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RESEARCH ARTICLE

Unicorn paths and hyperfiniteness for the mapping class group

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Abstract
Let S be an orientable surface of finite type. Using Pho-on's infinite unicorn paths, we prove the hyperfiniteness of orbit equivalence relations induced by the actions of the mapping class group of S on the Gromov boundaries of the arc graph and the curve graph of S. In the curve graph case, this strengthens the results of Hamenstädt and Kida that this action is universally amenable and that the mapping class group of S is exact.

1. Introduction

An equivalence relation E on a standard Borel space X is *Borel* if E is a Borel subset of $X \times X$. An equivalence relation is *countable* (resp., *finite*) if every equivalence class is countable (resp., finite). Given a Borel action of a countable group on a standard Borel space X, the induced orbit equivalence relation is a countable Borel equivalence relation. A Borel equivalence relation E is *hyperfinite* if E can be written as an increasing union of a sequence of finite Borel equivalence relations.

Let S be an oriented surface of genus $g \geq 0$ with $n \geq 0$ punctures, of negative Euler characteristic. We denote by $\mathcal{A}(S)$ (for $n \geq 1$) and $\mathcal{C}(S)$ its arc graph and its curve graph, respectively, which are Gromov hyperbolic (see Section 2). The actions of the mapping class group $\mathrm{Mod}(S)$ on $\mathcal{A}(S)$ and $\mathcal{C}(S)$ by automorphisms extend to actions on their Gromov boundaries by homeomorphisms. Our main result is the following:

Theorem 1.1. *The orbit equivalence relation on $\partial\mathcal{A}(S)$ induced by the action of* $\mathrm{Mod}(S)$ *is hyperfinite.*

As a consequence we will derive the following:

Corollary 1.2. *The orbit equivalence relation on $\partial\mathcal{C}(S)$ induced by the action of* $\mathrm{Mod}(S)$ *is hyperfinite.*

This strengthens the results of Hamenstädt [Ham09, Cor 2] and Kida [Kid08, Thm 1.4(ii)] that this equivalence relation is universally amenable (for a definition, see Section 1.1).

We will also obtain the following (for the definition of $\mathcal{CL}(S)$, see Section 5):

Corollary 1.3. *The orbit equivalence relation on the space of complete geodesic laminations $\mathcal{CL}(S)$ induced by the action of* $\mathrm{Mod}(S)$ *is hyperfinite.*

In particular, this equivalence relation is universally amenable. Since $\mathcal{CL}(S)$ is compact and Hausdorff, and its point stabilisers are virtually abelian, this gives a new proof of [Ham09, Thm 1] that the action of Mod(S) on $\mathcal{CL}(S)$ is topologically amenable (see Section 1.1). This implies that Mod(S) is exact (which was proved independently to Hamenstädt by Kida [Kid08, Thm C.5]).

1.1. Amenability

Let μ be a Borel probability measure on a standard Borel space X. The notion of amenability for a measurable action of a countable group on (X, μ) was introduced by Zimmer [Zim78] (see also [Zim84]) and has many equivalent definitions (see, e.g., [AEG94]; for a more general definition for measured groupoids, see [ADR00]). It is closely related to the notion of amenability for countable Borel equivalence relations [Gao09, Def 7.4.5] (see also [Kec93, §3] or [Moo20, §3]). Namely, a measurable action on (X, μ) is μ-amenable if and only if μ-almost all stabilisers are amenable and the induced orbit equivalence relation on X is μ-amenable [AEG94, Thm 5.1]. A countable Borel equivalence relation on X or a Borel action of a countable group on X is *universally amenable* if it is μ-amenable for every quasi-invariant Borel probability measure μ on X (or, equivalently, for every Borel probability measure μ on X [KM04, Cor 10.2]). In particular, any Borel action of an amenable group is universally amenable.

By [ADR00, Thm 3.3.7], an action of a countable group G by homeomorphisms on a locally compact Hausdorff space X is universally amenable if and only if it is topologically amenable (for a definition, see [Oza06b, Def 2.1]). If X is compact Hausdorff, then the topological amenability of the action implies the exactness of G [AD02, Thm 7.2].

Boundary actions have been studied extensively from the point of view of amenability. Connes, Feldman and Weiss [CFW81, Cor 13] and, independently, Vershik [Ver78, Thm 2], showed that the tail equivalence relation discussed in Section 1.2 is universally amenable, which implies that the induced action of the finitely generated free group F_n on its Gromov boundary ∂F_n is universally amenable. This was later generalised by Adams [Ada94] to all hyperbolic groups (see also [Kai04]). Furthermore, Ozawa [Oza06a] proved that the action of a relatively hyperbolic group with amenable parabolic subgroups on the Gromov boundary of its coned-off Cayley graph is topologically amenable. Moreover, Nevo and Sageev proved that the action of a cocompactly cubulated group on a particular subset of its Roller boundary is universally amenable [NS13]. Lécureux proved that if a group G acts geometrically on a building X, then the action of G on the combinatorial boundary of X is topologically amenable [Léc10]. Finally, Bestvina, Guirardel and Horbez proved that the action of Out(F_n) on the Gromov boundary of its free factor complex is universally amenable (see [BGH21, Thm 6.4] and [GHL20, Prop 7.2]), which uses the description of the Gromov boundary of the free factor complex in [BR15] and [Ham14].

