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BEHNKE-STEIN THEOREM ON COMPLEX SPACES WITH SINGULARITIES

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§0. Introduction

Let X be a non-compact Riemann surface and $D \subset X$ an open subset. By a classical result due to Behnke and Stein [2] D is Runge in X (i.e. the restriction map $\mathcal{O}(X) \to \mathcal{O}(D)$ has dense image) iff $X \setminus D$ has no compact connected components. In other words the obstruction to holomorphic approximation is purely topological. This result has been generalized to 1-dimensional Stein spaces by Mihalache in [11].

Let now X be a connected non-compact complex manifold of dimension n and $D \subset X$ an open subset. The open subset D is said to be n-Runge in X if for every compact set $K \subset D$ there exists an n-convex exhaustion function $\varphi : X \to \mathbf{R}$ such that $K \subset \{\varphi < o)$ and $\{\varphi \leq o\} \subset D$. Clearly this notion extends the classical one in dimension 1 and, by the results of Andreotti and Grauert [1], if D is n-Runge in X then the natural restriction map $H^{n-1}(X, \mathcal{F}) \to H^{n-1}(D, \mathcal{F})$ has dense image for every $\mathcal{F} \in \mathrm{Coh}(X)$. Also from the results of Greene and Wu [7] it follows that D is n-Runge in X iff $X \setminus D$ has no compact connected components (see also [10]).

The main purpose of this paper is to generalize the above results to complex spaces with singularities. More precisely we prove:

THEOREM. *Let X be a complex space of pure dimension n with no compact irreducible components and $D \subset X$ an open subset. Then the following conditions are equivalent:*

1) D is n-Runge in X

2) *For every $\mathcal{F} \in \mathrm{Coh}(X)$ the restriction map $H^{n-1}(X, \mathcal{F}) \to H^{n-1}(D, \mathcal{F})$ has dense image*

3) *If Ω^n_X is the canonical sheaf of X then the restriction map $H^{n-1}(X, \Omega^n_X) \to H^{n-1}(D, \Omega^n_X)$ has dense image.*

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4) *The natural map* $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ *is injective.*
5) $X \setminus D$ *has no compact irreducible components.*

Let us remark that for $D = \phi$ (empty set) this Theorem implies that X is n-complete, a well-known result due to Ohsawa [13].

Our proof of the above theorem uses resolution of singularities [8], a theorem of Diederich and Fornaess [5] on the approximation of q-convex functions with corners, and a result due to Takegoshi [19] (see also Ohsawa [14]) on the vanishing of $R^i\pi_*\Omega^n_{\tilde{X}}$ where $\pi : \tilde{X} \to X$ is a resolution of singularities for the n-dimensional complex space X and $\Omega^n_{\tilde{X}}$ is the sheaf of germs of holomorphic differential n-forms on $\tilde{X}$.

§1. Preliminaries and definitions

All complex spaces considered in this paper are assumed to be reduced and with countable topology.

It is known [8] that any complex space X admits a resolution of singularities $\pi : \tilde{X} \to X$, i.e. $\tilde{X}$ is a complex manifold and π is a proper modification such that the induced map $\pi^{-1}(\mathrm{Reg}(X)) \to \mathrm{Reg}(X)$ is an isomorphism. Let $n = \dim(X)$ and let $\pi : \tilde{X} \to X$ be a resolution of singularities. If $\Omega^n_{\tilde{X}}$ denotes the sheaf of germs of holomorphic differential n-forms on $\tilde{X}$ then we define the canonical sheaf Ω^n_X of X to be the 0-direct image $\pi_*\Omega^n_{\tilde{X}}$. By Grauert's coherence theorem Ω^n_X is a coherent sheaf on X. Moreover, this definition does not depend on the chosen resolution of singularities [6]. When X is normal, an equivalent definition for Ω^n_X is the following: Let $S \subset X$ be the singular locus of X and for any open set $U \subset X$ define $\Gamma(U) = \left\{\text{the holomorphic } n\text{-fomrs } \varphi \text{ on } U \setminus S \text{ such that } \int_{U\setminus S} \varphi \wedge \bar{\varphi} < \infty\right\}$. Then the sheaf associated to the presheaf $\{\Gamma(U)\}$ is exactly Ω^n_X [6].

