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Extension of the Riemann ξ-Function's Logarithmic Derivative Positivity Region to Near the Critical Strip

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Abstract. If K is a number field with $n_k = [k:\mathbb{Q}]$, and ξ_k the symmetrized Dedekind zeta function of the field, the inequality

$$\Re \frac{\xi_k'(\sigma+it)}{\xi_k(\sigma+it)} > \frac{\xi_k'(\sigma)}{\xi_k(\sigma)}$$

for $t \neq 0$ is shown to be true for $\sigma \geq 1 + 8/n_k^{\frac{1}{3}}$ improving the result of Lagarias where the constant in the inequality was 9. In the case $k = \mathbb{Q}$ the inequality is extended to $\sigma \geq 1$ for all t sufficiently large or small and to the region $\sigma \geq 1 + 1/(\log t - 5)$ for all $t \neq 0$. This answers positively a question posed by Lagarias.

1 Introduction

The Riemann ξ function is $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$. In [8] Lagarias shows that, assuming the Riemann hypothesis,

$$\Re \frac{\xi'(\sigma+it)}{\xi(\sigma+it)} > \frac{\xi'(\sigma)}{\xi(\sigma)}$$

for all $\sigma > 1/2$ and for all $t \neq 0$. He also shows that this inequality holds unconditionally in case $\sigma \geq 10$ and remarks that it seems likely the inequality could be established unconditionally for $\sigma > 1 + \epsilon$ for any given fixed positive ϵ "by a finite computation".

The significance of the inequality is that the Riemann hypothesis is equivalent to the statement

$$\Re \frac{\xi'(s)}{\xi(s)} > 0 \text{ when } \Re s > \frac{1}{2}$$

(see [6, 8] and the use of an assumption weaker than the Rieman hypothesis [5]), so an approach to a proof of the Riemann hypothesis requires extending the positivity region of the $\xi(s)$ function to the left.

Lagarias does quite a lot besides addressing positivity for the logarithmic derivative of the Riemann zeta function. For example if k is a number field of degree n_K over $\mathbb{Q}$, then provided $\sigma \geq 1+9/n_k^{1/3}$ the infinum of $\Re\, \xi_k'(s)/\xi_k(s)$ on the vertical line $s = \sigma$ is attained at $t = 0$. Here $\xi_k(s)$ is the product of the appropriate zeta function

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for the number field ζ_k multiplied by an entire function (the gamma factor) which ensures it is entire and satisfies a functional equation.

In this paper we show, using essentially the same approach as Lagarias but summing an infinite series in terms of the polygamma function, that the constant can be reduced to $7.71542\cdots < 8$. This is Theorem 3.1

In Theorem 3.2 we derive the Riemann zeta form ($k = \mathbb{Q}$) of the inequality of Lagarias unconditionally up to $\sigma = 1$, for sufficiently small or large t, and for mid-range t to $\sigma \geq 1 + 1/(\log|t| - 5)$. This will be Theorem 3.2, following six lemmas. *Sufficiently small* means up to a value of t which satisfies $|t| \leq \sqrt{2-\sqrt{2}}\gamma$, where γ is the y-coordinate of the first off critical line non-trivial zero of $\zeta(s)$. *Sufficiently large* means greater than $e^{(e^{16c_1^3})}$, where c_1 is the absolute constant appearing in an inequality for the logarithmic derivative of $\zeta(s)$. This is presumably a very large number, a "finite" computation, but far beyond the reach of anything practical.

The new technique for the "sufficiently small" region involves structuring and bounding the derivative of a term, from the Mittag–Leffler expansion for the logarithmic derivative of $\xi(s)$, which consists of a sum of terms from four related off-critical line zeros.

2 Preliminary Lemmas

Lemma 2.1 *There exists an absolute constant c_1 such that for all $\sigma \geq 1$ and all $t \geq t_1 > 0$*

$$\Big|\frac{\zeta'(s)}{\zeta(s)}\Big| \leq c_1(\log t)^{2/3}(\log\log t)^{1/3}.$$

Proof This follows from Richert [10] or Cheng [3]. See also [11, Section 6.19]. ■

Lemma 2.2 *For $\sigma > 1$ let*

$$f(\sigma) := \frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{\sigma-1} - \gamma_o,$$

where γ_o is Euler's constant. Then there exists a positive absolute constant c_2 such that

$$-c_2(\sigma-1) < f(\sigma) < 0,$$

and c_2 can be taken to be $\gamma_o^2 - 2\gamma_1$ where

$$\gamma_1 = -\lim_{N\to\infty}\Big(\sum_{m=2}^{N}\frac{\log m}{m} - \frac{\log^2 N}{2}\Big) = -0.07235\cdots,$$

so $c_2 = 0.47789\cdots$.

