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ON THE CLASS OF FUNCTIONS CONVEX IN THE NEGATIVE DIRECTION OF THE IMAGINARY AXIS

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Abstract

In this paper we present a new proof of the equivalence of the analytic and the geometric characterization of the class of functions convex in the negative or positive direction of the imaginary axis.

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1. Introduction

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open disk in the plane, and let $\mathbb{T} = \partial\mathbb{D}$. For each $k > 0$, let

$$\mathbb{O}_k = \left\{ z \in \mathbb{D} : \frac{|1 - z|^2}{1 - |z|^2} < k \right\}$$

denote the disk in $\mathbb{D}$ called an *oricycle*, such that the boundary circle $\partial\mathbb{O}_k$ is tangent at $z = 1$ to the unit circle $\mathbb{T}$. The Julia Lemma ([4]; see also [1, page 56]) which is recalled below is the basis for our considerations.

LEMMA 1.1 (Julia). *Let ω be an analytic function in $\mathbb{D}$ with $|\omega(z)| < 1$, for $z \in \mathbb{D}$. Assume that there exists a sequence (z_n), $n \in \mathbb{N}$, of points in $\mathbb{D}$ such that* $\lim_{n\to\infty} z_n = 1$, $\lim_{n\to\infty} \omega(z_n) = 1$ *and*

$$\lim_{n\to\infty} \frac{1 - |\omega(z_n)|}{1 - |z_n|} = \alpha < \infty. \tag{1.1}$$

Then

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \leq \alpha \frac{|1-z|^2}{1-|z|^2}, \quad z \in \mathbb{D},$$

and hence, for every $k > 0$, $\omega(\mathbb{O}_k) \subset \mathbb{O}_{\alpha k}$.

REMARK 1.1. Since

$$\frac{1-|\omega(z)|}{1-|z|} \geq \frac{1-|\omega(0)|}{1+|\omega(0)|}, \quad z \in \mathbb{D},$$

for every function ω analytic in $\mathbb{D}$ with $|\omega(z)| < 1$ for $z \in \mathbb{D}$, the constant α defined in (1.1) is positive (see [1, page 43]).

2. Convexity in the negative direction of the imaginary axis

2.1. For $w \in \mathbb{C}$ and $\theta \in [0, 2\pi)$, let $l[w, \theta] = \{w + te^{i\theta} : t \in [0, +\infty)\}$. For $A, B \subset \mathbb{C}$ and $w \in \mathbb{C}$, let

$$A \pm B = \{u \pm v \in \mathbb{C} : u \in A \wedge v \in B\}, \quad A + w = A + \{w\}.$$

2.2. Let us start with the following definition.

DEFINITION 2.1. A domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, is called *convex in the negative direction of the imaginary axis* if and only if the half-line $l[w, 3\pi/2]$ is contained in $\mathbb{C} \setminus \Omega$ for every $w \in \mathbb{C} \setminus \Omega$. The set of all such domains will be denoted by $\mathscr{X}^-$.

Obviously, $\Omega \in \mathscr{X}^-$ if and only if the half-line $l[w, \pi/2]$ is contained in Ω for every $w \in \Omega$.

DEFINITION 2.2. Let $\mathscr{CV}^-$ denote the class of all analytic and univalent functions f in $\mathbb{D}$ such that $f(\mathbb{D})$ is in $\mathscr{X}^-$. Functions in the class $\mathscr{CV}^-$ will be called *convex in the negative direction of the imaginary axis.*

2.3. Now we introduce, for an arbitrary domain in $\mathscr{X}^-$, a special selected null-chain (C_n).

Construction of a prime end for the domain convex in the negative direction of the imaginary axis

Let us recall that a *crosscut* C of a domain $G \subset \overline{\mathbb{C}}$ is an open Jordan arc in G such that $\overline{C} = C \cup \{a, b\}$, where $a, b \in \partial G$. Let $\Omega \in \mathscr{X}^-$ be an arbitrary domain.