One has the following result due to Takegoshi [19] (see also Ohsawa [14]):

THEOREM 1. *If* X *is a comlex space of pure dimension* n *and* $\pi : \tilde{X} \to X$ *is a resolution of singularities then* $R^i\pi_*\Omega^n_{\tilde{X}} = 0$ *for* $i \geq 1$.

Let now X be a complex space and $D \subset X$ an open subset. Let $\nu : \hat{X} \to X$ be the normalization of X and $\hat{D} = \nu^{-1}(D)$. We say that $X \setminus D$ has no compact irreducible components iff $\hat{X} \setminus \hat{D}$ has no compact connected components.

If U is an open subset in $\mathbf{C}^n$, a function $\varphi \in C^\infty(U, \mathbf{R})$ is called q-convex iff the Levi form $L(\varphi)$ has at least $(n - q + 1)$ positive (> 0) eigenvalues at any point of U. Using local embeddings this notion can be easily extended to complex spaces [1]. A complex space X is called q-complete iff there exists a q-convex function $\varphi : X \to \mathbf{R}$ which is an exhaustion function on X, i.e. $X_c = \{\varphi < c\} \subset\subset X$ for any $c \in \mathbf{R}$.

If X is a complex space and $D \subset X$ is an open subset we say that D is q-Runge in X iff the following holds: for every compact set $K \subset D$ there exists a q-convex function φ on X which is an exhaustion function on X and such that $K \subset \{\varphi < 0\}$ and $\{\varphi \leqslant 0\} \subset D$. It follows from this definition that X is assumed to be q-complete.

One has the following topological property for q-Runge domains [20]: If X is an n-dimensional complex space and $D \subset X$ is q-Runge then the relative homology groups $H_i(X, D; \mathbf{C})$ vanish for every $i \geqslant n + q$.

In particular it follows:

Corollary 1. *Let X be an n-dimensional complex space and $D \subset X$ an n-Runge open set. Then the natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective.*

Let us now recall some basic facts about q-convex functions with corners [5]. If X is a complex space we denote by $F_q(X)$ the q-convex functions with corners on X i.e. those continuous functions f on X such that for any point $x \in X$ there is an open neighbourhood $U = U(x)$ of x and finitely many q-convex functions $f_1, \dots, f_s$ on U such that $f|_U = \max(f_1, \dots, f_s)$.

In [5] the following result is proved:

Theorem 2. *Let X be a complex space of dimension n, $f \in F_q(X)$, $1 \leq q \leq n$, and $\eta > 0$ a continuous function on X. Then there exists an $\tilde{q}$-convex function $\tilde{f}$ on X, where $\tilde{q} = n - [n/q] + 1$, such that $|f - \tilde{f}| < \eta$ on X.*

In particular, when $q = n$ we get from the above theorem the following:

Corollary 2. *Let X be a complex space of dimension n, $f \in F_n(X)$, and $\eta > 0$ a continuous function on X. Then there exists an n-convex function $\tilde{f}$ on X, such that $|f - \tilde{f}| < \eta$ on X.*

We shall need also the following result from [3]:

Lemma 1. *Let X be a complex space, $A \subset X$ a closed analytic subset, $f \in F_q(A)$ and $\eta > 0$ a continuous function on A. Then there exists an open neighbourhood V of A and $f \in F_q(V)$ such that $|f - \tilde{f}| < \eta$ on A.*

§2. Proof of the Theorem

Before going into the proof of the Theorem we shall establish some lemmas.

