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ON C^0-SUFFICIENCY OF COMPLEX JETS

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1. Introduction. In this paper we shall study the sufficiency of complex jets. Let $A(\mathbf{C}^n, \mathbf{C})$ be the set of all analytic functions $f : \mathbf{C}^n \to \mathbf{C}$ with $f(0) = 0$. We call two functions f and g of $A(\mathbf{C}^n, \mathbf{C})$ *equivalent of order r at* 0 if, at 0, their Taylor expansions up to and including the terms of degree $\leqq r$ are identical. This defines an equivalence relation on $A(\mathbf{C}^n, \mathbf{C})$. An *r-jet*, denoted $j^{(r)}(f)$, is the equivalence class of f, with f being called the *realization* of $j^{(r)}(f)$. The set of all r-jets is denoted by $J^r(\mathbf{C}^n, \mathbf{C})$.

Definition. An r-jet Z of $J^r(\mathbf{C}^n, \mathbf{C})$ is called *C^0-sufficient* if for any two realizations f and g of Z, there exists a local homeomorphism $h : \mathbf{C}^n \to \mathbf{C}^n$, $h(0) = 0$, such that $f[h(z)] = g(z)$ for all z in a neighborhood of $0 \in \mathbf{C}^n$.

The r-jet Z is said to be *analytic sufficient* if, in the above definition, h is a local diffeomorphism. Also, we say that Z is *v-sufficient*, or *v-insensitive* (v stands for variety), if the germs of the varieties $f^{-1}(0)$ and $g^{-1}(0)$ of f and g respectively are homeomorphic. It is obvious that C^0-sufficiency implies v-sufficiency.

In Section 2, the main result asserts that if there is a positive number ϵ such that

$$\left|\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\right| \geqq \epsilon |z|^{r-1}$$

for all small $|z|$, where $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, then $j^{(r)}(f)$ is C^0-sufficient.

For real jets from $\mathbf{R}^n$ to $\mathbf{R}$, this result was first obtained by Kuiper [**2**], and later independently by Kuo [**3**]. By generalizing the method used in [**3**] to the complex case, we establish the criterion of C^0-sufficiency for complex jets.

From our Theorem 1 in Section 2 and Theorem 1 in [**3**] we found that the criterion of the C^0-sufficiency for real jets and that for complex jets are of the same form. In Section 3 we consider some examples in order to distinguish between real and complex jets with respect to:

(1) C^1-sufficiency

and

(2) the degree of C^0-sufficiency (which is defined in [**5**, p. 120]).

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2. Main results.

THEOREM 1. *Let* $f(z) = f(z_1, \ldots, z_n)$ *be in* $A(\mathbf{C}^n, \mathbf{C})$. *If there exists a positive number* ϵ *such that*

$$\left|\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\right| \geqq \epsilon |z|^{r-1} \tag{1}$$

from some positive integer r *and for all* $z = (z_1, \ldots, z_n)$ *in a neighborhood of* $0 \in \mathbf{C}^n$, *then* $j^{(r)}(f)$ *is a* C^0*-sufficient jet in* $J^r(\mathbf{C}^n, \mathbf{C})$.

Proof. Let f_r be the polynomial obtained from the Taylor expansion of f about 0 up to and including the terms of degree $\leqq r$. From the definition it follows that $j^{(r)}(f) = j^{(r)}(f_r)$. We wish to prove that for any analytic function g in $j^{(r)}(f)$ there exists a local homeomorphism $h : \mathbf{C}^n \rightarrow \mathbf{C}^n$ with $h(0) = 0$ such that $g(h(z)) = f_r(z)$ for all z in a neighborhood of $0 \in \mathbf{C}^n$.

First, let us make the following remarks:

(1) We may assume that the linear terms of f_r are identically zero. Since otherwise, by the implicit function theorem the linear terms determine the local topological type of the mapping and hence $j^{(1)}(f)$ is already sufficient.