Proof Write

$$f(\sigma) = \frac{1}{\sigma-1} - \gamma_o - \sum_{p,m\geq 1}\frac{\log p}{p^{m\sigma}},$$

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so

$$f'(\sigma) = \sum_{p,m\geq 1} \frac{m \log^2 p}{p^{m\sigma}} - \frac{1}{(\sigma-1)^2}$$
$$= \sum_p \log^2 p \sum_{m\geq 1} \frac{m}{(p^\sigma)^m} - \frac{1}{(\sigma-1)^2}$$
$$= \sum_p \frac{\log^2 p . p^\sigma}{(p^\sigma - 1)^2} - \frac{1}{(\sigma-1)^2}.$$

Hence

$$f''(\sigma) = \frac{2}{(\sigma-1)^3} + \sum_p \frac{\log^3 p . p^\sigma}{(p^\sigma-1)^2} - 2\sum_p \frac{\log^3 p . p^\sigma}{(p^\sigma-1)^3}$$
$$= \frac{2}{(\sigma-1)^3} + \frac{\log^3 2 . 2^\sigma (2^\sigma - 3)}{(2^\sigma-1)^3} + \sum_{p\geq 3} \frac{\log^3 p . p^\sigma (p^\sigma - 3)}{(p^\sigma-1)^3}.$$

If $\sigma \geq \log_2 3$, each term is non-negative, so $f''(\sigma) > 0$. If $1 < \sigma < \log_2 3$, the sum of the first two terms is positive, so in all cases $f''(\sigma) > 0$. Hence $f(\sigma)$ is concave upwards on $(1, \infty)$.

Now the Laurent expansion of $\zeta(s)$ in the neighborhood of $s = 1$ [7, Theorem 1.4] is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \cdots$$

where, for $k \geq 0$

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{n\to\infty} \Big(\sum_{m=1}^{n} \frac{\log^k m}{m} - \frac{\log^{k+1} n}{k+1} \Big),$$

so γ_0 is Euler's constant and $\gamma_1 < 0$. From this it follows that in a neighborhood of $\sigma = 1$

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{\sigma-1} - \gamma_o = (2\gamma_1 - \gamma_0^2)(\sigma-1) + O(|\sigma-1|^2),$$

so $f'(1) = 2\gamma_1 - \gamma_0^2$. Therefore, by the concavity of $f(\sigma)$, $f(\sigma) > (2\gamma_1 - \gamma_0^2)(\sigma - 1)$ for $\sigma > 1$.

Now, by continuous extension, $f(1) = 0$ and $f'(1) < 0$. If there was a value $\sigma > 1$ with $f(\sigma) \geq 0$ then, by Rolle's theorem, there would be a value with $f'(\sigma) = 0$ and so, since $\lim_{\sigma\to\infty} f(\sigma) = -\gamma_0$, a point with $f''(\sigma) = 0$. But by what we have proved this is impossible. Hence $f(\sigma) < 0$ for all $\sigma > 1$. ■

Note that the constant c_2 is the best possible, since it is the absolute value of the slope of $f(\sigma)$ at $\sigma = 1$. Note also the interesting inequality of Delange [4], which apparently can be extended to about $\sigma = 0.9184\cdots$, *i.e.,* to the left of the line $\sigma = 1$.

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Lemma 2.3 *If* $1 \le \sigma < 10$ *and* $t \ge t_2 > 0$:

$$\Re\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)} \ge \log\frac{t}{2} - 2 - \frac{2}{5t^2}.$$

Proof This follows directly using the asymptotic expression

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + R, |R| \le \frac{1}{10|z|^2}, |z| \ge 2, \Re z > 0.$$

and the bound

$$\Big|\frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)}\Big| \le 2$$

which holds for $1 \le \sigma \le 10$. ■

Lemma 2.4 *Let* $\sigma \ge 1$, $0 < \beta < 1/2, \gamma > 0$ *be real numbers and define*

$$h(t) := \frac{(\gamma - t)^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + (\gamma - t)^2)((\sigma + b - 1)^2 + (\gamma - t)^2)},$$

and $f(t) := h(t) + h(-t)$. *Let* $c_4 = \sqrt{2 - \sqrt{2}}$. *Then for all* t *with* $|t| < c_4\gamma$, $f(t) > f(0)$.

