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THE SMOOTH VARIATIONAL PRINCIPLE AND GENERIC DIFFERENTIABILITY

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A modified version of the smooth variational principle of Borwein and Preiss is proved. By its help it is shown that in a Banach space with uniformly Gâteaux differentiable norm every continuous function, which is directionally differentiable on a dense G_δ subset of the space, is Gâteaux differentiable on a dense G_δ subset of the space.

The question of generic differentiability (that is, on a dense G_δ subset of the domain) of non-convex functions has been considered by many authors. First results in this direction were obtained by Kenderov in [7] where he proved that in a separable Banach space a continuous and quasi-differentiable in the sense of Pshenichniy function as well as a locally Lipschitz and directionally differentiable function is generic Gâteaux differentiable. Later, Lau and Weil [9], Fabian [4], Lebourg [10] have obtained generalisations of Kenderov's results in several directions, but only in the separable case. In the non-separable case analogous extensions are proved by Zhivkov [13, 14]. Other types of results about generic Frechet differentiability of non-convex functions are obtained by Ekeland and Lebourg [3], Zajicek [12], Fabian [4], de Barra, Fitzpatrick and Giles [1], Georgiev [5], *et cetera.*

In this paper, by a modification of the smooth variational principle of Borwein and Preiss [2], we establish a result stating that in a Banach space with uniformly Gâteaux differentiable norm, every continuous, directionally differentiable on a dense G_δ subset of its domain, function is generic Gâteaux differentiable.

Let E be a Banach space. The function $f\colon E \to \mathbf{R}$ is said to be directionally differentiable at x_0 if for every $h \in E$ the one-sided directional derivative $f'(x_0;h) = \lim_{t\downarrow 0}(f(x_0+th)-f(x_0))/t$ exists. The function f is Gâteaux differentiable at x_0 if the operator $f'(x_0;.)$ is continuous and linear. In this case $f'(x_0;.)$ is denoted by $\nabla f(x_0)$.

The following assertion is a slight modification of the Borwein-Preiss smooth variational principle [2] and is essential in the sequel.

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Theorem 1. *Let E be a Banach space, $X \subset E$ be a closed non-empty subset, $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $\varepsilon > 0$, $\lambda > 0$, $p \geqslant 1$ be given. Suppose that x_0 satisfies the condition*

$$\inf_{x \in X}\{f(x) + \frac{\varepsilon}{\lambda^p}\|x - x_0\|^p\} < f(x_0) < \inf f(X) + \varepsilon.$$

Then there exists $r_1 > 0$ such that for every $r_2 > 0$ there exist a point $v \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converging to v, a sequence $\{\mu_n\}_{n=0}^{\infty} \subset [0, 1]$ with $\sum_{n=0}^{\infty} \mu_n = 1$ such that

(1) $f(v) + (\varepsilon/\lambda^p)\Delta(v) \leqslant f(x) + (\varepsilon/\lambda^p)\Delta(x) \ \forall x \in X$, *where*

(2) $\Delta(x) = \sum_{n=0}^{\infty} \mu_n \|x - x_n\|^p$,

(3) $\|v - x_0\| < \lambda$,

(4) $\|x_n - v\| < r_2 \ \forall n \geqslant 1$,

(5) $\|x_1 - x_0\| \geqslant r_1$

Proof: Choose $\varepsilon' < \varepsilon$ such that $f(x_0) < \inf f(X) + \varepsilon'$ and put $\mu_n = (1 - q_1)q_1^n$, $n = 0, 1, 2 \ldots$, where $q_1 \in (0, \min\{1, (\varepsilon - \varepsilon')/\varepsilon\})$. For fixed $\delta \in (0, f(x_0) - \inf_{x \in X}\{f(x) + \mu_0\, \varepsilon/\lambda^p \|x - x_0\|^p\})$ denote

$$X_\delta = \left\{x \in X : f(x) + \mu_0 \frac{\varepsilon}{\lambda^p}\|x - x_0\|^p \leqslant \inf_{z \in X}\{f(z) + \mu_0 \frac{\varepsilon}{\lambda^p}\|z - x_0\|^p\} + \delta\right\}.$$

The set X_δ is closed (because f is lower semicontinuous) and since $x_0 \notin X_\delta$ we have $r_1 := \operatorname{dist}(x_0, X_\delta) := \inf_{x \in X_\delta}\|x_0 - x\| > 0$.

