SUBMETHODS OF REGULAR MATRIX SUMMABILITY METHODS

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1. Introduction By a submethod of a regular matrix method A we mean a method (see 1 or 3) whose matrix is obtained by deleting a set of rows from the matrix A. We establish a one-one correspondence between the submethods of A and the points of the interval $0 < \xi \leqslant 1$. We designate the submethod which corresponds to ξ by $A(\xi)$ and are accordingly able to speak of sets of submethods of measure 0, of the first category, etc. Now, every bounded sequence $\{s_n\}$ is summed by certain submethods of A. We find that if $\{s_n\}$ is not summed by A itself, then the set of submethods of A by means of which it is summed is of the first category, but may be either of measure 0 or 1. A submethod of A may be either equivalent to A or strictly stronger than A. We find that the set of submethods equivalent to A is always of the first category. On the other hand, every regular method A has equivalent methods B and C such that the set of submethods of B which are equivalent to B is of measure 0 and the set of submethods of C which are equivalent to C is of measure 1. However, certain important methods are equivalent to almost all of their submethods, but we prove this only for the $(C, 1)$ method. We consider only bounded sequences, so that equivalence, etc., are relative to the set of bounded sequences. There is some analogy between this work and work on the Borel property (4; 5).

2. Category. Let $A = (a_{mn})$ be an infinite matrix. We establish a one-one correspondence between the submethods of A and the points of the interval $0 < \xi \leqslant 1$ by associating with each point ξ in this interval the submatrix of A whose nth row is deleted if and only if $a_n = 0$ in the non-terminating binary expansion $.a_1a_2 \ldots a_n \ldots$ of ξ. We designate the submatrix corresponding to ξ as $A(\xi)$ and use the same notation for the corresponding summability method. We say that a set of submethods $A(\xi)$, $\xi \in E$, has a specific property whenever the set E has this property. We shall refer only to regular methods although it will be clear that our results hold for other methods as well.

THEOREM I. *If A is a regular method and $\{s_n\}$ is a bounded sequence which is not summable by means of A, then the set of submethods of A by means of which $\{s_n\}$ is summable is of the first category.*

Proof. Let ξ_0 be a real number for which $A(\xi_0)$ does not sum $\{s_n\}$, and let D be the set of ξ obtained by changing the binary expansion of ξ_0 in a finite

Received January 14, 1955.

https://doi.org/10.4153/CJM-1956-007-8 Published online by Cambridge University Press

number of places. D is everywhere dense, and there is a $k > 0$ such that, for every $\xi \in D$, the $A(\xi)$ transform $\{t_{n,\xi}\}$ of $\{s_n\}$ satisfies the condition

$$\limsup_{n\to\infty} t_{n,\xi} > \liminf_{n\to\infty} t_{n,\xi} + k.$$

We now consider the set S_n of all ξ such that there are $\mu, \nu > n$ for which $|t_{\nu,\xi}-t_{\mu,\xi}| > k$. For every n, the set S_n is open. For, if $\xi \in S_n$, $\mu > \nu > n$, and $|t_{\nu,\xi}-t_{\mu,\xi}| > k$, and if $|\eta-\xi| < 2^{-\mu-1}$, then $|t_{\nu,\eta}-t_{\mu,\eta}| = |t_{\nu,\xi}-t_{\mu,\xi}| > k$, so that $\eta \in S_n$. But $D \subset S_n$, for every n, so that the set

$$S = \bigcap_{n=1}^{\infty} S_n$$

is an everywhere dense set of type G_δ. Hence, its complement is of the first category. Finally, it is evident that for every $\xi \in S$ the sequence $\{s_n\}$ is not summable by means of $A(\xi)$.

Although the set of those $A(\xi)$ which sum $\{s_n\}$ is of the first category, it is non-denumerable. For, if $\{s_n\}$ is summable by means of $A(\xi)$ then it is summable by means of every submethod of $A(\xi)$.

We now show that the set of submethods of a regular method A which are equivalent to A is of the first category.