Proof Define $u := \sigma - \beta$ and $v := \sigma + \beta - 1$. Then $u > v > 0$ and we can write

$$h(t) = \frac{(\gamma - t)^2 + uv}{((\gamma - t)^2 + u^2)((\gamma - t)^2 + v^2)} = \frac{1}{u + v}\Big(\frac{u}{(\gamma - t)^2 + u^2} + \frac{v}{(\gamma - t)^2 + v^2}\Big).$$

Then

$$\begin{aligned} f(t) &= h(t) + h(-t) \\ &= \frac{1}{u(u + v)}\Big(\frac{1}{(\gamma - t)^2/u^2 + 1} + \frac{1}{(\gamma + t)^2/u^2 + 1}\Big) \\ &\quad + \frac{1}{v(u + v)}\Big(\frac{1}{(\gamma - t)^2/v^2 + 1} + \frac{1}{(\gamma + t)^2/v^2 + 1}\Big). \end{aligned}$$

Let

$$g_\gamma(t) := \frac{1}{(\gamma - t)^2 + 1} + \frac{1}{(\gamma + t)^2 + 1}.$$

Then the derivative

$$g'_\gamma(t) = \frac{4t(-t^4 - 2(1 + \gamma^2)t^2 + (3\gamma^4 + 2\gamma^2 - 1))}{((\gamma - t)^2 + 1)^2((\gamma + t)^2 + 1)^2}$$

and $g'_\gamma(0) = 0$, g'_γ is an odd function of t, and the numerator is positive if

$$0 < t < (\gamma^2 + 1)^{1/4}(2\gamma - (\gamma^2 + 1)^{1/2})^{1/2},$$

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or for the slightly smaller but more convenient range $0 < t < \sqrt{2-\sqrt{2}}\gamma = c_4\gamma$, so $g_\gamma'(t) > 0$ in this range. Hence

$$f'(t) = \frac{1}{u^2(u+v)} g'_{\frac{\gamma}{u}}\left(\frac{t}{u}\right) + \frac{1}{v^2(u+v)} g'_{\frac{\gamma}{v}}\left(\frac{t}{v}\right)$$

is positive for t with $0 < t/u < c_4\gamma/u$ and $0 < t/v < c_4\gamma/v$, that is the same range as before. Therefore $f(0) < f(t)$. But $f(t)$ is even, so the same inequality holds for t negative also. ■

Lemma 2.5 *Let c_0 be a positive real number representing the y coordinate of the first zeta zero which is off the critical line (assuming such a zero exists). If $0 < t < c_4 c_0$ and $1 \le \sigma < 10$, then*

$$\Re\frac{\xi'(\sigma + \mathrm{i}t)}{\xi(\sigma + \mathrm{i}t)} > \frac{\xi'(\sigma)}{\xi(\sigma)}.$$

Proof With the same notation as in Levinson and Montgomery [9], we can write

$$\Re\frac{\xi'(\sigma + \mathrm{i}t)}{\xi(\sigma + \mathrm{i}t)} - \frac{\xi'(\sigma)}{\xi(\sigma)} = (\sigma - 1/2) I(\sigma, t),$$

where $I(\sigma, t) = T_o + T_1$ and

$$\begin{aligned} T_o = \sum_{\beta<1/2} \Big[&\frac{(\gamma - t)^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + (\gamma - t)^2)((\sigma + \beta - 1)^2 + (\gamma - t)^2)} \\ &- \frac{\gamma^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + \gamma^2)((\sigma + \beta - 1)^2 + \gamma^2)} \Big], \\ T_1 = \sum_{\beta=1/2} \Big[&\frac{1}{(\sigma - 1/2)^2 + (t - \gamma)^2} - \frac{1}{(\sigma - 1/2)^2 + \gamma^2} \Big]. \end{aligned}$$

The proof of Lagarias [8], assuming the Riemann Hypothesis, shows that $T_1 > 0$ whether or not the Riemann hypothesis is assumed to be true. Lemma 2.4 shows that, since each $\gamma \ge c_0$, each term in the sum for T_o is positive for $t < c_4 c_0$ so the Lemma follows directly. ■

Lemma 2.6 *Let $\psi^{(n)}(z)$ be the polygamma function. Then for $n \ge 2$ and all real $x > 0$, $(n-1)! \le -(-1)^n x^n \psi^{(n)}(x)$.*

Proof This follows from an examination of the proof of [1, Theorem 4]. ■

Note that other properties of the digamma and polygamma functions can be found in [2] and the references in that paper.

