T. Kubota
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ON A CLASSICAL THETA-FUNCTION

TOMIO KUBOTA

To Professor Katuzi Ono on his 60th birthday

The purpose of this paper is to get a certain explicit expression of automorphic factors, formulated rather differently than usual, of the classically well known theta function[1)]

$$\vartheta(z) = \vartheta_3(0, z) = \sum_{m=-\infty}^{\infty} e^{\pi i m^2 z}, \quad (z = x + iy, \ y > 0). \tag{1}$$

The special linear group $G = SL(2, \boldsymbol{R})$ over the real field $\boldsymbol{R}$ has a 2-fold topological covering group $\tilde{G}$, and the maximal compact subgroup $T = SO(2)$ of G has also a naturally corresponding 2-fold covering group $\tilde{T}$ in $\tilde{G}$. While the upper half plane H is usually identified with the homogeneous space G/T, the properties discussed in §1 of the automorphic factors of $\vartheta(z)$, (13) among others, show directly that for the purpose of investigating $\vartheta(z)$ it is legitimate to identify the upper half plane H with $\tilde{G}/\tilde{T}$. Moreover, as we see in §2, the quadratic reciprocity law in the rational number field $\boldsymbol{Q}$ can be formulated as a multiplicativity of a number-theoretical function defined on a discrete subgroup of $\tilde{G}$. For a totally imaginalry number field this kind of result was already stated in [4] in a simpler form, but in general we need the covering group $\tilde{G}$.

It is famous in number theory that there is a close relationship between the quadratic reciprocity law and the function $\vartheta(z)$[2)]. The investigation in this paper, inclusive of all explicit calculations, may be regarded as a trial to catch as simply as possible the theoretical background of that interesting phenomenon.

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1) Called in many cases theta constant. It is an automorphic form with respect to the discontinuous group Γ defined in §1.

2) For example, see [2].

The contents of the present paper have various connections with [6], but can be read independently.

§1. Automorphic factors of the theta function.

Let Γ be the subgroup of the elliptic modular group $SL(2, \boldsymbol{Z})$ consisting of all $\sigma \in SL(2, \boldsymbol{Z})$ such that $\sigma \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ (mod 2). On the other hand, normalize the square root of a complex number $z \neq 0$ once for all by

$$\sqrt{z} = e^{\frac{1}{2} i \arg z} \sqrt{|z|}, \qquad -\pi \leqq \arg z < \pi. \tag{2}$$

Then, for the theta function in (1), we have

$$\vartheta(z) = \frac{1}{\sqrt{-iz}} \vartheta\left(-\frac{1}{\upsilon}\right) \tag{3}$$

and

$$\vartheta(z) = \vartheta(z+2). \tag{4}$$

The formula (3) is Poisson's summation formula. Since Γ is generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$, a consequence of (3), (4) is

$$\vartheta(z) = \frac{c_\sigma}{\sqrt{cz+d}} \vartheta(\sigma z), \qquad |c_\sigma| = 1, \tag{5}$$

for an arbitrary $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Of course, $\sigma z = \dfrac{az+b}{cz+d}$, c_σ is a constant depending upon σ, and is already studied in classical literatures[3]. But, here we propose to look for a convenient expression of c_σ for our purpose.

PROPOSITION 1. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* Γ *such that* $b \neq 0$ *and* $\sigma \equiv 1$ (mod. 4). *Then the constant* c_σ *in the transformation formula* (5) *is given by* $c_\sigma = (-c, d)\left(\dfrac{2b}{d}\right)$ *for* $c \neq 0$, *and* $c_\sigma = 1$ *for* $c = 0$. *Here,* $\left(\dfrac{2b}{d}\right)$ *is the Jacobi symbol, and* (c, d) *is the Hilbert symbol of degree* 2 *for* $\boldsymbol{R}$.

Proof.[4] Denoting by $\xi = \dfrac{b}{a}$, $(a, b \in \boldsymbol{Z})$, a rational number given by an irreducible fraction, we define a Gauss sum of exponential type by

$$G_0(\xi) = \sum_{c \bmod a} e^{2\pi i \xi c^2}, \qquad (c \in \boldsymbol{Z}), \tag{6}$$

3) [3], for example.

4) This proof is partly identical with the proof in [2] of the reciprocity law of the Gauss sum.

and put

(7) $$G(\xi) = G_0(\xi)/|G_0(\xi)|$$

whenever $G_0(\xi) \neq 0$. Now, if $t > 0$, then

$$\vartheta(2\xi + it) = \sum_{m=-\infty}^{\infty} e^{\pi i m^2(2\xi+it)}$$

$$= \sum_{c \bmod a} e^{2\pi i \xi c^2} \sum_{m=-\infty}^{\infty} e^{-\pi(am+c)^2 t},$$

and Poisson's summation formula yields

$$\sum_{m=-\infty}^{\infty} e^{-\pi(ma+c)^2 t} = \frac{1}{\sqrt{t}} \frac{1}{|a|} \sum_{m=-\infty}^{\infty} e^{-\pi\frac{m^2}{a^2 t} + \frac{2\pi i c}{a} m}.$$