1.2. Hyperfiniteness

As shown independently by Weiss and Slaman–Steel [Gao09, Thm 7.2.4], a Borel equivalence relation E is hyperfinite if and only if there is a Borel action of $\mathbb{Z}$ inducing E as its orbit equivalence relation. Since $\mathbb{Z}$ is amenable, the hyperfiniteness of a Borel equivalence relation implies its universal amenability. It is a well-known open problem whether the converse holds – that is, whether a universally amenable Borel equivalence relation is always hyperfinite. Connes, Feldman and Weiss [CFW81, Thm 10] (see also [KM04, Thm 10.1]) showed that a μ-amenable Borel equivalence relation on X becomes hyperfinite after removing from X a set of μ-measure 0.

The relative complexity of Borel equivalence relations is measured by Borel reducibility. Given two equivalence relations E and F on standard Borel spaces X and Y, respectively, a function $f : X \to Y$ is a *Borel reduction* from E to F if f is a Borel function and for every $a, b \in X$ we have $a \sim_E b$ if and only if $f(a) \sim_F f(b)$. A relation E is *Borel reducible* to F if there exists a Borel reduction from E to F. The relation E_0 is defined on $\{0,1\}^{\mathbb{N}}$ (with the product topology) as $(a_i)_{i=0}^\infty \sim_{E_0} (b_i)_{i=0}^\infty$ if $a_i = b_i$ for all i sufficiently large. It is easy to see that E_0 is hyperfinite. In fact, a countable Borel equivalence relation is hyperfinite if and only if it is Borel reducible to E_0 [Gao09, Thm 7.2.2].

Let Ω be a countable set with discrete topology. The *tail equivalence relation* E_t on $\Omega^{\mathbb{N}}$ is defined as $(a_i)_{i=0}^\infty \sim_{E_t} (b_i)_{i=0}^\infty$ if there exists $k \in \mathbb{Z}$ such that $a_i = b_{i+k}$ for all i sufficiently large. Dougherty,

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Jackson and Kechris showed that E_t is Borel reducible to E_0, and so it is hyperfinite [DJK94, Cor 8.2]. It is not hard to see that the orbit equivalence relation induced by the action of F_n on ∂F_n is Borel reducible to E_t with finite Ω. Hence that orbit equivalence relation on ∂F_n is hyperfinite, which we will shortly express by saying that the *boundary action of F_n is hyperfinite*.

More recently, Huang, Shinko and Sabok [HSS19] showed that for cocompactly cubulated hyperbolic groups, their boundary actions are hyperfinite. The proof relied on a study of geodesic ray bundles in hyperbolic groups. While Touikan [Tou18] showed that that approach does not work for arbitrary hyperbolic groups, Marquis [Mar19] used it to prove the hyperfiniteness of boundary actions of groups acting cocompactly on locally finite hyperbolic buildings with trivial chamber stabilisers. Very recently, Marquis and Sabok [MS20] showed the hyperfiniteness of the boundary action of an arbitrary hyperbolic group.

Organisation

In Section 2 we recall the basics on arcs, laminations and unicorn paths. In Section 3 we prove a pair of key lemmas: the local characterisation of Pho-on's infinite unicorn paths and the tail equivalence for asymptotic infinite unicorn paths. This allows for the proofs of Theorem 1.1 and Corollary 1.2 in Section 4. We prove Corollary 1.3 in Section 5.

2. Unicorn paths

2.1. Arcs and laminations

As in the introduction, S is obtained from a closed oriented surface of genus g by removing n points. Thus S has n topological ends, which are called *punctures*. An *oriented arc* on S is a map from $(0,1)$ to S that is proper. A proper map induces a map between topological ends of spaces, and in this sense each endpoint of $(0,1)$ is sent to a puncture of S. We will say that the oriented arc *starts* and *ends* at these punctures. A *homotopy* between oriented arcs a and b is a proper map $(0,1)\times[0,1]\to S$ whose restriction to $(0,1)\times\{0\}$ equals a and whose restriction to $(0,1)\times\{1\}$ equals b. In particular, a and b start at the same puncture and end at the same puncture. A *curve* on S is a map from a circle S^1 to S.

An oriented arc or a curve is *simple* if it is an embedding. In that case we can and will identify the oriented arc or the curve with its image in S. We record, however, the orientation of the arc, while for the curve we discard it. A curve is *essential* if it is not homotopically trivial. A curve $c\colon S^1\to S$ is *nonperipheral* if it cannot be homotoped into the puncture, in the sense that there is no proper map $S^1\times[0,1)\to S$ whose restriction to $S^1\times\{0\}$ is c. An oriented arc $a\colon (0,1)\to S$ is *essential* if there is no proper map $(0,1)\times[0,1)\to S$ whose restriction to $(0,1)\times\{0\}$ is a. Unless otherwise stated, all oriented arcs in this paper are simple and essential, and all curves are simple, essential and nonperipheral.