(2) Let $F(z, u) = f_r(z) + u[g(z) - f_r(z)]$, where $z = (z_1, \ldots, z_n)$ and u is a real number. Then there exists a positive real number α such that

$$\left|\left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial u}\right)\right| \geqq \frac{\epsilon}{2} |z|^{r-1} \tag{2}$$

for $0 \leqq |z| < \alpha$ and $0 \leqq u \leqq 1$, where ϵ was given in the hypothesis of the theorem.

(3) Note that $F(z, 0) = f_r(z)$ and $F(z, 1) = g(z)$. Therefore, what we really wish to do is to find a local homeomorphism $h : \mathbf{C}^n \rightarrow \mathbf{C}^n$ such that F is constant along some curve joining $(z, 0)$ and $(h(z), 1)$ for each z which is close enough to 0.

(4) For $0 < |z| < \alpha$, let

$$p(z, u) = \overline{[g(z) - f_r(z)]} \left|\left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial u}\right)\right|^{-2} \left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial u}\right),$$

where $\overline{g(z) - f_r(z)}$ is the complex conjugate of $g(z) - f_r(z)$.

(5) Let
$$\begin{aligned} X(z, u) &= (0, 1) - \overline{p(z, u)}, &&\text{if } 0 < |z| < \alpha, \\ &= (0, 1), &&\text{if } z = 0, \end{aligned}$$

where $\overline{p(z, u)}$ is the complex conjugate of $p(z, u)$. Then $X(z, u)$ has the following properties:

(i) $X(z, u)$ is continuous in (z, u) for $0 \leqq |z| < \alpha$ and $0 \leqq u \leqq 1$;

(ii) $\lim_{z \to 0} \dfrac{|X(z, u) - X(0, u)|}{|z|} = 0$ uniformly for $0 \leqq u \leqq 1$;

(iii) there exists a positive number α_1 such that the inner product $\langle X(z, u), (0, 1)\rangle$ is positive for $0 \leqq |z| < \alpha_1$. (We may assume $\alpha \leqq \alpha_1$.)

The proofs for (2) and (5) are similar to that of the real case which can be found in [**3**, pp. 168-169].

Now, consider the following system of ordinary differential equations:

$$\begin{bmatrix} \dfrac{dz}{dt} \\ \dfrac{du}{dt} \end{bmatrix} = X(z, u). \tag{3}$$

We will show that there exists one, and only one, solution passing through any point (z, u) for $0 \leqq |z| < \alpha$ and $0 \leqq u \leqq 1$. The existence of solutions to this system follows from the continuity of the vector field $X(z, u)$. We need only show the uniqueness.

From (ii) of Remark (5), we have

$$\frac{|X(z, u) - X(0, u)|}{|z|} \leqq \rho(|z|),$$

where ρ is a real-valued function and $\rho(s) \to 0$, as $s \to 0$. We may assume $\rho(s) < 1$ for $0 \leqq s < \alpha$.

Let $\varphi(t; z, u)$ denote a solution of the system (3) with $\varphi(0; z, u) = (z, u)$. Then it is clear that a solution passing through $(0, u)$ is given by $\varphi = \varphi(t; 0, u) = (0, t + u)$. We claim that this is the only solution passing through $(0, u)$. Suppose $\varphi_1 = \varphi_1(t; 0, u)$ is another solution passing through $(0, u)$. Then

$$\frac{d}{dt}[\varphi_1(t) - \varphi(t)] = X[\varphi_1(t)] - X[\varphi(t)],$$

and hence

$$\varphi_1(t) - \varphi(t) = \int_0^t (X[\varphi_1(\tau)] - X[\varphi(\tau)])d\tau.$$