LEMMA 1. *Let A be a regular method for which*

$$\lim_{m\to\infty} \max_n |a_{mn}| = 0.$$

There is a strictly increasing $F(n)$ such that if $\{s_n\}$ is A summable and $s_n = 1$, $n = n_\nu$ $(\nu = 1, 2, \ldots)$ and $s_n = 0$ for all other n, where $n_{\nu+1}-n_\nu > F(n_\nu)$ for an infinite number of values of ν, then

$$A - \lim_{n\to\infty} s_n = 0.$$

Proof. For each n, there is an $r > n$ and an $F(n)$ such that

$$\sum_{\mu=1}^{n} |a_{r\mu}| < \frac{1}{2n}, \qquad \sum_{\mu=n+F(n)}^{\infty} |a_{r\mu}| < \frac{1}{2n}. \tag{1}$$

Let $\{s_n\}$ satisfy the conditions of the Lemma, and let ν be such that $n_{\nu+1}-n_\nu > F(n_\nu)$. Let $r > n_\nu$ satisfy (1) for n_ν. (We write $\bar{n}$ for n_ν.)

$$\begin{aligned} t_r &= \sum_{\mu=1}^{\infty} a_{r\mu}s_\mu \\ &= \sum_{\mu=1}^{\bar{n}} a_{r\mu}s_\mu + \sum_{\mu=\bar{n}+1}^{\bar{n}+F(\bar{n})-1} a_{r\mu}s_\mu + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} a_{r\mu}s_\mu \\ &= \sum_{\mu=1}^{\bar{n}} a_{r\mu}s_\mu + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} a_{r\mu}s_\mu, \end{aligned}$$

and

$$|t_r| \leqslant \sum_{\mu=1}^{\bar{n}} |a_{r\mu}| + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} |a_{r\mu}| < \frac{1}{\bar{n}}.$$

Hence, if the A transform $\{t_n\}$ of $\{s_n\}$ converges, its limit must be 0.

For a detailed discussion of counting functions used in a different way see **(6)**.

LEMMA 2. *With the same restrictions on A as in Lemma 1, there is a $G(n)$ such that if the binary expansion of ξ is 1 for $n = n_\nu$ ($\nu = 1, 2, \ldots$) and is 0 everywhere else, and if $n_{\nu+1} - n_\nu > G(n_\nu)$ for an infinite number of values of ν, then $A(\xi)$ is strictly stronger than A.*

Proof. We need only choose the sequence $\{G(n)\}$ so that whenever $r - n > G(n)$ there is a $\nu = \nu(n)$ such that

$$\sum_{m=\nu}^{\infty} |a_{nm}| < \frac{1}{n} \text{ and } \sum_{m=1}^{\nu+F(n)} |a_{rm}| < \frac{1}{n}.$$

for every n, where $\nu(n)$ is strictly increasing. Suppose ξ satisfies the condition of the Lemma. Let

$$n_1 < m_1 \leqslant n_2 < m_2 \leqslant \ldots \leqslant n_k < m_k \leqslant \ldots$$

be places for which the binary expansion of ξ is 1, such that for every k, $m_k - n_k > G(n_k)$, and the binary expansion of ξ is 0 at all places between n_k and m_k. Supposing that s_n has been defined for all $n < \nu(n_k)$, we let $s_n = 0$ for

$$\nu(n_k) \leqslant n \leqslant \nu(n_k) + F(n_k)$$

and $s_n = 1$ for

$$\nu(n_k) + F(n_k) < n < \nu(n_{k+1}).$$

The construction of the sequence $\{s_n\}$ is then completed by induction. It is evidently summable to 1 by the $A(\xi)$ method. If $\{s_n\}$ were summable by the A method then, by Lemma 1, its limit would be 0. Hence $\{s_n\}$ is not A summable.

LEMMA 3. *If A is a regular row finite method for which*

$$\limsup_{m\to\infty} \max_{n} |a_{mn}| > 0$$

there is also a $G(n)$ for which the conclusion in Lemma 2 holds.

Proof. By hypothesis, there is a sequence m_ν ($\nu = 1, 2, \ldots$) and a $k > 0$ such that, for every ν, there is a k_ν for which

$$|a_{m_\nu k_\nu}| > k.$$

Evidently,

$$\lim_{\nu\to\infty} k_\nu = \infty.$$

We define $G(n)$ so that whenever $n_{\nu+1} - n_\nu > G(n_\nu)$, it follows that there is an m_μ with

$$n_\nu < m_\mu < n_{\mu+1}, \ |a_{n_{\nu+1}k_\mu}| < \frac{1}{n_\nu}, \ k_\mu > k(n_\nu),$$

https://doi.org/10.4153/CJM-1956-007-8 Published online by Cambridge University Press

where $k(m)$ is defined so that $|a_{m,k(m)}|$ is the last non-zero element of the mth row. Now, if ξ satisfies the conditions of the Lemma, we may define $\{s_n\}$ to be 1 at a certain infinite set of values of k_ν and 0 everywhere else such that $\{s_n\}$ is $A(\xi)$ summable to 0. But this $\{s_n\}$ is not A summable.