3 Proofs of the Two Main Results

First we show that the constant in Lagarias' Theorem 1.2 can be reduced from 9 to less than 8.

Theorem 3.1 *Let k be an algebraic number field of degree $n_k = [k, \mathbb{Q}]$. For $\sigma \geq 1 + 8/n_k^{\frac{1}{3}}$ and $t \neq 0$ we have*

$$\Re \frac{\xi_k'(\sigma + it)}{\xi_k(\sigma + it)} > \frac{\xi_k'(\sigma)}{\xi_k(\sigma)}.$$

Proof The proof of Lagarias implies that it suffices to show that for $\sigma_0 = 9$ the inequality

$$n_k \sum_{n=1}^{\infty} \frac{1}{(\sigma_0 + 2n)^3} \geq \frac{1}{(\sigma_0 - 1)^3}$$

holds. The infinite sum can be given explicitly in terms of the polygamma function, so the inequality becomes

$$n_k \frac{1}{16} |\psi^{(2)}(\frac{2 + \sigma_0}{2})| \geq \frac{1}{(\sigma_0 - 1)^3}.$$

By Lemma 2.6, this is true if

$$\frac{(\sigma_0 - 1)^3}{(1 + \sigma_0/2)^2} \geq \frac{16}{n_k}.$$

Lagarias' theorem shows we can assume $\sigma_0 \leq 10$ so, substituting the value $\sigma_0 = 10$ in the denominator, the inequality will hold if

$$(\sigma_0 - 1)^3 \geq \frac{16 \times 6^2}{n_k}.$$

This implies the result is true for $\sigma_0 \geq 1 + \sqrt[3]{576}/n_k^{1/3}$, so we can assume $\sigma_0 \leq 9.32043 \cdots$. Replacing the upper bound 10 with this lower value and iterating the procedure (or equivalently solving the inequality (1) with $n_k = 1$ for the smallest possible value of σ_0), lead to $\sigma_0 = 8.71542$ so

$$\sigma_0 \geq 1 + \frac{7.71542 \cdots}{n_k^{\frac{1}{3}}},$$

and the result of the theorem follows. ■

Note that in his proof, Lagarias neglects the positive contribution to the right hand side of the target inequality when he "shifts the contribution of the poles at

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odd negative integers to the neighboring even negative integers" using his inequality (2.21). In total this contribution is:

$$t^2 \sum_{n\ odd} \frac{1}{(\sigma+n)((\sigma+n)^2+t^2)} - \frac{1}{(\sigma+n+1)((\sigma+n+1)^2+t^2)} =$$

$$2\psi\Big(1+\frac{\sigma}{2}\Big) - 2\psi\Big(\frac{1+\sigma}{2}\Big) + 2\Re\,\psi\Big(\frac{1+\sigma+\mathrm{i}t}{2}\Big) - 2\Re\,\psi\Big(1+\frac{\sigma+\mathrm{i}t}{2}\Big)$$

and as $t \to \infty$ this expression tends to 0, so no improvement in the result can come from this approach.

Note also that if we simply solve the inequality (1) for explicit values of n_k we obtain the values for σ_0:

n_k	σ_0	n_k	σ_0	n_k	σ_0
1	8.71542	2	6.06835	3	5.04472
4	4.4734	5	4.09879	6	3.82964
7	3.62447	8	3.46148	9	3.32799
10	3.21606	11	3.12045	12	3.03753
13	2.96473	14	2.90012	15	2.84228
16	2.7901	17	2.74269	18	2.69938
19	2.65959	20	2.62287	21	2.58885
22	2.5572	23	2.52766	24	2.5
25	2.47403	26	2.44958	27	2.4265
28	2.40468	29	2.38399	30	2.36435