Suppose that the Euler characteristic $\chi=2-2g-n$ of S is negative. If $n\geq 1$, the *arc graph* $\mathcal{A}(S)$ is the graph whose vertex set A is the set of homotopy classes of oriented arcs on S. Two vertices in A are connected by an edge if they can be realised disjointly. Note that since our arcs are oriented, our $\mathcal{A}(S)$ differs from the usual arc graph by replacing each vertex with two.

Allow now $n=0$, but suppose that we are not in one of the exceptional cases where $g=0$ and $n=3$ or 4, or $g=1$ and $n=1$. Then the *curve graph* $\mathcal{C}(S)$ is the graph whose vertices are the homotopy classes of curves on S. Again, two vertices are connected by an edge if they can be realised disjointly. In the exceptional cases the edges of $\mathcal{C}(S)$ are defined differently, but we will not be appealing to that definition. By [MM99] and [MS13], the graphs $\mathcal{C}(S)$ and $\mathcal{A}(S)$ are Gromov-hyperbolic.

We fix an arbitrary complete hyperbolic metric on S. A *geodesic lamination* on S is a compact subset of S that is a disjoint union of *leaves* that are geodesic lines and circles in S that do not self-intersect. A geodesic lamination L is *minimal* if its every leaf is dense in L. Let $Y\subseteq S$ be a subsurface whose boundary components are all geodesic circles. We say that a geodesic lamination $L\subset Y$ *fills* Y if every curve on Y intersects L. Analogously, a pair of oriented arcs $a,b\subset Y$ *fills* Y if every curve on Y intersects the geodesic representative of a or b.

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A *peripherally ending lamination* is a minimal geodesic lamination that fills a subsurface Y containing all the punctures of S. An *ending lamination* is a minimal geodesic lamination that fills the entire S. Let $\mathcal{EL}(S) \subset \mathcal{EL}_0(S)$ denote the sets of ending and peripherally ending laminations on S, respectively, with the topology given by the following *coarse Hausdorff* convergence. Namely, $L_n \xrightarrow{\text{CH}} L$ if for any subsequence L_{n_k} Hausdorff converging to a geodesic lamination L', we have $L \subset L'$ [Ham06]. By [Kla99] and [Sch13] (see also Theorem 3.2), the spaces $\mathcal{EL}(S), \mathcal{EL}_0(S)$ can be equivariantly identified with the Gromov boundaries of $\mathcal{C}(S)$ and $\mathcal{A}(S)$.

2.2. *Unicorns*

As in Section 2.1, let A denote the set of homotopy classes of oriented arcs on S.

Definition 2.1. Set $a, b \in A$ and keep the notation a, b for the geodesic oriented arcs representing them. A *unicorn arc* for a and b is the homotopy class of an oriented arc that is a concatenation $a' \cup b'$ for a' an initial segment of a and b' a terminal segment of b, possibly $a' = a, b' = \emptyset$ or $a' = \emptyset, b' = b$. Note that orienting the arcs replaces the choice of endpoints in [HPW15, Def 3.1].

The set of all oriented arcs that are such concatenations $a' \cup b'$ can be ordered into a sequence $(a'_i \cup b'_i)_{i=0}^n$ so that for all $0 \leq i < n$, we have $a'_{i+1} \subset a'_i$ and $b'_{i+1} \supset b'_i$. We denote by $c_i \in A$ the homotopy class of $a'_i \cup b'_i$ and we call the sequence $P(a, b) = (c_i)_{i=0}^n \in A^{n+1}$ the *unicorn path* from a to b.

Note that we have $c_0 = a$ and $c_n = b$. Moreover, the unicorn path is indeed an edge path in $\mathcal{A}(S)$:

Remark 2.2 ([HPW15, Rm 3.2])**.** For each $0 \leq i < n$, the unicorn arcs c_i, c_{i+1} are adjacent in $\mathcal{A}(S)$.

Let L_0 be a peripherally ending lamination. Let l be a geodesic line on S that does not self-intersect and ends at a puncture in the sense that l contains a geodesic ray properly embedded in S. We say that l is *asymptotic to* L_0 if $l \subset S \setminus L_0$. Since each puncture of S lies in a once-punctured ideal polygon of $S \setminus L_0$, the number of such l is bounded by the total number of their ideal vertices, which is at most $2|\chi|$.

Definition 2.3 ([PO17, §3.1])**.** Set $a \in A$ and keep the notation a for the geodesic oriented arc representing it. Let l be a geodesic line asymptotic to $L_0 \in \mathcal{EL}_0(S)$. A *unicorn arc* for a and l is the homotopy class of an oriented arc that is a concatenation $a' \cup l'$ for a' an initial segment of a and l' a terminal segment of l, possibly $a' = a$ and $l' = \emptyset$.

