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A NOTE ON AN ITERATIVE TEST OF EDELSTEIN(1)

BY
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1. **Introduction.** Let (X_1, d_1) and (X_2, d_2) be metric spaces. A mapping $f: X_1 \to X_2$ is said to be a *Lipschitz mapping* (with respect to d_1 and d_2) if and only if (*)$d_2(f(x), f(y)) \leq \lambda \cdot d_1(x, y)$ for all $x, y \in X_1$, where λ is a fixed real number. The constant λ is called a *Lipschitz constant for f*. If (*) is satisfied for $\lambda=1$, then f is called *non-expansive* (see, for example, [21]) and if (*), again with $\lambda=1$, is replaced by a strict inequality for all $x \neq y$, then f is called *contractive* [1]. If $x \in X_1$ and $X_1 = X_2$, then the sequence $\{f^n(x)\}_{n=1}^{\infty}$, where $f^1(x)=f(x)$ and $f^n(x)=f(f^{n-1}(x))$ for each $n > 1$, is called the *sequence of iterates of f at x*.

In [1] Edelstein proved that a contractive mapping f of a metric space X into itself has a fixed point provided there is a point $x \in X$ such that some subsequence of the sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to a point p of X (in fact, p is shown to be the fixed point). Moreover, it is subsequently shown (see [1], (3.2)) that, under the above conditions, the entire sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to the fixed point p.

Generally it may be just as difficult to verify that a subsequence of the iterates at some point of the contractive mapping converge as it would be to locate the fixed point itself. In particular, there are contractive mappings (with a fixed point) such that the iterates at any point of the space other than the fixed point do not converge. The following example, due to R. Orr, gives a simple illustration of this.

EXAMPLE. Let $X=\{0, 2, \frac{3}{2}, \ldots, (n+1)/n, \ldots\}$ with absolute value distance and define $f: X \to X$ by $f(0)=0$ and $f[(n+1)/n)]=(n+2)/(n+1)$ for each $n=1, 2, \ldots$.

We say that the *iterative test* (for contractive mappings) *is conclusive for a metric space* (X, d) if and only if, for any contractive mapping $f: X \to X$, the following implication is valid:
if, for some $x \in X$, $\{f^n(x)\}_{n=1}^{\infty}$ does not converge (or, equivalently, no subsequence of $\{f^n(x)\}_{n=1}^{\infty}$ converges) to a point of X, then f has no fixed point.

In this paper we are concerned with the following question: For what classes of spaces is the iterative test for contractive mappings conclusive?

In §2 we give two examples of complete spaces for which the iterative test is not conclusive. In §3 (see Theorem 1) we prove that the iterative test is conclusive for the class of locally compact, connected spaces. The examples in §2 show that

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neither the hypothesis of local compactness nor the hypothesis of connectedness can be omitted in Theorem 1. In §4 we give a class of subsets of finite dimensional Banach spaces for which the iterative test is conclusive. Several unsolved problems are stated in §5.

REMARK. It might appear, from the example above, that completeness of the space is related to conclusiveness of the iterative test. However, the examples in §2 together with Theorem 2 of §4 show that this is not the case.

2. **Some examples.** The examples in this section will be discussed further in §3 and 5.

The first example shows that the iterative test for contractive mappings is not conclusive for any infinite dimensional Hilbert space.

EXAMPLE 1. Let $l_2 = \{(x_1, x_2, \ldots, x_i, \ldots): x_i$ is a real number for each $i=1, 2, \ldots$ and $\sum_{i=1}^{\infty} x_i^2 < \infty\}$ with the usual norm $\| \;\|$, i.e., if $x=(x_1, x_2, \ldots, x_i, \ldots) \in l_2$, then