Write $\varphi_1(t; 0, u) = (z(t), u(t))$. Then

$$\begin{aligned} |z(t)| &\leqq |\varphi_1(t) - \varphi(t)| \\ &\leqq \int_0^t |X[\varphi_1(\tau)] - X(0, \tau + u)|d\tau \\ &= \int_0^t |X[z(\tau), u(\tau)] - X[0, u(\tau)]|d\tau \\ &\leqq \int_0^t \rho(|z(\tau)|)|z(\tau)|d\tau, \end{aligned}$$

and hence

$$|z(t)| \leqq \int_0^t |z(\tau)|d\tau.$$

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By Gronwall's inequality [**1**, p. 24], we have $z(t) = 0$. Thus $\varphi_1(t;\, 0,\, u) = (0,\, u(t))$. Since

$$\frac{d}{dt}[\varphi_1(t) - \varphi(t)] = X(0, u(t)) - X(0, t + u) = 0,$$

we have

$$\frac{d}{dt}(u(t) - (t + u)) = 0.$$

It follows immediately that $u(t) = t + u$. Hence, we have $\varphi_1 \equiv \varphi$.

If $z \neq 0$, there is a neighborhood of z, say $N(z)$, which is bounded away from 0. It is quite easy to see that $X(z, u)$ satisfies a Lipschitz condition in $N(z) \times [0,1]$. (Here $[0,1]$ is the closed interval $0 \leqq u \leqq 1$.) Thus there can be at most one solution passing through any point (z, u) for $0 \leqq |z| < \alpha$ and $0 \leqq u \leqq 1$. Also, it follows from uniqueness that the terminal value of a solution depends continuously on the initial value [**1**, p. 94].

By (iii) of Remark (5), we know that the u-component of any solution $\varphi(t;\, z,\, 0)$ increases monotonically with t for all z in a neighborhood of 0. Hence, $\varphi(t;\, z,\, 0)$ meets the hyperplane $u = 1$ at a unique point $h(z)$. The mapping $z \to h(z)$ is then clearly a local homeomorphism with $h(0) = 0$.

We observe that for $\varphi(t) = \varphi(t;\, z,\, u)$ with $0 < |z| < \alpha$

$$\begin{aligned}
\frac{d}{dt}F(\varphi(t)) &= \frac{\partial F}{\partial z_1}\frac{dz_1}{dt} + \ldots + \frac{\partial F}{\partial z_n}\frac{dz_n}{dt} + \frac{\partial F}{\partial u}\frac{du}{dt} \\
&= \left\langle \left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial u}\right), \overline{X(\varphi(t))} \right\rangle \\
&= \left\langle \left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial u}\right), (0, 1) - p(z(t), u(t)) \right\rangle \\
&= \frac{\partial F}{\partial u} - (g(z) - f_r(z)) \\
&= 0,
\end{aligned}$$

for all $t \geqq 0$. When $z = 0$,

$$\frac{d}{dt}F(\varphi(t)) = 0, \quad t \geqq 0,$$

also holds, since in this case $F(\varphi(t)) = F(0,\, t + u) \equiv 0$. We thus draw the conclusion that $F(\varphi(t)) \equiv$ constant along each solution curve φ. For the solution $\varphi(t) = \varphi(t;\, z,\, 0)$, we have $\varphi(0) = (z,\, 0)$ and $\varphi(t_1) = (h(z),\, 1)$ for some $t_1 > 0$. Therefore, we get

$$f_r(z) = F(z, 0) = F(\varphi(0)) = F(\varphi(t_1)) = F(h(z), 1) = g(h(z)).$$

This completes the proof of Theorem 1.

With obvious modifications of the proof of the above theorem, we have the following extension.

THEOREM 2. *Let* $f(z) = f(z_1, \ldots, z_n)$ *be in* $A(\mathbf{C}^n, \mathbf{C})$. *Assume that there exist positive numbers* ϵ *and* δ *such that*

$$\left|\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\right| \geqq \epsilon |z|^{r-\delta}$$

for some positive integer r *and for all* $z = (z_1, \ldots, z_n)$ *in a neighborhood of* $0 \in \mathbf{C}^n$. *Then* $j^{(r)}(f)$ *is a* C^0*-sufficient jet in* $J^r(\mathbf{C}^n, \mathbf{C})$.