Let $r_2 > 0$ be fixed. Choose $q_2 \in (0, \min\{q_1, \delta/\varepsilon'\})$ such that for $q := (q_2/q_1)^{1/p}$ to be fulfilled:

$$s := \left(\frac{1 + q_2}{1 - q_1}\right)^{1/p} \cdot \frac{1}{1 - q} \cdot \left(\frac{\varepsilon'}{\varepsilon}\right)^{1/p} < 1 \text{ and } \lambda s q < r_2. \tag{6}$$

Define inductively the functions $\{f_n\}_{n=0}^{\infty}$ and the points $\{x_n\}_{n=0}^{\infty}$ by:

$$f_{n+1}(x) = f_n(x) + \frac{\varepsilon}{\lambda^p}\mu_n\|x - x_n\|^p, \quad f_0 := f; \tag{7}$$

x_n is such that

$$f_n(x_n) < \inf f_n(X) + \varepsilon_n \text{ where } \varepsilon_n = \varepsilon' q_2^n,\ n = 0, 1, 2, \ldots. \tag{8}$$

Since $x_1 \in X_\delta$, we have $\|x_0 - x_1\| \geqslant \operatorname{dist}(x_0, X_\delta) = r_1$, which is (5).

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Using (7) and (8) we can write

$$\mu_n \frac{\varepsilon}{\lambda^p} \|x_{n+1} - x_n\|^p = f_{n+1}(x_{n+1}) - f_n(x_{n+1})$$
$$= f_{n+1}(x_{n+1}) - f_{n+1}(x_n) + f_n(x_n) - f_n(x_{n+1}) < \varepsilon_{n+1} + \varepsilon_n.$$

Hence

$$\|x_{n+1} - x_n\| < \lambda \left(\frac{\varepsilon_{n+1} + \varepsilon_n}{\varepsilon \mu_n}\right)^{1/p} = \lambda \left(\frac{q_2^n(q_2+1)}{q_1^n(1-q_1)}\right)^{1/p} \left(\frac{\varepsilon'}{\varepsilon}\right)^{1/p}$$
$$= \lambda q^n \left(\frac{q_2+1}{1-q_1}\right)^{1/p} \left(\frac{\varepsilon'}{\varepsilon}\right)^{1/p}$$

and having in mind the notion in (6), for $m > n$ we obtain

$$\|x_m - x_n\| < \lambda s(1 - q^{m-n}) q^n. \tag{9}$$

This shows that $\{x_n\}_{n=0}^{\infty}$ is a fundamental sequence, therefore there exists a point v such that $x_n \to v$. Now assertions (3) and (4) follow by (9).

To establish (1), let $\gamma > 0$ be given. Since f is lower semicontinuous and Δ (defined by (2)) is continuous, there exists $\delta > 0$ such that

$$f(v) + \frac{\varepsilon}{\lambda^p}\Delta(v) < f(x) + \frac{\varepsilon}{\lambda^p}\Delta(x) + \gamma/3 \text{ whenever } \|x - v\| < \delta. \tag{10}$$