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}.$$

If $x=(x_1, x_2, \ldots, x_i, \ldots) \in l_2$, then let

$$f(x) = \left(0, \tfrac{3}{4}x_1, \tfrac{8}{9}x_2, \ldots, \frac{i(i+2)}{(i+1)^2}x_i, \ldots\right),$$

where $[i(i+2)/(i+1)^2]\, x_i$ appears in the $(i+1)^{st}$ coordinate of $f(x)$. It is easy to verify that f is a contractive mapping of l_2 into l_2 (note also that f is linear) with fixed point $(0, 0, \ldots, 0, \ldots)$. We now show that if $x \neq (0, 0, \ldots, 0, \ldots)$, then the sequence of iterates of f at x do not converge (this is a stronger assertion than needed to show that the iterative test is not conclusive for l_2). To see this let $x=(x_1, x_2, \ldots, x_i, \ldots) \in l_2$ such that $x \neq (0, 0, \ldots, 0, \ldots)$. Then there is some coordinate of x, say x_k, which is not zero. A straightforward induction on n shows that the $(k+n)^{th}$ coordinate of $f^n(x)$ is $[k(k+n+1)/(k+1)(k+n)] \cdot x_k$ for each $n=1, 2, \ldots$. Since

$$\left|\frac{k(k+n+1)}{(k+1)(k+n)} \cdot x_k\right| > \frac{k}{k+1} \cdot |x_k|$$

for each $n=1, 2, \ldots$, it follows that the sequence $\{f^n(x)\}_{n=1}^{\infty}$ does not converge to zero and, therefore, does not converge. This completes the proof that the iterative test is not conclusive for l_2. Clearly the same construction can be used to prove that the iterative test is not conclusive for the complex analogue of l_2. The fact that the iterative test is not conclusive for any Hilbert space now follows from the above and (1) the fact that any separable infinite dimensional real (respectively, complex)

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Hilbert space is isometrically isomorphic to l_2 (respectively, the complex analogue of l_2) and (2) the existence of a projection of norm one from any infinite dimensional Hilbert space onto a separable infinite dimensional subspace.

Next we give an example of a locally compact and complete metric space for which the iterative test is not conclusive.

EXAMPLE 2. Let f denote the contractive mapping of Example 1 and let $x \in l_2 - \{(0, 0, \ldots, 0, \ldots)\}$. Let $X=\{(0, 0, \ldots, 0, \ldots)\} \cup \{f^n(x): n=1, 2, \ldots\}$ and let $g: X \to X$ be f restricted to X. Clearly X is both locally compact and complete and g shows that the iterative test for X is not conclusive.

3. **Locally compact, connected spaces.** In this section we prove that the iterative test is conclusive for the class of locally compact, connected spaces. We first prove two lemmas.

LEMMA 1. *If (X, d) is a metric space and $f: X \to X$ is a contractive mapping, then $C=\{x \in X$: the sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to a point of $X\}$ is a closed subset of X.*

Proof. Assume C is nonempty and let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in C such that $\{x_i\}_{i=1}^{\infty}$ converges to a point $x_0 \in X$. We show $x_0 \in C$. Obviously, each sequence $\{f^n(x_i)\}_{n=1}^{\infty}$ converges to a fixed point of f. But, since f is contractive, f has only one fixed point; call it p. Let $\varepsilon>0$. Choose and fix an integer j such that $d(x_j, x_0)<\varepsilon/2$. Since $\{f^n(x_j)\}_{n=1}^{\infty}$ converges to p, there exists N such that $d(f^n(x_j), p)<\varepsilon/2$ for all $n \geq N$. Therefore,

$$\begin{aligned} d(f^n(x_0), p) &\leq d(f^n(x_0), f^n(x_j)) + d(f^n(x_j), p) \\ &\leq d(x_0, x_j)+d(f^n(x_j), p) < \varepsilon \quad \text{for all } n \geq N. \end{aligned}$$

This proves that the sequence $\{f^n(x_0)\}_{n=1}^{\infty}$ converges to p and, hence, $x_0 \in C$. It now follows that C is a closed subset of X.

LEMMA 2. *Let (X, d) be a metric space and let $f: X \to X$ be a contractive mapping with fixed point p. If X is locally compact at p, then the set C of Lemma 1 is an open subset of X.*

Proof. First note that $C=\{x \in X$: the sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to $p\}$. Let $x_0 \in C$. Choose $r>0$ such that $K=\{x \in X: d(p, x) \leq r\}$ is compact. Now let $y \in X$ such that $d(x_0, y)<r/2$. There is an integer N such that $n \geq N$ implies $d(f^n(x_0), p)<r/2$. Hence, $d(f^n(y), p) \leq d(f^n(y), f^n(x_0))+d(f^n(x_0), p) \leq d(y, x_0)+ d(f^n(x_0), p)<r$ for all $n \geq N$. This proves that, for $n \geq N$, $f^n(y) \in K$. Therefore, since K is compact, the sequence of iterates of f at y has a subsequence which converges to a point of K. By 3.2 of [1] it now follows that the entire sequence $\{f^n(y)\}_{n=1}^{\infty}$ converges to p. This completes the proof of the lemma.