The set of all oriented arcs that are such concatenations $a' \cup l'$ can be ordered into a sequence $(a'_i \cup l'_i)_{i=0}^\infty$ so that for all $i \geq 0$, we have $a'_{i+1} \subset a'_i$ and $l'_{i+1} \supset l'_i$. We denote by $c_i \in A$ the homotopy class of $a'_i \cup l'_i$ and we call the sequence $P(a, l) = (c_i)_{i=0}^\infty \in A^{\mathbb{N}}$ the *infinite unicorn path* from a to l.

3. Key lemmas

Definition 3.1. Set $n \in \{3, 4, \ldots, \infty\}$. A sequence $(c_i)_{i=0}^n \in A^{n+1}$ is a *locally unicorn path* if for each $0 \leq j < k \leq n$, with $j + 3 \leq k < \infty$, the sequence $(c_i)_{i=j}^k$ is the unicorn path from c_j to c_k.

By Remark 2.2, a locally unicorn path is an edge path in $\mathcal{A}(S)$. Moreover, by [HPW15, Lem 3.5] each finite unicorn path of length ≥ 3 is a locally unicorn path. Furthermore, by [PO17, Lem 3.4] an infinite unicorn path is also locally unicorn.

By [HPW15, Prop 4.2] there is a universal constant C such that each finite unicorn path $P(a, b)$ is at Hausdorff distance $\leq C$ from a geodesic edge path in $\mathcal{A}(S)$ from a to b. Consequently, each locally unicorn path is bounded or converges with respect to the Gromov product [GdlH90, §7.2] to a point in $\partial\mathcal{A}(S)$. This leads to the following result of Pho-on (the existence of an equivariant homeomorphism was announced earlier by Schleimer [Sch13]):

Theorem 3.2 ([PO17, §3.2-3])**.** *Set $a \in A$. Set $L_0 \in \mathcal{EL}_0(S)$ and let l be a geodesic line asymptotic to L_0. Then $P(a, l)$ is not bounded and its limit $F(L_0) \in \partial\mathcal{A}(S)$ with respect to the Gromov product depends only on L_0. Furthermore, $F \colon \mathcal{EL}_0(S) \to \partial\mathcal{A}(S)$ is a* Mod(S)*-equivariant homeomorphism.*

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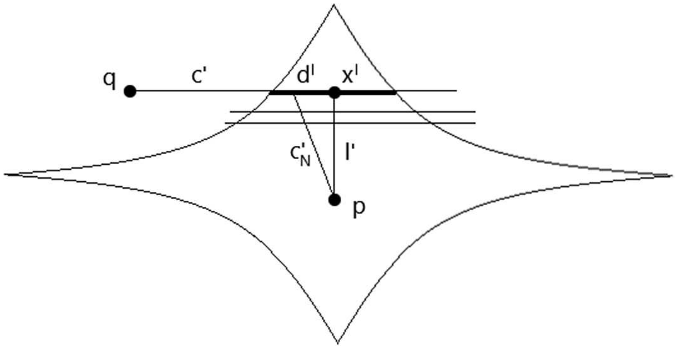

Figure 1. *Ideal polygon D.*

In fact, the local condition characterises infinite unicorn paths:

Lemma 3.3. *Let P be a locally unicorn path that is not bounded in $\mathcal{A}(S)$. Then P is an infinite unicorn path.*

Proof. Denote $P = (c_i)_{i=0}^{\infty} \in A^{\mathbb{N}}$, and keep the notation c_i for the geodesic oriented arcs representing them. Since P is not bounded in $\mathcal{A}(S)$, it converges to some point $F(L_0) \in \partial\mathcal{A}(S)$. By [PO17, Lem 3.9], we have that c_i coarse Hausdorff converge to $L_0 \in \mathcal{EL}_0(S)$. Denote $c = c_0$. We claim that for each $n \geq 1$ there is a geodesic line l asymptotic to L_0 such that for each $i \leq n$, the unicorn arc a_i on the infinite unicorn path from c to l coincides with c_i. Since there are only finitely many l asymptotic to L_0, the lemma follows from the claim.

To justify the claim, note that since P is a locally unicorn path, all c_i with $i \geq 1$ end at a common puncture p. Let D be the ideal polygon of $S \setminus L_0$ containing p. Let l be a geodesic line asymptotic to L_0 ending at p. Let $c' \cup l'$ represent the nth unicorn arc on the unicorn path from c to l, let $x^l = c' \cap l'$ and let d^l be the segment of c that is the component of $D \cap c$ containing x^l (see Figure 1). Let D_p be the component of $D - \bigcup_l d^l$ containing p, where the union is taken over all the geodesic lines l asymptotic to L_0 and ending at p. Let $\alpha > 0$ be the minimum possible angle that makes with L_0 a geodesic ray in D_p starting on L_0 and ending at p. Since $(c_i)_{i=0}^{\infty} \xrightarrow{\text{CH}} L_0$, there is $N \geq n$ such that c_N does not intersect L_0 at angle $\geq \alpha$. Consequently, the component c'_N of $c_N \cap D_p$ ending at p starts on d^l for some l (see Figure 1).