LEMMA 2. *Let X be a complex manifold of pure dimension n with no compact connected components and $K \subset X$ a compact subset such that $X \setminus K$ has no relatively compact connected components. Then for every open neighbourhood U of K there exists an n-convex exhaustion function $\varphi : X \to \mathbf{R}$ with $K \subset \{\varphi < 0\}$ and $\{\varphi \leqslant 0\} \subset U$.*

Proof. Clearly we may assume X connected. If h is any hermitian metric on X then it induces a riemannian metric on X_{real} and by the results in [7] there is a strongly sub-harmonic function φ on X_{real}, with respect to the riemannian induced metric, such that φ is an exhaustion function on X, $K \subset \{\varphi < 0\}$ and $\{\varphi \leqslant 0\} \subset U$. Clearly φ is n-convex with respect to the complex structure of X.

LEMMA 3. *Let X be a complex manifold of pure dimension n with no compact connected components, Ω^n the sheaf of germs of holomorphic differential n-forms on X and $D \subset X$ an open subset. If the restriction map $H^{n-1}(X, \Omega^n) \to H^{n-1}(D, \Omega^n)$ has dense image then the natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective.*

Proof. On X we have the resolution of the constant sheaf $\mathbf{C}$:

$$0 \to \mathbf{C} \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$$

This resolution can be decomposed into short exact sequences:

$$\begin{gathered} 0 \to \mathbf{C} \to \mathcal{O} \to \mathcal{F}_1 \to 0 \\ 0 \to \mathcal{F}_1 \to \Omega^1 \to \mathcal{F}_2 \to 0 \\ \cdots \\ 0 \to \mathcal{F}_{n-2} \to \Omega^{n-2} \to \mathcal{F}_{n-1} \to 0 \\ 0 \to \mathcal{F}_{n-1} \to \Omega^{n-1} \to \Omega^n \to 0 \end{gathered}$$

Since X has no compact connected components it follows $H^i(X, \Omega^j) = 0, j \geqslant 0$, $i \geqslant n$ $(\Omega^0 = \mathcal{O})$. So we get the isomorphisms:

$$H^n(X, \mathcal{F}_{n-1}) \simeq \cdots \simeq H^{2n-2}(X, \mathcal{F}_1) \simeq H^{2n-1}(X, \mathbf{C})$$

and the exact sequence:

$$H^{n-1}(X, \Omega^n) \to H^n(X, \mathcal{F}_{n-1}) \to H^n(X, \Omega^{n-1}) = 0$$

It follows that the map $H^{n-1}(X, \Omega^n) \to H^{2n-1}(X, \mathbf{C})$ is surjective. This map is defined as follows: if ω is an $(n, n-1)$ differential form $\bar{\partial}$-closed representing a cohomology class in $H^{n-1}(X, \Omega^n)$, then ω is also d-closed and therefore defines a cohomology class in $H^{2n-1}(X, \mathbf{C})$. Similarly the map $H^{n-1}(D, \Omega^n) \to H^{2n-1}(D, \mathbf{C})$ is surjective. We have a commutative diagram of continuous maps

$$\begin{array}{ccc} H^{n-1}(X, \Omega^n) & \to & H^{2n-1}(X, \mathbf{C}) \\ \downarrow & & \downarrow \\ H^{n-1}(D, \Omega^n) & \to & H^{2n-1}(D, \mathbf{C}). \end{array}$$

Hence, from the density of the map $H^{n-1}(X, \Omega^n) \to H^{n-1}(D, \Omega^n)$ it follows that the map $H^{2n-1}(X, \mathbf{C}) \to H^{2n-1}(D, \mathbf{C})$ has also a dense image. Since $H^{2n-1}(X, \mathbf{C})$, $H^{2n-1}(D, \mathbf{C})$ are Fréchet spaces whose topological duals are canonically isomorphic to $H_{2n-1}(X, \mathbf{C})$ and $H_{2n-1}(D, \mathbf{C})$ respectively [17], we get the desired conclusion.