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1. Assume first that Ω is neither a vertical strip nor a half-plane with the boundary straight line parallel to the imaginary axis. Then there exists $w_0 \in \partial\Omega$ such that the vertical half-line $l[w_0, \pi/2] \setminus \{w_0\}$ starting from w_0 is contained in Ω. For each $t \in (0, \infty)$, let us denote $C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $\Omega \cap C(t) \neq \emptyset$ for every $t \in (0, \infty)$. By Proposition 2.13 in [6, page 28], for each $t \in (0, \infty)$ there are countably many crosscuts $C_k(t) \subset C(t)$, $k \in \mathbb{N}$, of Ω each of which is an arc of the circle $C(t)$. By $\Omega_0(t) \subset \Omega$ we denote the component of $\Omega \setminus C(t)$ containing the half-line $l[w_0 + it, \pi/2] \setminus \{w_0 + it\}$ and by $Q(t) \in \bigcup_{k\in\mathbb{N}} C_k(t)$ we denote the crosscut containing the point $w_0 + it$. So $Q(t) \subset \partial\Omega_0(t)$. Let now (t_n), $n \in \mathbb{N}$, be a strictly increasing sequence of points in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and let $(Q(t_n))$ be the corresponding sequence of crosscuts of Ω. It is easy to observe that

(1) $\overline{Q(t_n)} \cap \overline{Q(t_{n+1})} = \emptyset$ for every $n \in \mathbb{N}$.

(2) $\Omega_0(t_{n+1}) \subset \Omega_0(t_n)$ for every $n \in \mathbb{N}$.

(3) $\operatorname{diam}^\# Q(t_n) \to 0$ as $n \to \infty$, where $\operatorname{diam}^\# B$ means the spherical diameter of the set $B \subset \mathbb{C}$.

Therefore $(C_n) = (Q(t_n))$ forms a null chain of Ω (see [6, page 29]). Notice also that the null chain (C_n) is independent of the choice of the sequence (t_n).

The equivalence class of the null chain (C_n) defines the prime end denoted by $p_\infty(\Omega)$. We can also show that infinity is a unique principal point of the prime end $p_\infty(\Omega)$.

2. (a) Let Ω be a vertical strip of width $d > 0$. Let $w_0 \in \partial\Omega$ be an arbitrary point. For each $t \in (d, \infty)$, set $C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $\Omega \cap C(t) \neq \emptyset$ for every $t \in (d, \infty)$. Observe that $\Omega(t)$ is a sum of two disjoint circular arcs, denoted by $Q^+(t)$ and $Q^-(t)$. Let $Q^+(t)$ be the circular arc which lies above $Q^-(t)$. Precisely, $Q^+(t)$ cuts the boundary straight lines of Ω at two points $w_1(t)$ and $w_2(t)$, and together with two half-lines $l[w_1(t), \pi/2]$ and $l[w_2(t), \pi/2]$ is a boundary of a domain denoted by $\Omega^+(t)$. Moreover, $\Omega^+(t) \subset \Omega$ and $\Omega^+(t) \cap \operatorname{Int} C(t) = \emptyset$.

Let now (t_n), $n \in \mathbb{N}$, be a strictly increasing sequence of points in (d, ∞) such that $\lim_{n\to\infty} t_n = \infty$, and let $(Q^+(t_n))$ be the corresponding sequence of crosscuts of Ω. It is easy to observe that the conditions (1)–(3) listed in Part 1 of this construction are fulfilled. Therefore $(C_n^+) = (Q^+(t_n))$ forms a null chain of Ω. The null chain (C_n^+) is independent of the choice of the sequence (t_n).

The equivalence class of the null chain (C_n^+) defines the prime end denoted by $p_\infty^+(\Omega)$. We can also say that infinity is a unique principal point of the prime end $p_\infty^+(\Omega)$

In a similar way the sequence $(Q^-(t_n))$ is a null chain which represents the second prime end $p_\infty^-(\Omega)$, different than $p_\infty^+(\Omega)$.