Let q be the puncture at which c starts. We have a bijection $h\colon l' \cap c \to c'_N \cap c$ such that each pair $x, h(x)$ lies in the same component of $D_p \cap c$. Furthermore, for each $x \in l' \cap c$, the segments $qx \subset c$ and $xp \subset l$ intersect only at x if and only if the segments $qh(x) \subset c$ and $h(x)p \subset c_N$ intersect only at $h(x)$. In other words, the concatenation $qx \cup xp$ represents a unicorn arc for c and l if and only if the concatenation $qh(x) \cup h(x)p$ represents a unicorn arc for c and c_N. Moreover, these two oriented arcs are homotopic. Finally, this correspondence preserves the order of unicorn arcs. Thus, for $0 \leq i \leq n$, we have $a_i = c_i$, justifying the claim. □

Set $L_0 \in \mathcal{EL}_0(S)$. We define an equivalence relation $\sim_{L_0}$ on A by declaring $a \sim_{L_0} b$ if the geodesic representatives of a, b start at the same puncture and their first points in L_0 lie on the same side of the ideal polygon of $S \setminus L_0$ containing that puncture. Note that $\sim_{L_0}$ has at most $2|\chi|$ equivalence classes.

We now prove a tail equivalence lemma that will later allow us to reduce the orbit equivalence on $\partial\mathcal{A}(S)$ to E_t.

Lemma 3.4. *Set $L_0 \in \mathcal{EL}_0(S)$, and set $a, b \in A$ with $a \sim_{L_0} b$. Then for each geodesic line l asymptotic to L_0, the unicorn path $(a_i)_{i=0}^{\infty}$ from a to l and the unicorn path $(b_i)_{i=0}^{\infty}$ from b to l satisfy $a_i = b_{i+k}$ for some $k \in \mathbb{Z}$ and all i sufficiently large.*

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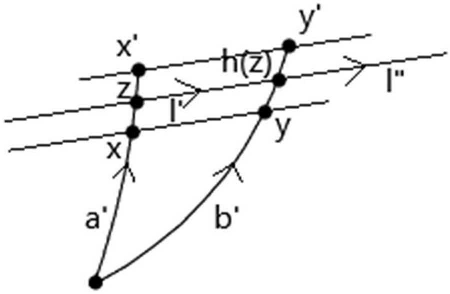

Figure 2. *Rectangle B.*

Proof. Set $a, b \in A$ with $a \sim_{L_0} b$ and keep the notation a, b for the geodesic oriented arcs representing them. Let x, y be the first points on a, b in L_0. Since $a \sim_{L_0} b$, there is a geodesic segment $xy \subset L_0$. Furthermore, since L_0 is minimal, there are segments xx', yy' in a, b such that $x'y'$ is a geodesic segment in L_0 and $xx'y'y$ bounds a topological disc B embedded in S (see Figure 2).

Consequently, the components of the intersection $B \cap l$ are geodesic segments joining xx' to yy', which yields a bijection $h\colon xx' \cap l \to yy' \cap l$. Let $a' \subset a$ be an initial segment of a ending in $z \in xx' \cap l$, and let b' be the initial segment of b ending in $h(z)$. Furthermore, let l', l'' be the terminal segments of l starting in $z, h(z)$, respectively. Assume without loss of generality $l'' \subset l'$, as in Figure 2.

Note that xz intersects $zh(z) \subset l'$ only at z. Furthermore, xz is disjoint from l'' if and only if $yh(z)$ intersects l'' only at $h(z)$. Consequently, the concatenation $a' \cup l'$ represents a unicorn arc if and only if $b' \cup l''$ represents a unicorn arc. Moreover, these two oriented arcs are homotopic. Finally, this correspondence preserves the order of unicorn arcs, and all a_i for i sufficiently large are accounted for in this way. □

4. Hyperfiniteness

We fix a base point $a_0 \in A$.

Definition 4.1. Let $\mathcal{P} \subset A^{\mathbb{N}}$ be the set of infinite unicorn paths from a_0 to any geodesic line asymptotic to any $L_0 \in \mathcal{EL}_0(S)$. Let $f\colon \mathcal{P} \to \partial\mathcal{A}(S)$ be the map assigning to such path its limit $F(L_0) \in \partial\mathcal{A}(S)$ with respect to the Gromov product (see Theorem 3.2).

Note that f is finite-to-one, since a_0 is fixed and there are finitely many geodesic lines asymptotic to a given $L_0 \in \mathcal{EL}_0(S)$.