LEMMA 4. *Let X be a complex space of dimension n, $\pi : \tilde{X} \to X$ a resolution of singularities, $D \subset X$ an open subset and $\tilde{D} = \pi^{-1}(D)$. If the natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective then the natural map $H_{2n-1}(\tilde{D}, \mathbf{C}) \to H_{2n-1}(\tilde{X}, \mathbf{C})$ is injective too.*

Proof. Let $S = \mathrm{Sing}(X)$ and $\tilde{S} = \pi^{-1}(S)$. One has the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc} \cdots \to & 0 = H_{2n-1}(\tilde{S}, \mathbf{C}) & \to & H_{2n-1}(\tilde{X}, \mathbf{C}) & \to & H_{2n-1}(\tilde{X}, \tilde{S}; \mathbf{C}) & \to & H_{2n-2}(\tilde{S}, \mathbf{C}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow \wr & & \downarrow & \\ \cdots \to & 0 = H_{2n-1}(S, \mathbf{C}) & \to & H_{2n-1}(X, \mathbf{C}) & \to & H_{2n-1}(X, S; \mathbf{C}) & \to & H_{2n-2}(S, \mathbf{C}) & \to \cdots \end{array}$$

$H_{2n-1}(\tilde{S}, \mathbf{C}) = H_{2n-1}(S, \mathbf{C}) = 0$ since $\tilde{S}$, S have complex dimension $\leq n-1$. Also, from the triangulation theorem for analytic sets [9] it follows that the map $H_{2n-1}(\tilde{X}, \tilde{S}; \mathbf{C}) \xrightarrow{\sim} H_{2n-1}(X, S; \mathbf{C})$ is an isomorphism. It follows that the map $H_{2n-1}(\tilde{X}, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective. Similarly the map $H_{2n-1}(\tilde{D}, \mathbf{C}) \to H_{2n-1}(D, \mathbf{C})$ is injective. If we consider the commutative diagram:

$$\begin{array}{ccc} H_{2n-1}(\tilde{D}, \mathbf{C}) & \to & H_{2n-1}(\tilde{X}, \mathbf{C}) \\ \downarrow & & \downarrow \\ H_{2n-1}(D, \mathbf{C}) & \to & H_{2n-1}(X, \mathbf{C}) \end{array}$$

it follows that the map $H_{2n-1}(\tilde{D}, \mathbf{C}) \to H_{2n-1}(\tilde{X}, \mathbf{C})$ is injective, as desired.

In [15] the following result is proved:

LEMMA 5. *Let X be a complex manifold and $A \subset X$ a closed analytic subset. Then there exists $h \in C^{\infty}(X, \mathbf{R})$ such that*:

1) $h \geqslant 0$, $A = \{h = 0\}$

2) for any point $x \in X$ there is an open neighbourhood $U = U(x)$ of x and $\theta \in C^{\infty}(U, \mathbf{R})$ such that $\log(h|_{U\setminus A}) + \theta|_{U\setminus A}$ is 1-convex. We shall need also:

LEMMA 6. *Let Y be a complex space of dimension k, $K \subset Y$ a compact subset and $D \subset Y$ an open subset such that $K \subset D \subset Y$. Then there exists a $(k+1)$-convex exhaustion function $\psi : Y \to \mathbf{R}$ such that $K \subset \{\psi < 0\}$ and $\{\psi \leqslant 0\} \subset D$.*

Proof. It is clear that for smooth Y it is nothing to prove. For the general case the proof is by induction on $\dim(Y)$.

If $\dim(Y) = 0$ the statement of the lemma is obvious. So we assume that the statement of Lemma 6 holds for complex spaces of dimension $\leqslant k-1$. If Y has dimension k then $Z = \operatorname{Sing}(Y)$ has dimension $\leqslant k-1$ and, by the induction hypothesis, there exists a k-convex exhaustion function φ on Z with $\varphi < 0$ on $K \cap Z$ and $\{\varphi \leqslant 0\} \subset D \cap Z$. By Lemma 1 and Corollary 2 we may assume that φ can be extended to a k-convex function $\tilde{\varphi}$ in a neighbourhood of Z in Y (replacing φ by an approximation of it). Modifying suitably this function outside a neighbourhood of Z we get the desired function on Y (noting, of course, that the $(k+1)$-convexity means nothing on $\operatorname{Reg}(Y)$).
The proof of Lemma 6 complete.