For the next considerations, the prime end $p_\infty^+(\Omega)$ will be denoted by $p_\infty(\Omega)$.

(b) Let now Ω be a half-plane with the boundary straight line parallel to the imaginary axis. Let $w_0 \in \partial\Omega$ be an arbitrary point. For each $t \in (0, \infty)$, let

$C(t) = \{w \in \mathbb{C} : |w - w_0| = t\}$. It is clear that $Q(t) = \Omega \cap C(t)$ is a half-circle for every $t > 0$. Repeating considerations similar to those above we see that the sequence $(C_n) = (Q(t_n))$, for an arbitrary strictly increasing sequence (t_n), $n \in \mathbb{N}$, of points in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$, forms a null chain of Ω which represents a prime end denoted by $p_\infty(\Omega)$.

In this way we construct for every domain Ω in $\mathscr{X}^-$, in a unique way, a prime end $p_\infty(\Omega)$.

Let f be a conformal mapping $\mathbb{D}$ onto Ω, this is, let $f \in \mathscr{C}\mathscr{V}^-$. By the prime end theorem there exists a bijective mapping $\widehat{f}$ of the unit circle $\mathbb{T}$ onto the set of all prime ends of Ω ([6, page 30]). Hence there is a unique $\zeta_\infty \in \mathbb{T}$ such that $p_\infty(\Omega) = \widehat{f}(\zeta_\infty)$. We can also show that infinity is a unique principal point of the prime end $p_\infty(\Omega)$.

3. An analytic characterization of the class of function convex in the negative direction of the imaginary axis

3.1. In the proof of the main theorem, which analytically characterizes the class $\mathscr{C}\mathscr{V}^-$, we will need the following lemma.

LEMMA 3.1. *Let* (a_n), $n \in \mathbb{N}$, *be a sequence such that* $a_n > 0$, $n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n) = 0. \tag{3.1}$$

Then there exists a convergent subsequence (a_{n_k}), $k \in \mathbb{N}$, *of the sequence* (a_n). *Moreover* $0 \le \lim_{k\to\infty} a_{n_k} = a \le 1$.

PROOF. Suppose that only finitely many elements of the sequence (a_n) lie in the interval $(0, 1]$. Then $a_n > 1$ for sufficiently large n, which contradicts (3.1). This means that infinitely many elements of the sequence (a_n) lie in the interval $(0, 1]$. Taking a convergent subsequence (a_{n_k}), $k \in \mathbb{N}$, of the sequence (a_n) completes the proof. □

3.2. Now we will prove the theorem which says that every function f in the class $\mathscr{C}\mathscr{V}^-$, with $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$, preserves convexity in the negative direction of the imaginary axis on every oricycle $\mathbb{O}_k$.

THEOREM 3.1. *Let f be an analytic and univalent function in* $\mathbb{D}$. *Then* $f \in \mathscr{C}\mathscr{V}^-$ *and* $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$, *if and only if* $f(\mathbb{O}_k) \in \mathscr{X}^-$ *for every* $k > 0$.

PROOF. 1. Assume that $f \in \mathscr{C}\mathscr{V}^-$ and $\zeta_\infty = 1$ corresponds to the prime end $p_\infty(f(\mathbb{D}))$. For each $t \in (0, \infty)$, let us define the function

$$\omega_t(z) = f^{-1}(f(z) + it), \quad z \in \mathbb{D}.$$

Since $f(\mathbb{D})$ is a domain convex in the negative direction of the imaginary axis, $f(z) + it \in f(\mathbb{D})$ for every $t \in (0, \infty)$ and $z \in \mathbb{D}$. Hence, from the univalence of f, it follows that the function ω_t is well defined for each $t \in (0, \infty)$.

For every domain $\Omega \in \mathscr{L}^-$, we select two points $w_0 \in \partial\Omega$ and $w_1 \in \Omega$, in the following way. If Ω is not a vertical strip or a half-plane with the boundary straight line parallel to the imaginary axis, then there exists w_0 in $\partial\Omega$ such that the half-line $l[w_0, \pi/2] \setminus \{w_0\}$ lies in Ω. Let $w_1 \in \Omega$ be an arbitrary point lying on this half-line.