Theorem 3.2 *Let $1 \le \sigma < 10$ and $t \neq 0$. Then there exist absolute constants c_1 and c_2, so that (unconditionally) for $\{\sigma+\mathrm{i}t : |t| \le c_1, 1 \le \sigma < 10\}$ or $\{\sigma+\mathrm{i}t : |t| \ge c_2, 1 \le \sigma < 10\}$ or $\{\sigma+\mathrm{i}t : \log|t| \ge 5+1/(\sigma-1), 1 < \sigma < 10\}$, we have*

$$\Re\frac{\xi'(\sigma+\mathrm{i}t)}{\xi(\sigma+\mathrm{i}t)} > \Re\frac{\xi'(\sigma)}{\xi(\sigma)}.$$

Proof Let $s = \sigma + \mathrm{i}t$ and $1 < \sigma$. Then since $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$, we can write

$$\Delta_0 := \Re\frac{\xi'(s)}{\xi(s)} - \frac{\xi'(\sigma)}{\xi(\sigma)} = \Delta_1+\Delta_2+\Delta_3+\Delta_4, \textit{ where}$$

$$\Delta_1 := -\frac{t^2}{\sigma(\sigma^2+t^2)},$$

$$\Delta_2 := -\frac{t^2}{(\sigma-1)((\sigma-1))^2+t^2)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)},$$

$$\Delta_3 := \Re\frac{\zeta'(s)}{\zeta(s)},$$

$$\Delta_4 := \frac{1}{2}\Big(\Re\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)}\Big).$$

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First, $\Delta_1 > -1/\sigma$. From Lemma 2.2 it follows that

$$\Delta_2 \geq -\gamma_0 + \frac{\sigma - 1}{(\sigma-1)^2 + t^2}.$$

By Lemma 2.1 we can write

$$\Delta_3 = \Re \frac{\zeta'(s)}{\zeta(s)} \geq -c_1 (\log t)^{\frac{2}{3}} (\log\log t)^{\frac{1}{3}}.$$

By Lemma 2.3 we can write $\Delta_4 = \log t - c_5\theta$ for some small positive constant c_5 (we can take $c_5 = 4$) and real θ with $|\theta| < 1$. Hence $\Delta_0 > 0$ if

$$\log t - c_1 (\log t)^{2/3} (\log\log t)^{1/3} > 4.$$

This is true if and only if

$$1 - c_1 \Big(\frac{\log\log t}{\log t}\Big)^{1/3} > \frac{4}{\log t}.$$

If we assume $t \geq t_3 := e^8$, then $1 - 4/\log t \geq 1/2$, so with this restriction we require

$$\frac{\log\log t}{\log t} < \frac{1}{8c_1^3}.$$

This inequality holds if $t \geq t_4 := e^{(e^{16c_1^3})}$. So provided $\gamma_0 c_4 \geq t_4$ the two regions $(0, \gamma_0 c_4]$, $[t_4, \infty)$ overlap and the Lagarias inequality holds for all $\sigma > 1$. If however $t_4 > \gamma_0 c_4$, we argue differently. First let

$$\Delta_2' := -\frac{t^2}{(\sigma-1)((\sigma-1))^2 + t^2)} \quad \text{and} \quad \Delta_3' := \Re\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)},$$

so $\Delta_0 = \Delta_1 + \Delta_2' + \Delta_3' + \Delta_4$. Since $|\frac{\zeta'(s)}{\zeta(s)}| \leq -\frac{\zeta'(\sigma)}{\zeta(\sigma)}$, $\Delta_3' \geq 0$ for all $t \geq 0$. Therefore

$$\Delta_0 > -\frac{1}{\sigma} - \frac{1}{\sigma - 1} + \log t - 4,$$

so $\Delta_0 > 0$ if $\log t > 4 + 1/\sigma + 1/(\sigma - 1)$, and this is true if $\log t > 5 + 1/(\sigma - 1)$. The best uniform value of σ which may be obtained using this method is given approximately by

$$\sigma_0 = 1 + \frac{1}{\log c_4 c_0 - 5}. \qquad \blacksquare$$

If we assume $c_4 c_0 = 10^8$, this leads to $\sigma_0 = 14/13$. Strengthening the above approach to the Lagarias problem requires the derivation of a good explicit value for the constant c_1 (compare [3]) and knowledge of the best current value value for c_0 (currently 3.2×10^9) [12].

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