We now give the main result of this section.

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THEOREM 1. *If (X, d) is a locally compact, connected metric space, then the iterative test is conclusive for (X, d).*

Proof. Let $f: X\to X$ be a contractive mapping with fixed point p and define C as in Lemma 1. By Lemma 1 C is closed, by Lemma 2 C is open, and, since $p \in C$, C is nonempty. From the connectedness of X we may now conclude that $C=X$. This proves the theorem.

REMARK. Example 1 shows that the iterative test is not conclusive for the class of connected spaces and Example 2 shows that it is not conclusive for the class of locally compact (complete) spaces. However, there are metric spaces which are neither locally compact nor connected and for which the iterative test is conclusive. We discuss this further in the next section.

4. **Subsets of finite dimensional Banach spaces.** In this section we give another class of spaces for which the iterative test for contractive mappings is conclusive. We first need some preliminary results and definitions.

LEMMA 3. *Let X be a Banach space, let Y be a subset of X, and let $f: Y\to X$ be a Lipschitz mapping with Lipschitz constant λ. If p and q are points in Y such that $\|f(p)-f(q)\|=\lambda \cdot \|p-q\|$, then $\|f(y_1)-f(y_2)\|=\lambda \cdot \|y_1-y_2\|$ for all points y_1 and y_2 in Y and on the line segment joining p and q.*

Proof. Let p and q be points in Y such that $\|f(p)-f(q)\|=\lambda \cdot \|p-q\|$ and let $y \in Y$ such that y is on the line segment joining p and q. Since

$$\begin{aligned}\lambda \cdot \|p-q\| &= \|f(p)-f(q)\| \leq \|f(p)-f(y)\| + \|f(y)-f(q)\| \\ &\leq \lambda \cdot \|p-y\| + \lambda \cdot \|y-q\| = \lambda \cdot (\|p-y\| + \|y-q\|) \\ &= \lambda \cdot \|p-q\|,\end{aligned}$$

we obtain that $\|f(p)-f(y)\|=\lambda \cdot \|p-y\|$ and $\|f(y)-f(q)\|=\lambda \cdot \|y-q\|$. The result now follows.

We now make a few comments about the well known natural extension of a Lipschitz mapping. Let (X_1, d_1) be a metric space, let (X_2, d_2) be a complete metric space, let Z be a subset of X_1, and let $f: Z\to X_2$ be a Lipschitz mapping with Lipschitz constant λ. Then f has a unique continuous extension $\bar{f}: \bar{Z}\to X_2$ to the closure $\bar{Z}$ of Z into X_2. The mapping $\bar{f}$ is defined by $\bar{f}(z)=\lim_{i\to\infty} f(z_i)$ for any $z \in \bar{Z}$, where $\{z_i\}_{i=1}^{\infty}$ is any sequence of points in Z converging to z. It is well known that $\bar{f}$ has Lipschitz constant λ. However, as the example in the introduction shows (also see Example 3 below), even if $(**) d_2(f(z_1), f(z_2))<\lambda \cdot d_1(z_1, z_2)$ for all distinct z_1 and z_2 in Z, it does not follow that the same strict inequality holds for $\bar{f}$ and points of $\bar{Z}$. The next theorem places conditions on Z in order that $(**)$ be preserved by $\bar{f}$. This will be the principal tool for the main result (Theorem 3) of this section. We first give a definition.

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Let X be a Banach space and let Y be a subset of X. A subset Z of Y is said to be *line segment dense in* Y if and only if any line segment (in X) joining two distinct points p and q of Y contains a point of Z different from both p and q.