In the case when Ω is a vertical strip or a half-plane with the boundary straight line parallel to the imaginary axis, let $w_1 \in \Omega$ be an arbitrary point and $w_0 \in \partial\Omega$ be such that $\operatorname{Im} w_1 = \operatorname{Im} w_0$.

Assume now that for the domain $f(\mathbb{D})$ the points w_0 and w_1 are chosen as above. Of course $l[w_1, \pi/2]$ lies in $f(\mathbb{D})$. Let us fix $t \in (0, \infty)$ and let us consider the sequence $(w_n) = (w_1 + it_n)$ of points in $l[w_1, \pi/2]$ and the corresponding sequence $(z_n) = (f^{-1}(w_n))$ of points in $\mathbb{D}$, where $t_n = (n-1)t$, $n \in \mathbb{N}$.

With the same notation as in the construction of a prime end for the domain in the class $\mathscr{L}^-$, let $C(t_n) = \{w \in \mathbb{C} : |w - w_0| = |w_n - w_0|\}$ and let $Q(t_n) \subset C(t_n)$, for $n \in \mathbb{N}$, denote the crosscut of $f(\mathbb{D})$ containing the point w_n. From the method of choosing w_0 and w_1 we see that the conditions (1)–(3) are satisfied and $(Q(t_n))$ is a null-chain representing the prime end $p_\infty(f(\mathbb{D}))$. By the prime end theorem $(f^{-1}(Q(t_n)))$ is a null-chain in $\mathbb{D}$ that separates the origin from $\zeta_\infty = 1$ for large n. Since $z_n = f^{-1}(w_n) \in f^{-1}(Q(t_n))$ and $\operatorname{diam} f^{-1}(Q(t_n)) \to 0$ for $n \to \infty$, we conclude that $\lim_{n\to\infty} z_n = 1$. Observe that $\omega_t(z_n) = f^{-1}(w_n + it) = z_{n+1}$. Let now

$$a_n = \frac{1 - |\omega_t(z_n)|}{1 - |z_n|}, \quad n \in \mathbb{N}.$$

Hence

$$a_n = \frac{1 - |\omega_t(z_n)|}{1 - |z_n|} = \frac{1 - |z_{n+1}|}{1 - |z_n|}$$

for all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} (a_1 a_2 \cdots a_n) &= \lim_{n\to\infty} \left(\frac{1 - |z_2|}{1 - |z_1|} \frac{1 - |z_3|}{1 - |z_2|} \cdots \frac{1 - |z_n|}{1 - |z_{n-1}|} \frac{1 - |z_{n+1}|}{1 - |z_n|} \right) \\ &= \lim_{n\to\infty} \frac{1 - |z_{n+1}|}{1 - |z_1|} = 0.\end{aligned}$$

By Lemma 3.1 there exists a convergent subsequence (a_{n_k}), $k \in \mathbb{N}$, of the sequence (a_n) such that $0 \le \lim_{k\to\infty} a_{n_k} = \alpha(t) \le 1$. Hence we conclude that there exists a convergent subsequence (z_{n_k}) of the sequence (z_n) such that

$$\lim_{k\to\infty} \frac{1 - |\omega_t(z_{n_k})|}{1 - |z_{n_k}|} = \alpha(t) \le 1$$

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for every fixed $t \in (0, \infty)$. In fact, in view of Remark 1.1, $\alpha(t) > 0$ for every $t \in (0, \infty)$.

In this way, for each $t \in (0, \infty)$, the function ω_t satisfies the assumptions of the Julia Lemma. Hence, and by the fact that $\alpha(t) \leq 1$ for every $t \in (0, \infty)$, we have

$$\omega_t(\mathbb{O}_k) \subset \mathbb{O}_{\alpha(t)k} \subset \mathbb{O}_k \tag{3.2}$$

for every $k > 0$.