Choose n sufficiently large such that $\varepsilon_n < \gamma/3$, $\|x_n - v\| < \delta$ and $(\varepsilon/\lambda^p) \sum_{k=n}^{\infty} \mu_k \|x_n - x_k\|^p < \gamma/3$. For every $x \in X$, using (10), (7) and (8), we can write

$$f(v) + \frac{\varepsilon}{\lambda^p}\Delta(v) < f(x_n) + \frac{\varepsilon}{\lambda^p}\Delta(x_n) + \gamma/3$$
$$= f_n(x_n) + \frac{\varepsilon}{\lambda^p} \sum_{k=n}^{\infty} \mu_k \|x_n - x_k\|^p + \gamma/3$$
$$< f_n(x) + \varepsilon_n + \gamma/3 + \gamma/3 < f(x) + \frac{\varepsilon}{\lambda^p}\Delta(x) + \gamma$$

and (1) is proved. ∎

THEOREM 2. *Let the Banach space E have a uniformly Gâteaux differentiable norm (this means that the norm is Gâteaux differentiable on $E \setminus \{0\}$ and the limit $\lim_{t \downarrow 0}(\|x + th\| - \|x\|)/t$ is uniform with respect to $x \in S := \{x \in E \colon \|x\| = 1\}$). Then every continuous function defined on an open subset $D \subset E$, which is directionally*

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differentiable on a dense G_δ subset of D, is Gâteaux differentiable on a dense G_δ subset of D.

PROOF: By Proposition 2.1 of [**14**], f is locally Lipschitzian on a dense and open subset D_1 of D. Let $U \subset D_1$ be an open subset such that f is Lipschitz on U. If we prove that f is Gâteaux differentiable on a dense G_δ subset of U, then the theorem would be proved, having in mind the localisation principle (see [**8**], Chapter I, Section 10, V) stating that a subset P of a topological space is of first Baire category if for every point $p \in P$ there exists an open set $H \ni p$ such that $P \cap H$ is of first Baire category in H.

Define the sets:

$$X'_n = \Bigg\{ x \in U \colon \exists p_n \in (1, 2), \quad \exists t_n \in \left(0, \left(\frac{1}{n}\right)^{\frac{1}{p_n - 1}}\right), \quad \exists \{x_{n,m}\}_{m=0}^{\infty} \subset U,$$

$$\exists \{\mu_{n,m}\}_{m=0}^{\infty} \subset [0, 1], \sum_{m=0}^{\infty} \mu_{n,m} = 1, \exists x_n \subset U \colon x_{n,m} \to x_n, \ \|x - x_n\| < t_n^2,$$

$$B(x_n; 2t_n) \subset U, \ \|x_{n,m} - x_n\| < t_n^2 \forall m \geqslant 1, \ 2t_n < \|x_{n,0} - x_{n,1}\|^2 < 1/n^2 \ \text{and}$$

$$f(x) + 2\Delta_n(x) < \inf_{z \in B(x_n; 2t_n)} \{f(z) + 2\Delta_n(z)\} + t_n^2, \ \text{where}$$

$$\Delta_n(y) = \sum_{m=0}^{\infty} \mu_{n,m} \|y - x_{n,m}\|^{p_n} \Bigg\},$$

$$X''_n = \Bigg\{ x \in U \colon \exists p_n \in (1, 2), \ \exists t_n \in \left(0, \left(\frac{1}{n}\right)^{\frac{1}{p_n - 1}}\right), \exists x_n \subset U \colon \ \|x - x_n\| < t_n^2,$$

$$B(x_n; 2t_n) \subset U, \ f(x) < \inf_{z \in B(x_n; 2t_n)} \{f(z) + 2\|x - z\|^{p_n}\} + t_n^2 \Bigg\}.$$

Since f is continuous, the sets X'_n and X''_n are open. We shall prove that the set $X_n := X'_n \cup X''_n$ is dense in U.

Let $n \geqslant 2$, $x_{n,0} \in U$ be fixed. Choose $\varepsilon \in (0, 1/n)$ in such a way that $B[x_{n,0}; \varepsilon] \subset U$. For $p_n \in (1, (\ln \varepsilon/2)/(\ln \varepsilon))$ we put $\lambda = (\varepsilon/2)^{1/p_n}$. So we have $\lambda < 2\lambda^{p_n} = \varepsilon$. Having in mind that if αf for some $\alpha > 0$ is Gâteaux differentiable at a point x then f is also Gâteaux differentiable at x, we may assume without loss of generality that the Lipschitz constant of f is less than 1. We can write

$$f(x_{n,0}) \leqslant \inf_{z \in B[x_{n,0}; \varepsilon]} \{f(z) + L\|x_{n,0} - z\|\} < \inf_{z \in B[x_{n,0}; \varepsilon]} f(z) + \varepsilon.$$