LEMMA 7. *Let X be a complex manifold of pure dimension n with no compact connected components, let $A \subset X$ be a closed analytic subset and $K \subset X$ a compact subset such that $X \setminus K$ has no relatively compact connected components. Let V be any open neighbourhood of $A \cup K$. Then there exists a function $\varphi : X \to [-\infty, \infty)$ with the following properties*:

1) $A = \{\varphi = -\infty\}$, $\exp \varphi$ *is smooth*, $\varphi|_{X\setminus A}$ *is n-convex*

2) $A \cup K \subset \{\varphi < 0\}$ *and* $\{\varphi \leqslant 0\} \subset V$

3) $\varphi|_{X\setminus V}$ *is an exhaustion function*

Proof. By Lemma 2 there exists a n-convex exhaustion function $\psi : X \to \mathbf{R}$ such that $K \subset \{\psi < 0\}$ and $\{\psi \leqslant 0\} \subset V$. Let also $h \in C^{\infty}(X, \mathbf{R})$ be a function with the properties stated in Lemma 5. Then it is easy to see that one can take $\varphi = \chi \circ \psi + \log h$ where $\chi : \mathbf{R} \to \mathbf{R}$ is a suitable strictly increasing smooth convex function with $\chi(t) = At$ if $t \leqslant 0$ and $A > 0$ is sufficiently large.

We can now prove the main lemma needed in the proof of the Theorem:

LEMMA 8. *Let X be a complex space of pure dimension n with no compact irreducible components. Let also $\pi : \tilde{X} \to X$ be a resolution of singularities, $D \subset X$ an open subset and $\tilde{D} = \pi^{-1}(D)$. If $\tilde{X} \setminus \tilde{D}$ has no compact connected components then D is n-Runge in X.*