Fixing now $k > 0$ we see from (3.2) that $f^{-1}(f(\mathbb{O}_k) + it) \subset \mathbb{O}_k$, so $f(\mathbb{O}_k) + it \subset f(\mathbb{O}_k)$ for every $t \in (0, \infty)$. Therefore, $f(\mathbb{O}_k) \in \mathscr{X}^-$ for every $k > 0$.

2. Let us now assume that $f(\mathbb{O}_k) \in \mathscr{X}^-$ for every $k > 0$. Since $\infty \in \partial f(\mathbb{O}_k)$ for every $k > 0$ and $f(\mathbb{D}) = \bigcup_{k>0} f(\mathbb{O}_k)$, $\infty \in \partial f(\mathbb{D})$ and $f(\mathbb{D})$ is convex in the negative direction of the imaginary axis. Observe also that there exists a prime end $p_\infty(f(\mathbb{D}))$ which corresponds to some point $\zeta_\infty \in \mathbb{T}$. We need to show that $\zeta_\infty = 1$. To this end, let $k > 0$ be fixed and suppose that $\zeta_\infty \neq 1$. Let $(Q(t_n))$, $n \in \mathbb{N}$, be an arbitrary sequence of crosscuts of $f(\mathbb{D})$ which represents the prime end $p_\infty(f(\mathbb{D}))$ corresponding in a unique way to a point $\zeta_\infty \in \mathbb{T}$, that is, $(Q(t_n))$ is a null-chain of $f(\mathbb{D})$. By the prime end theorem $(f^{-1}(Q(t_n)))$ is a null-chain that separate in $\mathbb{D}$ the points 0 from ζ_∞ for large n. Since $\zeta_\infty \neq 1$ and $\operatorname{diam} f^{-1}(Q(t_n)) \to 0$ for $n \to \infty$ we see that

$$f^{-1}(Q(t_n)) \cap \mathbb{O}_k = \emptyset \tag{3.3}$$

for large n.

On the other hand, $f(\mathbb{O}_k)$ is in $\mathscr{X}^-$, which implies that $Q(t_n) \cap f(\mathbb{O}_k) \neq \emptyset$ for large $n \in \mathbb{N}$. This contradicts (3.3) and shows that $\zeta_\infty = 1$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$. The proof of the theorem is finished. □

Using Theorem 3.1 we are able to find an analytic characterization of functions in the class $\mathscr{C}\mathscr{V}^-$.

THEOREM 3.2. *If $f \in \mathscr{C}\mathscr{V}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$, then*

$$\operatorname{Im}\left\{(1-z)^2 f'(z)\right\} \geq 0, \quad z \in \mathbb{D}. \tag{3.4}$$

PROOF. Let $f \in \mathscr{C}\mathscr{V}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$. By Theorem 3.1 the domain $f(\mathbb{O}_k)$ for every $k > 0$, is in the class $\mathscr{X}^-$. This means geometrically that the function

$$\gamma_k \ni z \to \operatorname{Re} f(z) \tag{3.5}$$

is monotonic on the analytic arc $\gamma_k = \partial\mathbb{O}_k \setminus \{1\}$ for every $k > 0$. We will use the following parametrization of γ_k

$$\gamma_k : z = z(\theta) = \frac{1 + ke^{i\theta}}{1+k}, \quad \theta \in (0, 2\pi). \tag{3.6}$$

Hence in place of (3.5) we consider the function

$$(0, 2\pi) \ni \theta \to \operatorname{Re} f(z(\theta)). \tag{3.7}$$

We have

$$\begin{aligned}(1 - z(\theta))^2 &= \frac{k^2}{(1+k)^2}\left(1 - e^{i\theta}\right)^2 = -\frac{4k\sin^2(\theta/2)}{(k+1)i}\left(\frac{k}{k+1}e^{i\theta} i\right) \\ &= \frac{4k\sin^2(\theta/2)}{k+1} z'(\theta) i = 2\operatorname{Re}\{1 - z(\theta)\} z'(\theta) i, \quad \theta \in (0, 2\pi).\end{aligned}$$