Remark 4.2. We equip the countable set A with the discrete topology and $A^{\mathbb{N}}$ with the product topology. Then the set $\mathcal{P} \subset A^{\mathbb{N}}$ is Borel. Indeed, by Lemma 3.3, $\mathcal{P}$ is the set of locally unicorn paths that are not bounded. The set of locally unicorn paths is closed in $A^{\mathbb{N}}$, since each of the conditions on $(c_i)_{i=j}^{k}$ to be a unicorn path is closed. Furthermore, for each $n \geq 0$, the set of sequences in $A^{\mathbb{N}}$ at distance $\leq n$ from a_0 is closed, so the set of sequences in $A^{\mathbb{N}}$ at bounded distance from a_0 is a countable union of closed sets. Consequently, $\mathcal{P}$ is a countable intersection of open sets.

Since locally unicorn paths are uniformly Hausdorff close to geodesic edge paths, and the function f assigns their limits in $\partial\mathcal{A}(S)$, we have that f is continuous with respect to the metric on $\partial\mathcal{A}(S)$ defined using the Gromov product.

Let $\mathcal{T}$ be the countable set of finite-length edge paths in $\mathcal{A}(S)$, up to the action of $\mathrm{Mod}(S)$, equipped with the discrete topology. For an infinite unicorn path $P = (c_i)_{i=0}^{\infty}$, given $i \geq 0$ and $j = i+1, \ldots$, the subsurfaces $\Sigma_{i,j} \subseteq S$ filled by c_i and c_j form an ascending sequence $\Sigma_{i,i+1} \subseteq \Sigma_{i,i+2} \subseteq \cdots$ that stabilises with some subsurface, which we call $\Sigma_i \subseteq S$. For each $i \geq 0$, let $m(i) > i+1$ be minimal satisfying $\Sigma_{i+1,m(i)} = \Sigma_{i+1}$. Let $T_i = \left(c_j\right)_{j=i}^{m(i)}$, and let $[T_i]$ be the equivalence class of T_i in $\mathcal{T}$. Let $g\colon \mathcal{P} \to \mathcal{T}^{\mathbb{N}}$ be

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the map defined by $g(P) = ([T_i])_{i=0}^{\infty}$. Let E_t be the tail equivalence relation on $\mathcal{T}^{\mathbb{N}}$ described in Section 1 (with $\Omega = \mathcal{T}$).

Note that the definition of g can be analogously extended to infinite unicorn paths $P \notin \mathcal{P}$ (that is, to infinite unicorn paths that start at points distinct from a_0), which we will make use of later on.

Remark 4.3. We equip $\mathcal{T}^{\mathbb{N}}$ with the product topology. Then the map $g\colon \mathcal{P} \to \mathcal{T}^{\mathbb{N}}$ is Borel. Indeed, for all $0 \le i < j$, the maps $P \to \Sigma_{i,j}$ are continuous maps from $\mathcal{P}$ to the countable discrete set of subsurfaces of S, and hence their limits $P \to \Sigma_i$ are Borel. Thus, for all $0 \le i < j$, the subset of $\mathcal{P}$ defined by the identity $\Sigma_{i,j} = \Sigma_i$ is Borel, and so the maps $m(i)\colon \mathcal{P} \to \mathbb{N}$ are Borel. Consequently, all the maps $[T_i]\colon \mathcal{P} \to \mathcal{T}$ are Borel, as desired.

Lemma 4.4. *Set $P, P' \in \mathcal{P}$. If $g(P) \sim_{E_t} g(P')$, then there is $\psi \in \mathrm{Mod}(S)$ satisfying $\psi f(P) = f(P')$. Conversely, for each orbit ω of the action of $\mathrm{Mod}(S)$ on $\partial\mathcal{A}(S)$, there are finitely many equivalence classes of E_t on $\mathcal{T}^{\mathbb{N}}$ containing all $g(P)$ for $P \in \mathcal{P}$ with $f(P) \in \omega$.*

Proof. Denote $P = (c_i)_{i=0}^{\infty}, P' = \left(c'_i\right)_{i=0}^{\infty}$. Let T'_i be defined for P' analogously as T_i was for P. If $g(P) \sim_{E_t} g(P')$, then there are $k \in \mathbb{Z}, j \in \mathbb{N}$ such that $[T_i] = \left[T'_{i+k}\right]$ for all $i \ge j$. In particular, there is $\psi \in \mathrm{Mod}(S)$ with $\psi T_j = T'_{j+k}$. We will show inductively that $\psi T_i = T'_{i+k}$ for all $i \ge j$, so in particular $\psi c_i = c'_{i+k}$ implying $\psi f(P) = f(P')$.

Suppose that we have established $\psi T_i = T'_{i+k}$ for some $i \ge j$. If $m(i+1) \le m(i)$, then $\psi T_{i+1} = T'_{i+1+k}$ is immediate, so we can assume $m(i+1) > m(i)$. Let $\rho \in \mathrm{Mod}(S)$ be such that $\rho T_{i+1} = T'_{i+1+k}$. Then $\rho^{-1}\psi$ fixes all $c_{i+1}, \ldots, c_{m(i)}$. Thus the restriction of $\rho^{-1}\psi$ to the subsurface $\Sigma_{i+1} \subset S$, which c_{i+1} and $c_{m(i)}$ fill, is the identity map. By the definition of Σ_{i+1}, we have that $c_{m(i)+1}, \ldots, c_{m(i+1)}$ all lie in Σ_{i+1}. This implies that $\rho^{-1}\psi$ fixes them, and so $\psi T_{i+1} = T'_{i+1+k}$, completing the induction.

