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A BOUND ON THE NUMBER OF INVARIANT MEASURES

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For τ a piecewise C^2 transformation, we present a method for obtaining an upper bound for the number of independent absolutely continuous measures invariant under τ.

Let $I=[0,1]$ and let $\tau: I \to I$ be a piecewise C^2 transformation with $\inf_{I_1} |d\tau/dx|>1$, where $I_1=I-P$ and P denotes the points of discontinuity of τ and τ'. A measure μ is invariant (under τ) if for all measurable sets $S \subset I$, $\mu(S)=\mu(\tau^{-1}(S))$, where $\tau^{-1}(S)=\{x \in I: \tau(x) \in S\}$. If there exists an integrable function $f(x)$, $f(x) \geq 0$, such that $\theta(S)=\int_S f(x)\,dx$ for all measurable S, μ is said to be absolutely continuous (with respect to Lebesque measure). The function f is referred to as an invariant function.

In [1] it is shown that τ admits an absolutely continuous invariant measure. Let $\{x_1, x_2, \ldots, x_n\} \subset (0, 1)$ denote the points where τ' does not exist. Let F denote the space of invariant functions. In [2] it is shown that the dimension of F, N_τ, is less than or equal to n.

In this note we will consider the partition $0=b_0<b_1<\cdots<b_m<b_{m+1}=1$, where τ is continuous and monotonic on each interval (b_{i-1}, b_i). Clearly $m \leq n$.

THEOREM. $N\tau \leq m$.

Proof. Let $\{f_1, f_2, \ldots, f_n\}$ and $L_1, L_2, \ldots, L_n$ be as in Theorem 1 of [2]. We claim that for each $i=1,2,\ldots,n$, M_i contains some b_j, $j \in \{1,2,\ldots,m\}$ in its interior. Suppose this is not true for some i, and let $[a, b]$ be the largest interval in $L_1=$ support of f_i. Then τ is monotonic and continuous on $[a, b]$. Since $\inf|\tau'|>1$, $\tau[a, b]$ is an interval with length strictly greater than $[a, b]$. But L_i is invariant under τ, i.e., $\tau(L_i)=L_i$ a.e. Thus $\tau(a, b) \subset \tau(L_i)=L_i$ a.e., and L_i contains an interval a.e. larger than $[a, b]$. This is a contradiction. Therefore each L_i has to contain at least one b_j, $1 \leq j \leq m$, in its interior. Since the L_i's have disjoint interior [2, Theorem 1], $N_\tau \leq m$. Q.E.D.

REMARKS. (1) We can reword the theorem as follows: the number of independent absolutely continuous measures invariant under τ is at most one less than the number of incontinuous monotonic pieces in the graph of τ.

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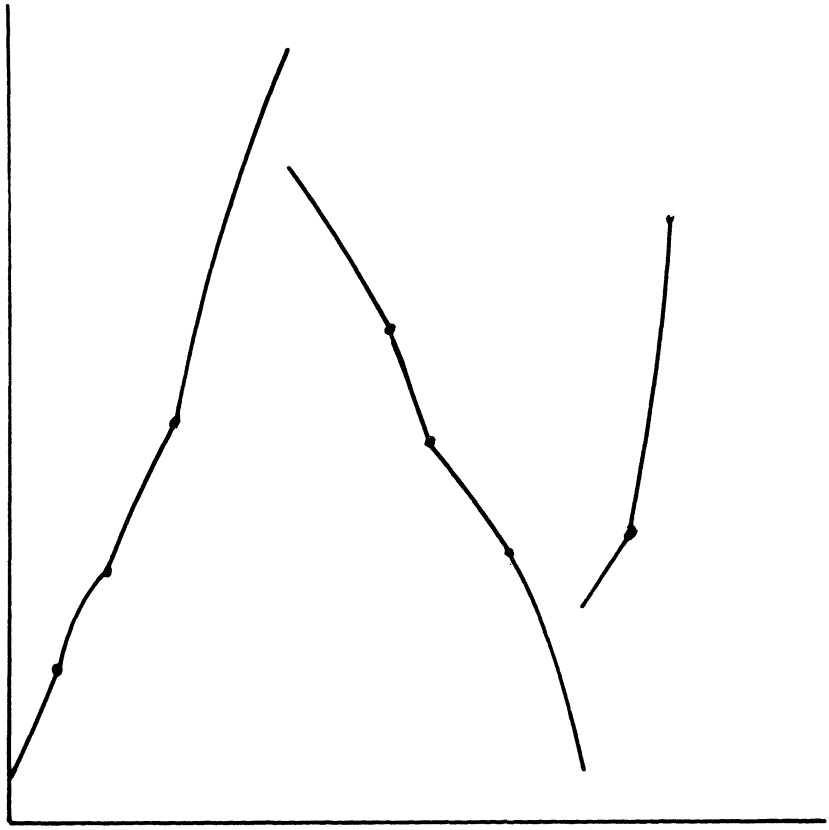

(2) In the special case where τ is continuous on I, the total number of peaks and valleys (relative maxima and minima in $(0,))$ in the graph of τ constitutes an upper bound for the dimension of F.

Example. The transformation shown below has 10 pieces but at most only 2 independent absolutely continuous invariant measures.

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