In view of the fact that the arc γ_k is positively oriented, the same is true of the arc $f(\gamma_k)$, since f is a conformal mapping. Hence

$$\begin{aligned}\frac{d}{d\theta}\operatorname{Re} f(z(\theta)) &= \operatorname{Re}\left\{z'(\theta) f'(z(\theta))\right\} \\ &= \frac{k+1}{4k\sin^2(\theta/2)}\operatorname{Re}\left\{-i(1 - z(\theta))^2 f'(z(\theta))\right\} \\ &= \frac{1}{2\operatorname{Re}\{1 - z(\theta)\}}\operatorname{Im}\left\{(1 - z(\theta))^2 f'(z(\theta))\right\} \ge 0\end{aligned} \tag{3.8}$$

for $\theta \in (0, 2\pi)$, so (3.4) holds. □

Now we will prove the converse theorem.

THEOREM 3.3. *If f is an analytic function in $\mathbb{D}$ and*

$$\operatorname{Im}\left\{(1 - z)^2 f'(z)\right\} \ge 0, \quad z \in \mathbb{D}, \tag{3.9}$$

then $f \in \mathscr{CV}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$.

PROOF. Let f be analytic in $\mathbb{D}$ and satisfy (3.9).

1. If there exists a point $z_0 \in \mathbb{D}$ such that the equality in (3.9) holds, then by the maximum principle for harmonic functions the equality in (3.9) holds in the whole disk $\mathbb{D}$. This implies that there exists a real number $a \in \mathbb{R} \setminus \{0\}$ so that $(1 - z)^2 f'(z) \equiv a$, $z \in \mathbb{D}$. This is satisfied only for the function

$$f(z) = f_0(z) = b + \frac{a}{1 - z}, \quad z \in \mathbb{D},$$

where $b \in \mathbb{C}$. In this case we conclude at once that $f_0(\mathbb{D})$ is a half-plane with a straight line as the boundary parallel to the imaginary axis. Hence $f_0 \in \mathscr{CV}^-$ and, as is immediately apparent, $p_\infty(f_0(\mathbb{D})) = \widehat{f_0}(1)$. Observe also that in this case the function defined in (3.7) is constant on every arc γ_k, $k > 0$, that is, every disk $\mathbb{O}_k$

is mapped onto the half-plane with a boundary straight line parallel to the imaginary axis.

2. Assume now that in (3.9) strong inequality holds. Since f satisfying (3.9) is close-to-convex with respect to the convex function

$$h(z) = \frac{-iz}{1-z}, \quad z \in \mathbb{D},$$

f is univalent in $\mathbb{D}$ ([5]).

Let us now consider again the function (3.7) defined on the analytic arcs γ_k : $\partial\mathbb{O}_k \setminus \{1\}$ for each $k > 0$, parametrized by (3.6). Repeating again the calculations (3.8) we see that the condition (3.9) implies

$$\frac{d}{d\theta} \operatorname{Re} f(z(\theta)) > 0, \quad \theta \in (0, 2\pi).$$

This means that every positively oriented arc γ_k which is mapped by the function f satisfying (3.9) onto the positively oriented arc $f(\gamma_k)$, $k > 0$, is the boundary of the domain $f(\mathbb{O}_k)$ convex in the direction of the negative imaginary half-axis. By Theorem 3.1, $f \in \mathscr{CV}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$ which completes the proof of the theorem. □

The following theorems are immediate consequences of Theorem 3.2 and Theorem 3.3 by applying them to the function $f(z) = g(e^{-i\mu}z)$, $z \in \mathbb{D}$, where $g \in \mathscr{CV}^-$ and $p_\infty(g(\mathbb{D})) = \widehat{g}(1)$.