Proof. Let $S = \mathrm{Sing}(X)$ and $\tilde{S} = \pi^{-1}(S)$, hence $\pi : \tilde{X} \setminus \tilde{S} \to X \setminus S$ is an isomorphism. Let $K \subset D$ be any compact subset. We have to find a n-convex exhaustion function $\varphi : X \to \mathbf{R}$ such that $K \subset \{\varphi < 0\}$ and $\{\varphi \leqslant 0\} \subset D$. Let $Q_1 = \pi^{-1}(K) \subset \tilde{D}$ and define Q to be the union of Q_1 and of those components of $\tilde{X} \setminus Q_1$ which are relatively compact in $\tilde{X}$. Then Q is a compact set and $Q \subset \tilde{D}$ because $\tilde{X} \setminus \tilde{D}$ has no compact connected components. We set $K' = \pi(Q)$ and $L = K' \cap S$. By Lemma 6 for the inclusion $L \subset D \cap S \subset S$ we can find a n-convex exhaustion function $\varphi_1 : S \to \mathbf{R}$ such that $L \subset \{\varphi_1 < 0\}$ and $\{\varphi_1 \leqslant 0\} \subset D \cap S$. By Lemma 1 and Corollary 2 we may assume that φ_1 can be extended to an n-convex function φ_2 defined in an open neighbourhood W of S (replacing φ_1 by an approximation of it). Let V_1 be a sufficiently small open neighbourhood of S, $\bar{V}_1 \subset W$, such that $\varphi_2 |_{\bar{V}_1}$ is an exhaustion function, $\bar{V}_1 \cap \{\varphi_2 \geq -\varepsilon_0\} \cap K' = \phi$ for some $\varepsilon_0 > 0$ and $\bar{V}_1 \cap \{\varphi_2 \leqslant 0\} \subset D$. Let V_2 be an open neighbourhood of S with $\bar{V}_2 \subset V_1$. We choose now an open neighbourhood U of K', $U \subset\subset D$, $\bar{U} \cap \bar{V}_1 \cap \{\varphi_2 \geqslant -\varepsilon_0\} = \phi$ and we define $H = U \cup V_2$ and $V = \pi^{-1}(H) \subset \tilde{X}$. Then V is an open neighbourhood of $\tilde{S} \cup Q$ and using Lemma 7 for $\tilde{X}$, Q, $\tilde{S}$ ($\tilde{X} \setminus Q$ has no connected components which are relatively compact in $\tilde{X}$) we obtain a function $\psi : X \to [-\infty, \infty)$ (induced by the isomorphism $\pi : \tilde{X} \setminus \tilde{S} \to X \setminus S$) such that: $S = \{\psi = -\infty\}$, $\exp \psi$ is continuous, ψ is n-convex outside S, $S \cup K' \subset \{\psi < 0\}$, $\{\psi \leqslant 0\} \subset H$ and $\psi |_{X \setminus H}$ is an exhaustion function. In particular $\psi |_{\bar{V}_1 \setminus V_2}$ is also an exhaustion function. Moreover, on the set $(\bar{V}_1 \setminus V_2) \cap \{\psi \leqslant 0\}$ we have $\varphi_2 \leqslant -\varepsilon_0$. It follows that we can find a smooth strictly increasing convex function $\theta : \mathbf{R} \to \mathbf{R}$, $\theta(t) = \alpha t$ if $t \leqslant 0$ ($\alpha > 0$ small enough) such that setting $\psi_1 = \theta \circ \psi$ one has $\psi_1 > \varphi_2$ on $\bar{V}_1 \setminus V_2$ and $\psi_1 |_{X \setminus H}$ is an exhaustion function. Clearly $\psi_1(x) \to -\infty$ if $x \to y \in S$.

We define now a function $\beta : X \to \mathbf{R}$ which is locally the maximum of two n-convex functions, hence $\beta \in F_n(X)$, in the following way: If $x \in V_1$ we put $\beta(x) = \max(\varphi_2(x), \psi_1(x))$. In a neighbourhood of S β is equal to φ_2 since $\psi_1(x) \to -\infty$ if $x \to y \in S$. Since φ_2 is n-convex on V_1 and ψ_1 is n-convex outside S it follows that on V_1 β is locally the maximum of two n-convex functions.

If $x \in X \setminus \bar{V}_2$ we define $\beta(x) = \psi_1(x)$. On $V_1 \setminus \bar{V}_2$ the two definitions of β agree because $\psi_1(x) > \varphi_2(x)$ if $x \in V_1 \setminus \bar{V}_2$. Hence $\beta \in F_n(X)$. Since $\varphi_2 |_{\bar{V}_1}$ is an exhaustion function and $\psi_1 |_{X \setminus H}$ is an exhaustion function it follows that β is an exhaustion function on X. Also one has $\{\beta \leqslant 0\} \subset \bar{U} \cup [\bar{V}_1 \cap \{\varphi_2 \leqslant 0\}] \subset D$, hence $\{\beta \leqslant 0\}$ is a compact subset of D. From the definition of β it follows that $K \subset K' \subset \{\beta < 0\}$. Lemma 8 follows now from Corollary 2 because $\beta \in F_n(X)$.

To prove the Theorem we need also some simple topological lemmas.

In [12] the following topological result is proved:

LEMMA 9. *Let M be a locally compact Hausdorff space, $A \subset M$ a closed subset and P a connected compact component of A. Then there exists a fundamental system of open neighbourhoods V of P in M such that $(\partial V) \cap A = \phi$ where ∂V denotes the boundary of V in M.*

From this result it follows immediately:

LEMMA 10. *Let M be a locally compact Hausdorff space and $A \subset M$ a closed subset. Then A has compact connected components iff A can be written $A = L \cup F$ where $L \neq \phi$ is a compact set, F is a closed set and $L \cap F = \phi$.*

Proof. If A has a compact connected component P, then by Lemma 9 there is an open neighbourhood V of P with $\bar{V} =$ compact and $(\partial V) \cap A = \phi$. We define then $L = \bar{V} \cap A$ and $F = (\complement V) \cap A$.