For the converse, let ω be the orbit under $\mathrm{Mod}(S)$ of some $F(L_0) \in \partial\mathcal{A}(S)$. Let $l_1, \ldots, l_n$ be the finitely many geodesic lines asymptotic to L_0. Choose $a_1, \ldots, a_p \in A$ that are representatives of the equivalence classes of $\sim_{L_0}$ distinct from the one containing a_0. For $0 \le q \le p$ and $1 \le j \le n$, let $P_{qj} = P\left(a_q, l_j\right)$.

Set $P = (c_i)_{i=0}^{\infty} \in \mathcal{P}$ with $f(P) \in \omega$. By Theorem 3.2, we have $P = P(a_0, l)$, where l is a geodesic line asymptotic to ψL_0 for some $\psi \in \mathrm{Mod}(S)$. In particular, for some $1 \le j \le n$ we have $l = \psi l_j$. Thus $\psi^{-1}P = P\left(\psi^{-1}a_0, l_j\right)$. Choose $0 \le q \le p$ so that $a_q \sim_{L_0} \psi^{-1}a_0$. By Lemma 3.4, writing $P_{qj} = (b_i)_{i=0}^{\infty}$, we have $\psi^{-1}c_i = b_{i+k}$ for some $k \in \mathbb{Z}$ and all i sufficiently large. We have then that $g\left(P_{qj}\right)$ and $g(\psi^{-1}P)$, hence also $g(P)$, are tail equivalent. □

Proof of Theorem 1.1. Write E for the equivalence relation on $\partial\mathcal{A}(S)$ induced by the action of $\mathrm{Mod}(S)$ and write E^* for the equivalence relation on $\mathcal{P}$ that is the pullback of E via f – that is, $P \sim_{E^*} P'$ if $f(P) \sim_E f(P')$. Since E is Borel and countable, and f is Borel and finite-to-one, we have that E^* is also Borel and countable.

Since f is a Borel finite-to-one function, it has a Borel right inverse by the Lusin–Novikov uniformisation theorem [Kec95, Thm 18.10]. Consequently, E is Borel reducible to E^*. Thus it is enough to show that E^* is hyperfinite.

Write E_t^* for the equivalence relation on $\mathcal{P}$ that is the pullback of E_t via g. Since E_t is Borel, and g is Borel, we have that E_t^* is Borel. By Lemma 4.4, we have $E_t^* \subseteq E^*$ and every equivalence class of E^* contains finitely many equivalence classes of E_t^*. (In particular, E_t^* is countable.) Thus by [JKL02, Prop 1.3(vii)], it is enough to show that E_t^* is hyperfinite.

Note that g is a Borel reduction of E_t^* to E_t. Thus since E_t is hyperfinite [DJK94, Cor 8.2], we have that E_t^* is hyperfinite as well. □

Proof of Corollary 1.2. Assume first that S has $n \ge 1$ punctures. Then by [Kla99, Thm 1.3], Theorem 1.1 and [JKL02, Prop 1.3(iii)], it suffices to prove that $\mathcal{EL}(S)$ is a Borel subset of $\mathcal{EL}_0(S)$. Indeed, $L_0 \in \mathcal{EL}_0(S)$ is a minimal filling lamination if and only if each geodesic representative of a curve c on S intersects L_0 and does it transversally. Given c, this is an open condition, and so $\mathcal{EL}(S)$ is a countable intersection of open sets.

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Second, assume $n = 0$ and let S' be the surface obtained from S by adding one puncture at a point outside the closure of the union of all embedded geodesic circles and lines, which exists by [BS85, Thm I]. This induces a closed embedding $e\colon \mathcal{EL}(S) \to \mathcal{EL}(S')$, which is a section for the map $r\colon \mathcal{EL}(S') \to \mathcal{EL}(S)$ defined by forgetting the puncture (see [PO17, §4.2] for details). Thus for each $L_1, L_2 \in \mathcal{EL}(S)$, with $\psi' e(L_1) = e(L_2)$ for some $\psi' \in \mathrm{Mod}(S')$, the image $\psi \in \mathrm{Mod}(S)$ of ψ' under the puncture-forgetting map $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$ satisfies $\psi(L_1) = L_2$.

Conversely, set $L \in \mathcal{EL}(S)$ and let $R_1, \ldots, R_n \subset S$ be the components of $S \setminus L$. For $1 \le j \le n$, let L_j be a lamination in $\mathcal{EL}(S')$ obtained from L by adding a puncture in R_j, under an arbitrary identification with S'. All such identifications differ by $\mathrm{Mod}(S')$, so the resulting orbit $\left[L_j\right]$ in $\mathcal{EL}(S')$ does not depend on our choice. Since e is a section for r, we have $e(L) \in \bigcup_{j=1}^{n} \left[L_j\right]$. Analogously, for any $\psi \in \mathrm{Mod}(S)$, we have $e(\psi(L)) \in \bigcup_{j=1}^{n} \left[L_j\right]$.

