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A computer aided study of a group defined by fourth powers: Addendum

M.F. Newman

The fact that the group studied in the original paper, M.F. Newman, *Bull. Austral. Math. Soc.* 14 (1976), 293-297, is infinite follows immediately from a result in a 1940 paper of Coxeter. The computer aided methods give more detailed information about the group and some related groups.

4. Priority

The result that the group G presented by

(3) $\langle a, b;\ a^4 = b^4 = (ab)^4 = \left(a^{-1}b\right)^4 = \left(ab^2\right)^4 = \left(a^2b\right)^4 = \left(a^{-1}b^{-1}ab\right)^4 = e\rangle$

is infinite goes back to a paper [8] of Coxeter of 1940. He proved that

(4) $\langle a, b;\ a^4 = b^4 = (ab)^4 = \left(a^{-1}b\right)^4 = \left(a^{-1}b^{-1}ab\right)^2 = e\rangle$

presents an infinite group (see Section 8 of [8]) and that it also satisfies the relations $\left(ab^2\right)^4 = e$ and $\left(a^2b\right)^4 = e$ (see Section 4 of [8]). I am indebted to Professor John Leech for drawing my attention to Coxeter's paper via a reprint, dated June 1967, of his 1963 paper [4].

5. More detailed information

Coxeter's method for proving that a group given by a presentation P is infinite is to construct a family of groups G_n, one for each positive

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integer n, such that the set of orders $|G_n|$ is not bounded and such that for each group he can obtain a presentation which is visibly a presentation of a quotient of the group presented by P. For the presentation (4) the n-th group is an extension of a direct product of four cyclic groups of order n by a quaternion group of order 8 (constructed by two successive cyclic extensions). The same construction starting from four cyclic groups of infinite order gives an extension of a free abelian group of rank 4 by a quaternion group. Coxeter's argument (without having to be concerned about elements of order n) gives that this extension has the presentation (4). Clearly the matrix group H, at the end of Section 3 of this paper [9], also satisfies the relation $\left(a^{-1}b^{-1}ab\right)^2 = e$, so it is a quotient group of the group presented by (4). A straight-forward calculation shows that the elements $d = [a, b, a]$, $a^{-1}da$, $b^{-1}db$, $(ab)^{-1}dab$ generate a free abelian subgroup of rank 4 of H, so H is also an extension of a free abelian group of rank 4 by a quaternion group. It follows that the group presented by (4) is (isomorphic to) H.

The outputs, up to class 14, of the nilpotent quotient algorithm applied to the presentations (3) and (4) suggest that G is an extension of a cyclic group of order 2 by H. The corresponding output for the presentation

$$\langle a, b;\ a^4 = b^4 = (ab)^4 = \left(a^{-1}b\right)^4 = \left(a^2b\right)^4 = \left(ab^2\right)^4 = e \rangle$$

shows that as far as nilpotent quotients are concerned the group K with this presentation is not much larger than G; the maximal class 14 quotient has order 2^{35}. It is reasonable to guess from the outputs that the maximal class $2c$ quotient of G has order 2^{4c+2} (for $c \geq 2$) and that of K has order 2^{5c}, and, moreover, that, if D is the intersection of all normal subgroups of K with nilpotent quotient, then K/D is an extension of a cyclic group of infinite order by H. Also one asks whether D is the identity subgroup of K.

I am indebted to Professor Gilbert Baumslag for encouraging me to do the "obvious", namely apply the Reidemeister-Schreier algorithm to obtain a presentation for the kernel N of the mapping of K onto the quaternion

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group. After some manipulation, guided by the outputs of the nilpotent quotient algorithm, it can be seen that N has a presentation

$$\langle f, g, p, q;\ [f, p] = [f, q] = [g, p] = [g, q] = e,$$
$$[f, g, f] = [f, g, g] = e,\ [f, g]^2 = [p, q]\rangle .$$

Hence N is the central product of the subgroups generated by $\{f, g\}$ and $\{p, q\}$ amalgamating $[f, g]^2$ with $[p, q]$. It follows from the presentation that these two subgroups are free nilpotent of class 2 and therefore that N is torsion-free nilpotent. Thus N , and consequently K , is residually a finite 2-group. So D is the identity subgroup. The commutator subgroup N' is cyclic of infinite order generated by $[b, a]^2$. Therefore K is an extension of a cyclic group of infinite order by H and G is a central extension of a cyclic group of order 2 by H . Put differently G is an extension of the direct product of $\langle f, g;\ [f, g, f] = [f, g, g] = [f, g]^2 = e\rangle$ with two cyclic groups of infinite order by a quaternion group. The subgroup N' of K is central, so K is also a central extension by H .

Additional references

[8] H.S.M. Coxeter, "A method for proving certain abstract groups to be infinite", *Bull. Amer. Math. Soc.* **46** (1940), 246-251.

[9] M.F. Newman, "A computer aided study of a group defined by fourth powers", *Bull. Austral. Math. Soc.* **14** (1976), 293-297.

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