Ann.* **253** (1980), 89–101.

5. N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler* (London Math. Soc., Lecture Notes **89,** 1984).

6. R. Kaufman, Co-analytic sets and extreme points, *Bull. London Math. Soc.* **19** (1987), 72–74.

7. R. Kaufman, Extreme points and descriptive sets (manuscript).

8. E. T. Poulsen, A simplex with dense extreme points, *Ann. Inst. Fourier* (*Grenoble*) **11** (1961), 83–87.

9. J. W. Roberts, A compact convex set with no extreme points, *Studia Math.* **60** (1971), 255–266.

10. M. Talagrand, Un espace de Banach réticulé qui a presque la proprieté de Radon–Nikodým, *Studia Math.* **76** (1983), 255–266.

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