THEOREM 3.4. *If $f \in \mathscr{CV}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(e^{i\mu})$, $\mu \in \mathbb{R}$, then*

$$\operatorname{Im}\left\{e^{i\mu}(1 - e^{-i\mu}z)^2 f'(z)\right\} \geq 0, \quad z \in \mathbb{D}. \tag{3.10}$$

THEOREM 3.5. *If f is an analytic function in $\mathbb{D}$ and* (3.10) *is true for $\mu \in \mathbb{R}$, then $f \in \mathscr{CV}^-$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(e^{i\mu})$.*

4. Convexity in the positive direction of the imaginary axis

The results presented in Section 3 can be applied at once to the functions called *convex in the positive direction of the imaginary axis.*

DEFINITION 4.1. A domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, will be called *convex in the positive direction of the imaginary axis* if and only if the half-line $l[w, \pi/2]$ is contained in $\mathbb{C} \setminus \Omega$ for every $w \in \mathbb{C} \setminus \Omega$ or equivalently if the half-line $l[w, 3\pi/2]$ is contained in Ω for every $w \in \Omega$. The set of all such domains will be denoted by $\mathscr{X}^+$.

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DEFINITION 4.2. Let $\mathscr{CV}^+$ denote the class of all analytic and univalent functions f in $\mathbb{D}$ such that $f(\mathbb{D})$ is in $\mathscr{X}^+$. Functions in the class $\mathscr{CV}^+$ will be called *convex in the positive direction of the imaginary axis.*

We can repeat exactly the construction as in Section 2 and find the prime end $p_\infty(\Omega)$ for every $\Omega \in \mathscr{X}^+$.

Finally, in view of at Theorem 3.1, Theorem 3.4 and Theorem 3.5 we can formulate the following theorems.

THEOREM 4.1. *Let f be an analytic and univalent function in $\mathbb{D}$. Then $f \in \mathscr{CV}^+$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(1)$, if and only if $f(\mathbb{O}_k) \in \mathscr{X}^+$ for every $k > 0$.*

THEOREM 4.2. *If $f \in \mathscr{CV}^+$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(e^{i\mu})$, $\mu \in \mathbb{R}$, then*

$$\operatorname{Im}\left\{e^{i\mu}(1 - e^{-i\mu}z)^2 f'(z)\right\} \leq 0, \quad z \in \mathbb{D}. \tag{4.1}$$

THEOREM 4.3. *If f is an analytic function in $\mathbb{D}$ and* (4.1) *is true for $\mu \in \mathbb{R}$, then $f \in \mathscr{CV}^+$ and $p_\infty(f(\mathbb{D})) = \widehat{f}(e^{i\mu})$.*

5. Remarks

The class of functions convex in the direction of the imaginary axis, denoted by $\mathscr{CV}$, was introduced by Robertson [7]. A function f, analytic and univalent in $\mathbb{D}$, belongs to $\mathscr{CV}$ if and only if the domain $f(\mathbb{D})$ is convex in the direction of the imaginary axis, that is, $[w_1, w_2] \subset f(\mathbb{D})$ for every w_1 and w_2 in $f(\mathbb{D})$ such that $\operatorname{Re} w_1 = \operatorname{Re} w_2$. Robertson proposed an analytic condition to characterize the class $\mathscr{CV}$ and proved it under some additional assumptions on functions in $\mathscr{CV}$ connected with the regularity on the unit circle. In the papers [3] and [8] it was shown that the Robertson condition is correct for the whole class $\mathscr{CV}$.

In fact, the classes $\mathscr{CV}^+$ and $\mathscr{CV}^-$ are the subclasses of the class $\mathscr{CV}$ distinguished by Hengartner and Schober [3], where also the analytic conditions (3.4) and (4.1) with $\mu = 0$ were demonstrated.

We use the name *convex in the negative or positive direction of the imaginary axis* following Ciozda [2], where she studied the so-called class L_0 of functions *convex in the direction of the negative real half-axis*. To be precise, a function f analytic and univalent in $\mathbb{D}$ is convex in the direction of the negative real half-axis if and only if for every $w \in f(\mathbb{D})$ the half-line $l[w, 0]$ is contained in $f(\mathbb{D})$.

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