Conversely, if A can be written $A = L \cup F$ with $L \neq \phi$ compact set and F closed set, $L \cap F = \phi$, we choose a connected component P of A with $P \cap L \neq \phi$. From the equality $P = (P \cap L) \cup (P \cap F)$ it follows $P \subset L$ and therefore P is a compact connected component of A.

LEMMA 11. *Let X, Y be locally compact Hausdorff spaces and $\pi : X \to Y$ a continuous map which is proper and surjective. Let $A \subset Y$ be a closed subset and $\tilde{A} = \pi^{-1}(A)$. The following statements hold:*

1) *If $\tilde{A}$ has no compact connected components then A has no compact connected components.*

2) *Assume additionally that π has connected fibers. Then the condition "A has no compact connected components" implies "$\tilde{A}$ has no compact connected components".*

Proof. 1) follows immediately by Lemma 10. So let us verify 2). Assume that $\tilde{A}$ has compact connected components. Then by Lemma 10 $\tilde{A}$ can be written $\tilde{A} = L \cup F$ with $L \neq \phi$ compact set, F closed set and $L \cap F = \phi$. Therefore $A = \pi(L) \cup \pi(F)$, $\pi(\Gamma) \neq \phi$ is a compact set and $\pi(F)$ is a closed set. Since π has connected fibers it follows easily that $\pi(L) \cap \pi(F) = \phi$, and by Lemma 10 A has compact connected components. The proof of Lemma 11 is complete.

LEMMA 12. *Let X be a complex manifold of pure dimension n with no compact connected components and let $D \subset X$ be an open subset. Then the natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective iff $X \setminus D$ has no compact connected components.*

Proof. We have the exact sequence:

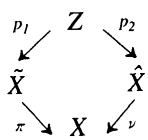
$$\cdots \to H_{2n}(X, \mathbf{C}) \to H_{2n}(X, D; \mathbf{C}) \to H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C}) \to \cdots$$

Since X has no compact connected components it follows that $H_{2n}(X, \mathbf{C}) = 0$, therefore the injectivity of the map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is equivalent to the vanishing of the relative homology group $H_{2n}(X, D; \mathbf{C})$. By Alexander duality [18] we have: $H_{2n}(X, D; \mathbf{C}) \cong \bar{H}^0_c(X \setminus D, \mathbf{C}) = \{f : X \setminus D \to \mathbf{C} \mid f$ is locally constant and $f = 0$ outside a compact subset of $X \setminus D\}$. Using this isomorphism Lemma 12 is a direct consequence of Lemma 10.

LEMMA 13. *Let X be a pure dimensional complex space, $\pi : \tilde{X} \to X$ a resolution of singularities, $D \subset X$ an open subset and $\tilde{D} = \pi^{-1}(D)$. Then the following conditions are equivalent:*

1) *$X \setminus D$ has no compact irreducible components*
2) *$\tilde{X} \setminus \tilde{D}$ has has no compact connected components*

