DERIVATION OF A GENERAL LORENTZ TRANSFORMATION WITHOUT ROTATION

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The purpose of this note is to show that the standard form of a general Lorentz transformation without special rotation [1], [2], [3] can be derived from simple algebraic hypotheses.

Let Greek indices go from 1 to 4 and Latin indices from 1 to 3. Summation convention over repeated indices in a product is used throughout and the velocity of light c is taken to be unity.

A Lorentz transformation in a four dimensional space-time is defined by the relations

$$x'_\mu = a_{\mu\nu} x_\nu, \qquad x_\mu = a_{\nu\mu} x'_\nu,$$

where

$$a_{\mu\lambda} a_{\mu\nu} = a_{\lambda\mu} a_{\nu\mu} = \delta_{\lambda\nu}, \tag{1}$$

$\delta_{\lambda\nu}$ being the Kronecker delta tensor.

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Let

$$\begin{aligned} u_i &= \frac{x_i}{x_4} \quad \text{when} \quad x'_i = 0, \quad \text{and} \\ u'_i &= \frac{x'_i}{x'_4} \quad \text{when} \quad x_i = 0. \end{aligned} \tag{2}$$

Then

$$u'_i = \frac{a_{i4}}{a_{44}}, \qquad u_i = \frac{a_{4i}}{a_{44}}. \tag{3}$$

A Lorentz transformation without spatial rotation is defined by the condition that

$$u'_i = -u_i, \qquad \text{or} \qquad a_{i4} = -a_{4i}. \tag{4}$$

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Let us write

$$a_{i4} = i\beta v_i, \qquad a_{44} = \beta,$$

where $i = \sqrt{-1}$ and v_i are all real in the usual interpretation of the coordinates x_μ in a Minkowski space. If $\lambda = \nu = 4$ in (1), it follows that

(5) $$\beta^2 = (1-v^2)^{-1}, \qquad v^2 = v_i v_i.$$

Similarly

(6) $$a_{ij}a_{ik}+a_{4j}a_{4k} = \delta_{jk} = a_{ij}a_{ik}-\beta^2 v_j v_k,$$

and

(7) $$a_{i4}a_{ij}+a_{44}a_{4j} = 0.$$

Hence

(8) $$a_{ij}v_i = \beta v_j,$$

and

(9) $$a_{ij}a_{ik}-a_{ij}a_{lk}v_i v_l = \delta_{jk} = a_{ij}a_{lk}p_{il},$$

say, where

(10) $$p_{il} = p_{li} = \delta_{il}-v_i v_l.$$

Replacing the coefficients in (9) by their transposed (this is justified in view of (1)), multiplying both sides by v_j and using the symmetry of p_{ij},

$$a_{ji}v_j p_{il}a_{kl} = v_k = a_{kl}p_{il}v_i.$$

Also, from (10),

$$p_{il}v_i = v_l-v^2 v_l = v_l\beta^{-2}.$$

Hence

(11) $$\beta v_k = a_{kl}v_l.$$

Comparison of (8) and (11) shows that the skewsymmetric part $a_{\underset{\vee}{ij}}$ of a_{ij}, satisfies the equations

(12) $$a_{\underset{\vee}{ij}}v_j = 0.$$

Hence, $a_{\underset{\vee}{ij}}$ may be written in the form

(13) $$a_{\underset{\vee}{ij}} = \alpha \begin{Bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{Bmatrix} = \alpha s_{ij}, \text{ say,}$$

where α is a parameter which may depend on v_i. Let us assume that the symmetric part $a_{\underline{ij}}$ of a_{ij} is given by

(14) $$a_{\underline{ij}} = \delta_{ij}+qv_i v_j,$$

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where q may again depend on v_i. Hence, from (8),

$$q = \frac{\beta-1}{v^2}. \tag{15}$$

Thus, the most general Lorentz transformation of the type required is

$$a_{ij} = \delta_{ij}+qv_iv_j+\alpha s_{ij}.$$

By (1) we now have

$$\alpha^2 s_{ij}s_{ik} = 0, \tag{16}$$

and this cannot be satisfied by an arbitrary real vector

$$\begin{aligned} &\underline{v} = (v_1, v_2, v_3) \text{ unless} \\ &\alpha = 0. \end{aligned} \tag{17}$$

Therefore the transformation is given by

$$a_{ij} = \delta_{ij}+\frac{\beta-1}{v^2}v_iv_j, \quad a_{i4} = -a_{4i} = i\beta v_i, \quad a_{44} = \beta, \tag{18}$$

as required (ref. 1).

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Consider the transformation (18) followed by an infinitesimal Lorentz transformation without spacial rotation. The latter is defined by

$$b_{ij} = \delta_{ij}, \qquad b_{i4} = -b_{4i} = iu_i, \qquad b_{44} = 1. \tag{19}$$

The coefficients of the combined transformation are

$$c_{\mu\nu} = b_{\mu\lambda}a_{\lambda\nu}, \tag{20}$$

so that

$$\begin{aligned} c_{4j} &= b_{4i}a_{ij}+b_{44}a_{4j} \\ &= -iu_i\left(\delta_{ij}+\frac{v_i-v_j}{v^2}(\beta-1)\right)-i\beta v_j \\ &= -i\beta(u_j+v_j)+i\frac{\beta-1}{v^2}(v_iv_iu_j-u_iv_iv_j), \end{aligned} \tag{21}$$

$$c_{j4} = i\beta(u_j+v_j), \tag{22}$$

and

$$c_{44} = \beta(1+u_jv_j). \tag{23}$$

Since the condition for the absence of spatial rotation is still

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$$c_{j4} = -c_{4j},$$

it follows that successive transformations of the above type induce a rotation of the space axes of the original coordinate system. This rotation is of the amount (from (21))

$$\frac{\beta-1}{v^2} \frac{v_i v_i u_j - u_i v_i j_j}{1+u_i v_j} = \frac{\beta-1}{\beta v^2} \frac{\underline{v}_\wedge(\underline{u}_\wedge \underline{v})}{1+\underline{u}\cdot\underline{v}}. \tag{24}$$

For infinitesimal $\underline{u} = d\underline{u}$, this gives the precession operator

$$-\frac{\beta-1}{\beta v^2}\underline{v}_\wedge d\underline{u},$$

or the well known Thomas precession

$$-\frac{\underline{v}_\wedge\dot{\underline{v}}}{2}, \tag{25}$$

when v is such smaller than the speed of light.

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Summary

An elementary, purely algebraic derivation of the most general Lorentz transformations without spacial rotation and of the Thomas precession is given.

References

[1] C. Möller, *The Theory of Relativity* (Oxford, 1952).
[2] G. Stephenson and C. W. Kilmister, *Special Relativity for Physicists* (London, 1958).
[3] W. H. Furry, *Am. J. Phys.* 23 (8) (1955), 517—25.

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