Consequently, under the identification of $\mathcal{EL}(S)$ with $e(\mathcal{EL}(S))$, each orbit of $\mathrm{Mod}(S)$ on $\mathcal{EL}(S)$ consists of the intersections of finitely many orbits of $\mathrm{Mod}(S')$ on $\mathcal{EL}(S')$ with $e(\mathcal{EL}(S))$. Thus by [JKL02, Prop 1.3 (iii,vii)], the hyperfiniteness of the action of $\mathrm{Mod}(S)$ on $\mathcal{EL}(S)$ follows from the hyperfiniteness of the action of $\mathrm{Mod}(S')$ on $\mathcal{EL}(S')$. □

5. Complete geodesic laminations

A geodesic lamination L on S is *complete* if each component of $S \setminus L$ is an ideal triangle or a once-punctured monogon, and L lies in the closure (in the Hausdorff topology on the space of compact subsets of S) of the set of embedded geodesic circles. By $\mathcal{C}L(S)$ we denote the space of complete geodesic laminations with the Hausdorff topology, which is compact and Hausdorff [Ham09, §2.1].

Proof of Corollary 1.3. For a complete geodesic lamination L, let L' denote the union of minimal sublaminations of L that are not embedded geodesic circles. Let $Y(L)$ denote the subsurface of S filled by L'. Given a subsurface $Y \subseteq S$, let $\mathcal{C}L(S, Y) \subset \mathcal{C}L(S)$ denote the subspace of laminations L with $Y(L) = Y$. We claim that each $\mathcal{C}L(S, Y)$ is a Borel subset of $\mathcal{C}L(S)$, and hence $\mathrm{Mod}(S)\mathcal{C}L(S, Y)$ is Borel as well.

Indeed, for a lamination L in $\mathcal{C}L(S, Y)$, the union Z of the nonisolated leaves of L on $S \setminus Y$ is a union of disjoint (geodesic representatives of) curves on $S \setminus Y$. Thus a complete geodesic lamination L belongs to $\mathcal{C}L(S, Y)$ if and only if

- ◦ each curve c on Y intersects L transversally an infinite number of times (G_δ condition) and
- ◦ there exists a union of disjoint curves Z on $S \setminus Y$ such that each curve c on $S \setminus (Y \cup Z)$ or in $Z \cup \partial Y$ does not intersect L transversally an infinite number of times ($F_{\sigma\delta\sigma}$ condition).

Thus $\mathcal{C}L(S, Y)$ is an $F_{\sigma\delta\sigma}$ set, justifying the claim.

By [JKL02, Prop 1.3(v)], to prove that the orbit equivalence relation on $\mathcal{C}L(S)$ induced by the action on $\mathrm{Mod}(S)$ is hyperfinite, it suffices to show that its restriction to each $\mathrm{Mod}(S)\mathcal{C}L(S, Y)$ is hyperfinite. By [JKL02, Prop 1.3(vi)], it suffices to show that the orbit equivalence relation E on $\mathcal{C}L(S, Y)$ induced by the action of the stabiliser $\mathrm{Mod}(S)_Y$ of Y in $\mathrm{Mod}(S)$ is hyperfinite.

Let $Y_1, \ldots, Y_k$ be the components of Y, where we treat all geodesic boundary components as punctures. Let $g\colon \mathcal{C}L(S, Y) \to \mathcal{EL}(Y_1) \times \cdots \times \mathcal{EL}(Y_k)$ be the map assigning to each L the components of its sublamination L'. In the case where $Y = \emptyset$, the product $\mathcal{EL}(Y_1) \times \cdots \times \mathcal{EL}(Y_k)$ should be understood as a point. By Corollary 1.2 and [JKL02, Prop 1.3(iv)], the orbit equivalence relation on $\mathcal{EL}(Y_1) \times \cdots \times \mathcal{EL}(Y_k)$ induced by the action of $\mathrm{Mod}(Y_1) \times \cdots \times \mathrm{Mod}(Y_k)$ is hyperfinite. The group $\mathrm{Mod}(Y_1) \times \cdots \times \mathrm{Mod}(Y_k)$ is of finite index in $\mathrm{Mod}(Y_1 \sqcup \cdots \sqcup Y_k)$. Thus by [JKL02, Prop 1.3(vii)], the orbit equivalence relation on $\mathcal{EL}(Y_1) \times \cdots \times \mathcal{EL}(Y_k)$ induced by the action of $\mathrm{Mod}(Y_1 \sqcup \cdots \sqcup Y_k)$ is hyperfinite. Its pullback F under g is thus hyperfinite as well, since g has countable fibres. Since E is contained in F, it is hyperfinite by [JKL02, Prop 1.3(i)], as desired. □

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