Canad. Math. Bull. Vol. 30 (2), 1987

MIXED GROUPS OF FINITE NILSTUFE

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ABSTRACT. This paper constructs a class of examples to show that for torsion-free groups H with finite nilstufe $\nu(H) = n < \infty$ there can be divisible torsion groups D with $\nu(H \oplus D) = n + k$ for all $k \leq n + 1$. This answers a question of Feigelstock. The construction is based on a proposition which bounds $\nu(H \oplus D)$ in terms of $\nu(H)$ and rank (D).

DEFINITION. Let $A \neq 0$ be an abelian group. The nilstufe of A, $\nu(A)$, is the greatest positive integer n such that there is an associative ring R with additive group $R^+ \approx A$ and $R^n \neq 0$. If no such integer exists, $\nu(A) = \infty$.

Feigelstock [1] shows that mixed groups A with $\nu(A) < \infty$ are of the form $A = H \oplus D$ where H is a torsion-free group with $\nu(H) < \infty$ and D is a divisible torsion group. Further, if $\nu(H) = n$, then $\nu(H \oplus D) \leq 2n + 1$. Based on this result, he asks if it is true that for every $n > 1$ and every $0 \leq k \leq n + 1$ there is a divisible torsion group D and a torsion free group H with $\nu(H) = n$ such that $\nu(H \oplus D) = n + k$ [1, Question 3.1.11]. We will show that the answer is "yes". Our examples are based on the following result.

PROPOSITION. *Let $A = H \oplus D$ where H is a torsion-free group with $\nu(H) = n < \infty$ and D is a divisible torsion group. Writing $m = \max\{r_p(D) = p\text{-rank of } D\}$, we obtain $\nu(A) \leq n + m$.*

PROOF. Let R be any associative ring with $R^+ = A$. We know that D is the direct sum of its primary components D_p. It is easy to see that D and all the D_p are ideals in R. Further, $D^2 = 0$ because D is a nil group. Notice that $(R/D)^+ \approx H$, so $(R/D)^{n+1} = 0$ and $R^{n+1} \subseteq D$. One easily checks that if E is a nonzero divisible ideal of R, the ideal RE is also divisible. In addition, RE is properly contained in E, since otherwise $E = RE = R^{n+1}E \subseteq DD = 0$. Similarly, ER is divisible and properly contained in E. Since $r(D_p) \leq m$ for all p, the decreasing chain $D_p \geq RD_p \geq R^2D_p \geq \cdots$ can have at most m strict inclusions. Therefore, $R^mD_p = 0$ and $R^mD = \oplus R^mD_p = 0$. Now $R^{n+m+1} \subseteq R^mD = 0$, so $\nu(A) \leq n + m$.

We use this result to answer Feigelstock's question.

Received by the editors January 10, 1986.
AMS Subject Classification (1980): 20K99.

EXAMPLE. Let G_i be a rank one torsion-free group of type $t(G_i) = (i, i, i, \ldots)$ for $1 \leq i \leq n$. Letting $H = \overset{n}{\oplus} G_i$, we see that $\nu(H) = n$ as in Example 3.1.7 of [1]. Pick elements $e_i \in G_i$ with height sequence $(i, i \ldots i, 0, i \ldots)$ the zero occurring at a chosen prime p. Let $D = (Z_{p^\infty})^k$, $0 \leq k \leq n + 1$ be presented

$$\{{}_ia_j,\ 1 \leq i \leq k,\ 1 \leq j : p({}_ia_1) = 0\ p({}_ia_{j+1}) = {}_ia_j\}.$$

Define a multiplication on $H \oplus D$ by

$$({}_ia_j)({}_ha_m) = 0 \qquad e_i({}_ja_m) = \begin{cases} {}_{i+j}a_m & i + j \leq k \\ 0 & i + j > k \end{cases}$$

$$e_ie_j = \begin{cases} e_{i+j} & i + j \leq n \\ {}_{i+j-n}a_1 & n < i + j \leq n + k \\ 0 & i + j > n + k \end{cases}$$

This gives an associative multiplication with $(e_1)^{n+k} = {}_ka_1 \neq 0$, so $\nu(H \oplus D) = n + k$.

REFERENCES

1. S. Feigelstock, *Additive Groups of Rings*, Pitman Pub., London, 1983.

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