LEMMA 4. *For every regular A, there is a regular row finite B such that $A(\xi)$ is equivalent to $B(\xi)$ for all $0 < \xi \leqslant 1$.*

Proof. For every n, there is an m_n such that

$$\sum_{m=m_n+1}^{\infty} |a_{nm}| < \frac{1}{n}.$$

We let $b_{nm} = a_{nm}$, if $m \leqslant m_n$, and $b_{nm} = 0$ if $m > m_n$. The matrix $B = (b_{nm})$ has the required character.

We now prove:

THEOREM II. *For every regular method A, the set of ξ for which $A(\xi)$ is equivalent to A is of the first category.*

Proof. Because of Lemma 4, we need only prove the theorem for row finite methods. Suppose then that A is row finite. Let E_p be the set of all ξ such that

$$n_{\nu+1}(\xi) - n_\nu(\xi) < G\{n_\nu(\xi)\}$$

for all $n_\nu(\xi) \geqslant p$, where $n_1(\xi), n_2(\xi), \ldots$ are the places at which the binary expansion of ξ is 1. We show that E_p is a closed, nowhere dense set. For, if $\eta \notin E_p$, there is an $n_\mu > p$ for which

$$n_{\mu+1}(\eta) - n_\mu(\eta) > G\{n_\mu(\eta)\}$$

so that η is at a positive distance from E_p. Hence, the complement $C(E_p)$ of E_p is open. That $C(E_p)$ is everywhere dense is obvious since a point η can have arbitrary 0's and 1's in its first n places, for every n, and belong to $C(E_p)$. It follows that the set

$$E = \bigcup_{p=1}^{\infty} E_p$$

is of the first category. But, by Lemmas 2 and 3, this set contains all ξ for which $A(\xi)$ is equivalent to A.

3. Measure.

The situation is not as clear cut with respect to measure as it is with respect to category. Indeed, we have:

THEOREM III. *For every regular method A there exist methods B and C, equivalent to A, such that B is equivalent to $B(\xi)$ for almost all values of ξ and $C(\xi)$ is strictly stronger than C for almost all values of ξ.*

Proof. Let a_n, $(n = 1, 2, \ldots)$ be the rows of A, and let B be the matrix whose rows are $a_1, a_2, a_2, \ldots, a_n, \ldots, a_n, \ldots$ where a_n is repeated 2^{n-1} times

https://doi.org/10.4153/CJM-1956-007-8 Published online by Cambridge University Press

for every n. Let E_n be the set of ξ for which a_n is a row of $B(\xi)$. The measure of E_n is $1-2^{1-n}$. Let S be the set of ξ for which $B(\xi)$ contains all but a finite number of the rows of A. Then

$$S = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} S_n$$

is of measure 1. This little argument is sometimes called the Borel-Cantelli lemma (2, p. 201). Obviously, $B(\xi)$ is equivalent to B for every $\xi \in S$, and B is equivalent to A.

Now, let D be strictly stronger than A and let C be the matrix whose rows are $a_1, d_1, a_2, d_2, d_2, \ldots, a_n, d_n, \ldots, d_n, \ldots$ where the nth row, d_n, of D is repeated 2^{n-1} times. The method C is then equivalent to A. But for almost all values of ξ, all but a finite number of rows of $C(\xi)$ are taken from D. Hence, $C(\xi)$ is strictly stronger than C for almost all values of ξ.

We show next that if A is the $(C, 1)$ method, then $A(\xi)$ is equivalent to A for almost all ξ.

LEMMA 5. *There is an integer valued function $\phi(n)$ such that*

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = 0, \qquad \sum_{n=1}^{\infty} 2^{-\phi(n)} < \infty.$$

We omit the proof which can be easily supplied by the reader.