3. Complex jets versus real jets. Since the statement of our main theorem is similar to the corresponding one for real jets, we would like to point out some differences between complex and real jets.

Let Z be a real jet in $J^r(\mathbf{R}^n, \mathbf{R})$. We say that Z is *C^1-sufficient* if for any two realizations f and g (which are C^{r+1}-functions) of Z, there exists a local C^1-diffeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$, with $h(0) = 0$, such that $f(h(x)) = g(x)$ for all x in a neighborhood of $0 \in \mathbf{R}^n$. If $f(x) = f(x_1, \ldots, x_n)$ is a real homogeneous polynomial of degree r, it is easy to see that the vector field X, as constructed in the proof of Theorem 1, is of class C^1. Then the local homeomorphism h becomes a C^1-diffeomorphism. Hence, we have immediately the following corollary.

COROLLARY. *If* $f(x) = f(x_1, \ldots, x_n)$ *is a real homogeneous polynomial of degree* r *and if there exist positive numbers* ϵ *and* δ *such that*

$$\left|\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)\right| \geqq \epsilon |x|^{r-\delta}$$

for all $x = (x_1, \ldots, x_n)$ *in a neighborhood of* $0 \in \mathbf{R}^n$, *then* $j^{(r)}(f)$ *is a* C^1*-sufficient jet.*

Remark. For $\delta = 1$, this result was first obtained by Kuiper [**2**] using a different technique.

For complex jets, however, the above corollary does not hold. By this corollary, it is clear that the real jet $f(x, y) = x^5 + y^5$ is C^1-sufficient. It is also known (see [**2**]) that this jet is not C^2-sufficient. On the other hand, the corresponding complex jet given by $f(z_1, z_2) = z_1^5 + z_2^5$ is clearly C^0-sufficient by Theorem 1. However, this jet cannot be complex-C^1-sufficient by observing the two not-analytically conjugate realizations $z_1^5 + z_2^5$ and $z_1^5 + z_2^5 + z_1^3 z_2^3$. (Note that the real functions $x^5 + y^5$ and $x^5 + y^5 + x^3y^3$ are still real-C^1-equivalent.)

Next, let us consider the examples

$$g(x, y) = x^3 + 3xy^{2k},$$

and

$$f(z_1, z_2) = z_1^3 + 3z_1 z_2^{2k},$$

where k is an integer greater than one, in real and complex variables respectively. The real jet $g(x, y)$ has been studied by T. C. Kuo [**4**]. He found that the degree of C^0-sufficiency (see [**5**]), which is defined as the smallest integer r such that the jet $j^{(r)}(g)$ is C^0-sufficient, is $2k + 1$. However, the degree of C^0-sufficiency of the corresponding complex jet $f(z_1, z_2)$ is $3k$. This will be proved as follows.

First, we wish to find an integer r such that

$$\left|\frac{\partial f}{\partial z_1}\right| + \left|\frac{\partial f}{\partial z_2}\right| \geqq \epsilon(|z_1| + |z_2|)^{r-\delta},$$

where

$$\frac{\partial f}{\partial z_1} = 3(z_1 - iz_2{}^k)(z_1 + iz_2{}^k),$$

$$\frac{\partial f}{\partial z_2} = 6kz_1z_2{}^{2k-1}.$$

Let $S = \{(z_1, z_2) | |z_1|/|z_2| < \rho\}$ for some $\rho > 0$. Outside the set S, there is no problem; the above inequality is satisfied even for $r = 3$. Inside the set S, we consider the following three subsets:

$$H_1 = \{(z_1, z_2)|\ |z_1 - iz_2{}^k| < w|z_2|^k\},$$
$$H_2 = \{(z_1, z_2)|\ |z_1| < w|z_2|^k\},$$
$$H_3 = \{(z_1, z_2)|\ |z_1 + iz_2{}^k| < w|z_2|^k\},$$

where w is a small positive number. We claim that H_1, H_2, and H_3 are mutually disjoint when w is sufficiently small. In fact, H_1 and H_3 are contained in the set

$$H = \{(z_1, z_2)|(1 - w)|z_2|^k < |z_1| < (1 + w)|z_2|^k\},$$

and $H_2 \cap H = \emptyset$ if w is small. On the other hand, when a point $(z_1, z_2) \in H_1$, we have

$$|iz_2{}^k - z_1| < w|z_2|^k \leqq |z_1|,$$

where the second inequality follows from the fact that $(z_1, z_2) \notin H_2$. Suppose the point (z_1, z_2) is also in H_3. Then we would have

$$|iz_2{}^k + z_1| < w|z_2|^k \leqq |z_1|,$$

which is clearly a contradiction.

Hence, any point $(z_1, z_2) \in S$ lies inside at most one of the subsets H_1, H_2, and H_3. We have thus either

$$|\partial f/\partial z_1| = 3|z_1 - iz_2{}^k|\ |z_1 + iz_2{}^k| \geqq 3w^2|z_2|^{2k} \geqq 3w^2|z_2|^{3k-1},$$

or

$$|\partial f/\partial z_2| = 6k|z_1|\ |z_2|^{2k-1} \geqq 6kw|z_2|^{3k-1} \geqq 3w^2|z_2|^{3k-1}.$$

Also, inside S, we have $|z_1| < \rho|z_2|$. Hence

$$\epsilon(|z_1| + |z_2|)^{r-\delta} \leqq \epsilon(1 + \rho)^{r-\delta}|z_2|^{r-\delta} \leqq 3w^2|z_2|^{3k-1}$$

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if we choose $r = 3k$, some small $\epsilon > 0$, and $0 < \delta < 1$. It follows that

$$\left|\frac{\partial f}{\partial z_1}\right| + \left|\frac{\partial f}{\partial z_2}\right| \geqq \epsilon(|z_1| + |z_2|)^{3k-\delta}$$

for all (z_1, z_2) in a small neighborhood of $0 \in \mathbf{C}^2$. Thus we see that $j^{(3k)}(f)$ is C^0-sufficient.

Now, we prove that $j^{(3k-1)}(f)$ is v-sensitive, and hence not C^0-sufficient. We note that

$$\begin{aligned} q(z_1, z_2) &= z_1^3 + 3z_1z_2^{2k} - 2iz_2^{3k} \\ &= (z_1 - iz_2^k)^2(z_1 + 2iz_2^k) \end{aligned}$$

is a realization of $j^{(3k-1)}(f)$. For $N > 3k - 1$, both $q(z_1, z_2)$ and $q(z_1, z_2) - z_2^{2N}(z_1 + 2iz_2^k)$ are realizations of $j^{(3k-1)}(f)$; but they have different local topological types. Thus the degree of C^0-sufficiency of $f(z_1, z_2)$ is $3k$.

From the above discussion, we see that the sufficiency of real jets and the sufficiency of the corresponding complex jets are, in general, different with respect to the degree of C^0-sufficiency and C^1-sufficiency. However, with respect to analytic sufficiency, the degree of sufficiency of real jets and that of the corresponding complex jets are known to be the same [**6**, p. 155]. That is, if given $f : \mathbf{C}^n \to \mathbf{C}$ (or respectively $f : \mathbf{R}^n \to \mathbf{R}$) and if $j^{(r)}(f)$ is complex analytic sufficient (respectively real analytic sufficient) then for the same r, by replacing complex variables by real variables (respectively, replacing real variables by complex variables), $j^{(r)}(f)$ is also real analytic sufficient (respectively, complex analytic sufficient).

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