Proof. Let $\nu : \hat{X} \to X$ be the normalization of X and $\hat{D} = \nu^{-1}(D)$. By definition $X \setminus D$ has no compact irreducible components iff $\hat{X} \setminus \hat{D}$ has no compact connected components. Consider the commutative diagram:



where $Z = \tilde{X} \times_X \hat{X}$ is the reduced fiber product of $\tilde{X}$ and $\hat{X}$ over X and p_1, p_2 are the canonical projections given by $p_1 = \bar{p}_1 \circ i$ and $p_2 = \bar{p}_2 \circ i$. Here $\bar{p}_1 : \tilde{X} \times \hat{X} \to \tilde{X}$ is the projection on $\tilde{X}$, $\bar{p}_2 : \tilde{X} \times \hat{X} \to \hat{X}$ is the projection on $\hat{X}$ and $i : Z \to \tilde{X} \times \hat{X}$ is the inclusion map. Z is the reduced subspace given by $Z = \{(\tilde{x}, \hat{x}) \in \tilde{X} \times \hat{X} \mid \pi(\tilde{x}) = \nu(\hat{x})\}$. It is easy to see that p_1 and p_2 are proper modifications. Moreover, since $\tilde{X}$ and $\hat{X}$ are normal complex spaces, it follows from the Riemann extension theorem that $p_{1*}\mathcal{O}_Z \cong \mathcal{O}_{\tilde{X}}$ and $p_{2*}\mathcal{O}_Z \cong \mathcal{O}_{\hat{X}}$. in particular p_1 and p_2 have connected fibers. From these remarks Lemma 13 is a direct consequence of Lemma 11.

Proof of the Theorem.

1) ⇨ 2) follows from the paper of Andreotti and Grauert [1].

2) ⇨ 3) follows from the coherence of Ω_X^n

3) ⇨ 5) Let $\pi : \tilde{X} \to X$ be a resolution of singularities and $\tilde{D} = \pi^{-1}(D)$. Consider the commutative diagram:

$$\begin{array}{ccc} H^{n-1}(\tilde{X}, \Omega_{\tilde{X}}^n) & \xrightarrow{r_1} & H_{n-1}(\tilde{D}, \Omega_{\tilde{X}}^n) \\ \alpha \uparrow & & \uparrow \beta \\ H^{n-1}(X, \Omega_X^n) & \xrightarrow{r_2} & H_{n-1}(D, \Omega_X^n) \end{array}$$

where α, β, r_1, r_2 are continuous maps and r_2 has dense image by our hypothesis. By Theorem 1 the maps α and β are bijective, hence r_2 has also dense image. By Lemma 3 we get the injectivity of the map $H_{2n-1}(\tilde{D}, \mathbf{C}) \to H_{2n-1}(\tilde{X}, \mathbf{C})$ and by Lemma 12 $\tilde{X} \setminus \tilde{D}$ has no compact connected components. In view of Lemma 13 $X \setminus D$ has no compact irreducible components.

5) ⇨ 1) If $\pi : \tilde{X} \to X$ is a resolution of singularities and $\tilde{D} = \pi^{-1}(D)$ it follows from Lemma 13 that $\tilde{X} \setminus \tilde{D}$ has no compact connected components and by Lemma 8 D is n-Runge in X.

Therefore we have proved 1)⇔2)⇔3)⇔5).

1) ⇨ 4) follows from Corollary 1.

4) ⇨ 5) Let $\pi : \tilde{X} \to X$ be a resolution of singularities and $\tilde{D} = \pi^{-1}(D)$. By Lemma 4 the natural map $H_{2n-1}(\tilde{D}, \mathbf{C}) \to H_{2n-1}(\tilde{X}, \mathbf{C})$ is injective and by Lemma 12 $\tilde{X} \setminus \tilde{D}$ has no compact connected components. In view of Lemma 13 $X \setminus D$ has no compact irreducible components.

The proof of the Theorem is complete.

Remark. One can easily see that in the Theorem the condition "X has pure dimension n and has no compact irreducible components" can be weakened to "X has dimension n and has no compact irreducible components of dimension n". In this case the condition 5) must be replaced by

5′) $X \setminus D$ has no compact irreducible components of dimension n. This means the following: if $\nu : \hat{X} \to X$ is the normalization map and $\hat{D} = \nu^{-1}(D)$ then $\hat{X}_i \setminus \hat{D}$ has no compact connected components for every n-dimensional connected component $\hat{X}_i$ of $\hat{X}$.

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