We consider the one-one correspondence between the set of increasing sequences of positive integers and the set of points in the interval $0 < x \leqslant 1$, obtained by mating each sequence

$$n_1 < n_2 < \ldots < n_k < \ldots$$

with the point whose non-terminating binary expansion $.a_1 a_2 \ldots a_n \ldots$ has $a_n = 1$ for $n = n_k$ $(k = 1, 2, \ldots)$ and $a_n = 0$ everywhere else. The measure of a set of increasing sequences is defined as the measure of its set of images in the interval $0 < x \leqslant 1$.

LEMMA 6. *The set of increasing sequences $n_1 < n_2 < \ldots < n_k < \ldots$ of positive integers which satisfy the condition*

$$\lim_{k\to\infty} \frac{n_{k+1} - n_k}{n_k} = 0$$

is of measure 1.

Proof. For every k, the measure of the set for which

$$n_{k+1} - n_k > \phi(k)$$

is $2^{-\phi(k)}$. Thus, for every m, the measure of the set for which there is at least one $k > m$ for which $n_{k+1} - n_k > \phi(k)$ does not exceed

$$\sum_{k=m}^{\infty} 2^{-\phi(k)}.$$

It follows, by Lemma 5, that the set for which $n_{k+1}-n_k \leqslant \phi(k)$ for all but a finite number of values of k is of measure 1. But $n_{k+1}-n_k \leqslant \phi(k)$ implies

$$\frac{n_{k+1}-n_k}{n_k} \leqslant \frac{\phi(k)}{n_k} \leqslant \frac{\phi(k)}{k}.$$

Since

$$\lim_{k\to\infty} \frac{\phi(k)}{k} = 0,$$

it follows that the set of increasing sequences for which

$$\lim_{k\to\infty} \frac{n_{k+1}-n_k}{n_k} = 0$$

has measure 1.

Let $\xi \in (0, 1)$, and let $n_1 < n_2 < \ldots < n_k < \ldots$ be the sequence of integers at which 1 appears in its binary expansion.

LEMMA 7. *If A is the $(C, 1)$ method and ξ is such that*

$$\lim_{k\to\infty} \frac{n_{k+1}-n_k}{n_k} = 0$$

then $A(\xi)$ is equivalent to A.

Proof. Let $\{s_n\}$ be a bounded sequence, $|s_n| < M$, for all n, and let $\{t_n\}$ be the A transform of $\{s_n\}$. Then, for every n and k, we have

$$\begin{aligned}|t_n - t_{n+k}| &= \left|\frac{1}{n}\sum_{i=1}^{n} s_i - \frac{1}{n+k}\sum_{i=1}^{n+k} s_i\right| \\ &\leqslant \left(\frac{1}{n} - \frac{1}{n+k}\right)\sum_{i=1}^{n} |s_i| + \frac{1}{n+k}\sum_{i=n+1}^{n+k} |s_i| \leqslant \frac{2kM}{n+k}.\end{aligned}$$

Suppose $\{s_n\}$ is summable by the $A(\xi)$ method. Let $\epsilon > 0$. There is a k such that, for every $j > k$, $|t_{n_k}-t_{n_j}| < \frac{1}{2}\epsilon$ and $n_j \leqslant n < n_{j+1}$ implies

$$|t_{n_j} - t_n| < \frac{2(n-n_j)}{n} M < \frac{2(n_{j+1}-n_j)}{n_j} M < \tfrac{1}{2}\epsilon.$$

Hence, $|t_n - t_{n_k}| < \epsilon$, for every $n > n_k$, and so $\{s_n\}$ is summable by means of A.

By Lemmas 6 and 7, we have:

THEOREM IV. *The $(C, 1)$ summability method is equivalent to almost all of its submethods.*

Finally, we prove:

THEOREM V. *If A is a regular method, and $\{s_n\}$ is a bounded sequence not summable by means of A, then the set of submethods of A which sums $\{s_n\}$ is of measure either 0 or 1, and either value can occur.*

Proof. Let $\{s_n\}$ be a bounded sequence summed by A_1 to 1 and by A_2 to 0. Form A by intertwining the rows of A_1 with those of A_2 so that almost all submethods of A are equivalent to submethods of A_1. Then $\{s_n\}$ is not summed by A but is summed by almost all of its submethods. The $(C, 1)$ method is such that any sequence $\{s_n\}$ which it does not sum is also not summed by almost all of its submethods.

The set of submethods which sums a given sequence is homogeneous so that it must have measure 0 or 1.

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