If $f(x_{n,0}) \leqslant f(x) + 2\|x - x_{n,0}\|^{p_n}$ for every $x \in B[x_{n,0}; \varepsilon]$, then $x_{n,0} \in X''_n$ for $t_n \in \left(0, \min\left\{\varepsilon/2, (1/n)^{1/(p_n - 1)}\right\}\right)$. If this is not true, we apply Theorem 1 with λ, ε,

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p_n defined above and $X = B[x_{n,0};\varepsilon]$. So there exists $r_1 > 0$ such that for $r_2 = t_n^2$ where $t_n \in \left(0, \min\left\{(\varepsilon - \lambda)/2, (1/n)^{1/(p_n-1)}, r_1^2/2\right\}\right)$ we obtain a point x_n, sequence $\{x_{n,m}\}_{m=0}^{\infty} \subset X$, $x_{n,m} \to x_n$, $\{\mu_{n,m}\}_{m=0}^{\infty} \subset [0, 1]$, $\sum_{m=0}^{\infty} \mu_{n,m} = 1$ with the following properties:

$$f(x_n) + 2\Delta_n(x_n) = \inf_{z\in X}\{f(z) + 2\Delta_n(z)\} \tag{11}$$

where
$$\Delta_n(y) = \sum_{m=0}^{\infty} \mu_{n,m} \|y - x_{n,m}\|^{p_n},$$

$$\|x_n - x_{n,0}\| < \lambda < 2\lambda^p = \varepsilon, \tag{12}$$

$$\|x_{n,m} - x_n\| < t_n^2 \quad \forall m \geqslant 1, \tag{13}$$

$$\|x_{n,0} - x_{n,1}\| \geqslant r_1 > (2t_n)^{1/2}. \tag{14}$$

Regarding the proof of Theorem 1 we can see that $\|x_{n,0} - x_{n,1}\| < \lambda < \varepsilon < 1/n$. Also by the choice of t_n and by (12) we have $B(x_n; 2t_n) \subset X$. Now by (11), (12), (13) and (14) the denseness is proved.

By the Baire category theorem the set $X_0 = \bigcap_{n=2}^{\infty} X_n$ is dense and G_δ in U. We shall prove that f is Gâteaux differentiable on X_0. Let $x_0 \in X_0$. Consider the cases:

CASE 1. x_0 belongs to infinitely many X_n. Without loss of generality, we can assume that $x_0 \in X_n$ for every $n \geqslant 2$. Let p_n, t_n, x_n, $\{x_{n,m}\}_{m=0}^{\infty}$, $\{\mu_{n,m}\}_{m=0}^{\infty}$ be the elements from the definition of X_n' corresponding to x_0. It is easy to check that $x_0 \neq x_{n,0}$ for $n \geqslant 2$. Let $t_n' = t_n / \|x_0 - x_{n,0}\|$ and $z_n = (x_0 - x_{n,0}) / \|x_0 - x_{n,0}\|$. Since

$$\begin{aligned}\|x_0 - x_{n,1}\| &\leqslant \|x_0 - x_n\| + \|x_n - x_{n,1}\| < 2t_n^2 \\ &< \|x_{n,1} - x_{n,0}\|^4 /2 < \|x_{n,1} - x_{n,0}\| /2,\end{aligned}$$

we have

$$t_n' \leqslant \frac{t_n}{\|x_{n,1} - x_{n,0}\| - \|x_{n,1} - x_0\|} < \frac{2t_n}{\|x_{n,1} - x_{n,0}\|} < \|x_{n,1} - x_{n,0}\| < 1/n.$$

Since $\|.\|^p$, $p \geqslant 1$, is a convex function, it is locally Lipschitz. From the proof of this fact (see for instance [11], p.4) we can see that the functions $\|.\|^p$, $p \in (1, 2)$ are Lipschitz on the unit ball with one and the same Lipschitz constant L. Without loss of generality we may assume that $\nabla \|z_n\| \xrightarrow{\omega^*} z_0^*$, where $\nabla \|z\|$ denotes the Gâteaux derivative of the norm at z (choosing a convergent subsequence if it is necessary) because the closed dual unit ball is sequentially ω^*-compact (see [6]) and $\|\nabla\| z_n \| \|^* = 1$.