THEOREM 2. *Let X be a Banach space, let Y be a subset of X, and let $Z \subset Y$ be line segment dense in Y. If $f:Z \to X$ satisfies $\|f(z_1)-f(z_2)\| < \lambda \cdot \|z_1-z_2\|$ for all distinct points z_1 and z_2 in Z, then the unique continuous extension $\bar{f}:\bar{Z} \cap Y \to X$ of f to $\bar{Z} \cap Y$ satisfies $\|\bar{f}(p)-\bar{f}(q)\| < \lambda \cdot \|p-q\|$ for all distinct points p and q in $\bar{Z} \cap Y$.*

Proof. The proof is immediate from Lemma 3 and the fact that Z is line segment dense in Y.

REMARK. Lemma 3 and Theorem 2 could have been stated for mappings between subsets of different Banach spaces. Obviously the proofs would have been the same.

We are now in a position to state and prove the main result of this section.

THEOREM 3. *If A is a line segment dense subset of a finite dimensional Banach space, then the iterative test is conclusive for A (with the distance induced by the norm on the Banach space).*

Proof. Let $f: A \to A$ be a contractive mapping with fixed point p. Since A is line segment dense, we have from Theorem 2 (letting $\lambda = 1$) that $\bar{f}$, the unique continuous extension of f to the entire Banach space, is contractive. Since the Banach space is finite dimensional (thus locally compact), we may apply Theorem 1 to conclude that $\{\bar{f}^n(x)\}_{n=1}^{\infty}$ converges (to p) for each x in the Banach space. The result now follows from the fact that $\bar{f}$ is an extension of f.

Theorem 3 gives a large number of spaces which are neither locally compact nor connected but for which the iterative test is conclusive. For example, we have the following:

COROLLARY. *The iterative test is conclusive for any dense subset of the reals (with any norm).*

The Corollary above is valid because the natural extension of a contractive mapping defined on a dense subset of the reals is contractive. In the plane, with the usual norm, this is in general false and in fact the iterative test is not conclusive for some dense subsets of the plane. The next example illustrates this and, in addition, has the virtue that the set X is connected.

EXAMPLE 3. Let R^2 denote the plane with the usual Euclidean norm $\| \ \|$ and let $X = \{(x, y) \in R^2 : y \neq 0\} \cup \{(0, 0)\}$. We first decompose X into "rays" $r(a, b)$ as follows. Let $(a, b) \in X$. If $a \neq 0$, then let $r(a, b) = \{(x, y) \in X : y = (2b/a) \cdot x - b$ and sign $(y) =$ sign $(b)\}$. If $a = 0$, then let $r(a, b)$ be the y-axis. Now let $(x, y) \in X$. Then (x, y) is on one and only one "ray" $r(a_0, b_0)$ of the type just described. Let $f(x, y)$

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be the point on $r(a_0, b_0)$ with second coordinate equal to $y/2$. This defines a function $f:X\to X$. It is easy to see that f is contractive, only the points $z\in X$ of the form $z=(0, y)$ have the property that $\{f^n(z)\}_{n=1}^{\infty}$ converges (to a point of X), and the natural extension $\bar{f}:R^2\to R^2$ of f is not contractive. Thus, the example has all the properties promised above.

5. **Unsolved problems.** We list three questions related to the material in previous sections.

(1) For the class of Banach spaces, is finite dimensionality equivalent to conclusiveness of the iterative test? Theorem 1 shows that finite dimensionality implies conclusiveness. The technique employed in Example 1 can be used to show that the iterative test is not conclusive for many other infinite dimensional Banach spaces, notably the l_p spaces ($p\geq 1$) and the space c_0 of all sequences of real numbers which converge to zero. However, the author does not know, for example, if the iterative test is conclusive for the space m of all bounded sequences of real numbers or even for the separable space $c\subset m$ of all convergent sequences.

(2) For what spaces is conclusiveness of the iterative test for contractive mappings a topological invariant? If, under every remetrization of X which preserves the topology, the iterative test is conclusive, then what can be said about the topology of X? For example, must X, under these circumstances, be locally compact?

(3) Is the converse of Theorem 3 true for dense subsets of finite dimensional Banach spaces? In other words, if the iterative test is conclusive for a dense subset, A of a finite dimensional Banach space X, then is A line segment dense in X? In particular, is the iterative test conclusive for the planar set

$$Q^2 = \{(x, y)\in R^2 : x \text{ and } y \text{ are both rational}\}?$$

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