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Since the norm is uniformly Gâteaux differentiable, it is a routine matter to prove that for every $\varepsilon > 0$ and $h \in S$ there exists $\delta > 0$ such that

$$(15) \qquad \frac{\|x+th\|^p - \|x\|^p}{t} - p\langle \nabla \|x\|\,,\, h\rangle < \varepsilon \quad \forall x \in S, \forall t \in (0,\delta), \forall p \in [1,2].$$

For every $\varepsilon > 0$ and such δ, for $n > 1/\delta$, $h \in S$, since $x_0 + t_n h \in B(x_n; 2t_n)$, using (15), we can write

$$\begin{aligned}
\frac{f(x_0+t_nh)-f(x_0)}{t_n} &\geqslant -\frac{2\Delta_n(x_0+t_nh)-2\Delta_n(x_0)}{t_n} - t_n \\
&= -2\sum_{m=0}^{\infty} \mu_{n,m} \frac{\|x_0+t_nh-x_{n,m}\|^{p_n} - \|x_0-x_{n,m}\|^{p_n}}{t_n} - t_n \\
&\geqslant -2\mu_{n,0}\frac{\|x_0+t_nh-x_{n,0}\|^{p_n} - \|x_0-x_{n,0}\|^{p_n}}{t_n} \\
&\quad - 2\sum_{m=1}^{\infty} \mu_{n,m}\frac{\|t_nh\|^{p_n} + L\|x_0-x_{n,m}\|}{t_n} - t_n \\
&\geqslant -2\mu_{n,0}\frac{\|z_n+t'_nh\|^{p_n} - \|z_n\|^{p_n}}{t'_n}\|x_0-x_{n,0}\|^{p_n-1} - 2t_n^{p_n-1} - 4Lt_n - t_n \\
&> -2p_n\langle \nabla\|z_n\|\,,\, h\rangle - 2\varepsilon - 2/n - 4L/n - 1/n \\
&> -4\langle \nabla\|z_n\|\,,\, h\rangle - 2\varepsilon - 3/n - 4L/n.
\end{aligned}$$

Hence, for $z_1^* = -4z_0^*$, after passing to limits, we obtain

$$f'(x_0;h) \geqslant \langle z_1^*, h\rangle - 2\varepsilon$$

and since this is valid for every $\varepsilon > 0$ and $h \in S$, we have

$$f'(x_0;h) \geqslant \langle z_1^*, h\rangle \quad \forall h \in S.$$

CASE 2. If Case 1 is not fulfilled, then x_0 belongs to infinitely many X''_n. Without loss of generality we may assume that $x_0 \in X''_n$ for every $n \geqslant 1$. Let t_n, p_n and x_0 be the elements from the definition of X''_n corresponding to x_0. We can write

$$\frac{f(x_0+t_nh)-f(x_0)}{t_n} \geqslant -t_n^{p_n-1} - t_n > -2/n.$$

Hence, after passing to limits, we get $f'(x_0;h) \geqslant 0$.

Repeating this reasoning for the function $-f$, we obtain a dense G_δ subset in U at every point x of which it is fullfilled: there exist x_1^*, $x_2^* \in E^*$ such that $\langle x_2^*, h\rangle \geqslant f'(x;h) \geqslant \langle x_1^*, h\rangle$ for every $h \in S$. Hence $x_1^* = x_2^* = \nabla f(x)$ and the proof is completed. ∎

The proof of Theorem 2 shows that a Banach space with uniformly Gâteaux differentiable norm is a λ-space in the terminology of [14].

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