So, we obtain

(8) $$\lim_{t \to 0} \sqrt{t}\ \vartheta(2\xi + it) = G_0(\xi)/|a|$$

If, especially, this is applied to the both sides of (3), the so-called rescipro-city of Gauss sum

(9) $$\frac{G_0(\xi)}{\sqrt{|a|}} = \eta^{\operatorname{sgn}\xi} \frac{\sqrt{2|b|}}{|b_0|} G_0\left(-\frac{1}{4\xi}\right), \qquad \eta = e^{\frac{\pi i}{4}},$$

as stated in [2], Satz 161, is derived, where $\operatorname{sgn}\xi = \xi/|\xi|$. From (9) follows also

(10) $$G(\xi) = \eta^{\operatorname{sgn}\xi} G\left(-\frac{1}{4\xi}\right).$$

Next we put $z = it$ in the formula (5), and use (2), (3), (8) to have

$$1 = c_\sigma(c, d)\, \eta^{\operatorname{sgn} d - 1} G\left(\frac{b}{2d}\right), \quad (c \neq 0).$$

Furthermore, from (10) and from elementary properties of Gauss sums[5] follows

$$G\left(\frac{b}{2d}\right) = G\left(\frac{b/2}{d}\right) = \left(\frac{2b}{d}\right) G\left(\frac{1}{d}\right)$$

$$= \left(\frac{2b}{d}\right) \eta^{\operatorname{sgn} d} G\left(-\frac{d}{4}\right),$$

5) See [1], [2].

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and $d\equiv 1 \pmod 4$ implies $G\left(-\frac{d}{4}\right)=\eta^{-1}$. Hence, $c_\sigma=(-c,\,d)\left(\frac{2b}{d}\right)$ as asserted. If $c=0$, then $d=1$. So, the assertion is claer by (4).

Using fundamental properties of the Jacobi symbol, we can deduce from Proposition 1 immediately the following

COROLLARY. *Let* $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ *be an element of* Γ *such that* $\sigma\equiv 1 \pmod 4$. *Then,* $c_\sigma=(c,d)\left(\frac{2c}{d}\right)$ *for* $c\neq 0$, *and* $c_\sigma=1$ *for* $c=0$.

As shown in [5], the factor system of the 2-fold non-trivial covering group $\tilde{G}$ of $G=SL(2,\boldsymbol{R})$ is given by

$$a(\sigma,\tau)=(x(\sigma),\ x(\tau))\,(-x(\sigma)^{-1}x(\tau),\ x(\sigma\tau)),\ (\sigma,\tau\in G), \tag{11}$$

where, for $\sigma=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\in G$, $x(\sigma)=\gamma$ or δ according to $\gamma\neq 0$ or $=0$. Now, for the square root fixed by (2), the relation

$$\sqrt{c(\tau z)+d}\cdot\sqrt{c'z+d'}=a(\sigma,\tau)\sqrt{c''z+d''} \tag{12}$$

holds with $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\tau=\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}$, $\sigma\tau=\begin{pmatrix} a'' & b'' \\ c'' & d''\end{pmatrix}$, I $x(\sigma)<0$ is equivalent with the fact that $0\leqq\arg(cz+d)<\pi$, resp. $-\pi\leqq\arg(cz+d)<0$ is the case for all z in the upper half plane H. Therefore, if the elements of $\tilde{G}$ are denoted by $\tilde{\sigma}=(\sigma,\varepsilon)$, $(\sigma\in G,\ \varepsilon=\pm 1)$, and the operation of $\tilde{\sigma}$ on a function $f(z)$ of a complex variable is defined by $f^{\tilde{\sigma}}(z)=f(\sigma z)$, then

$$j(\tilde{\sigma},z)=\varepsilon\sqrt{cz+d}$$

becomes an automorphic factor over $\tilde{G}$, that is, j satisfies

$$j(\tilde{\sigma}\tilde{\tau},z)=j^{\tilde{\tau}}(\tilde{\sigma},z)\,j(\tilde{\tau},z),\qquad(\tilde{\sigma},\tilde{\tau}\in G). \tag{13}$$

Thus we get

PROPOSITION 2. *Let* $\tilde{\Gamma}$ *be the covering group of* Γ *determined by the factor set* (11), *denote by* $\tilde{\sigma}=(\sigma,\varepsilon)$, $(\sigma\in\Gamma,\ \varepsilon=\pm 1)$, *an element of* $\tilde{\Gamma}$, *and put* $\chi(\tilde{\sigma})=\chi(\sigma,\varepsilon)=c_\sigma\varepsilon$, c_σ *being as in* (5). *Then* χ *is a representation of degree* 1 *of* $\tilde{\Gamma}$, *i.e., we have* $\chi(\tilde{\sigma}\tilde{\tau})=\chi(\tilde{\sigma})\,\chi(\tilde{\tau})$, $(\tilde{\sigma},\tilde{\tau}\in\tilde{\Gamma})$.

In this way, the automorphic factor in (5) of $\vartheta(z)$ is decomposed into a representation of $\tilde{\Gamma}$ and an automorphic factor of $\tilde{G}$. Making use of this result, the following therorem is proved:

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THEOREM. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* Γ, *and put* $\eta = e^{\frac{\pi i}{4}}$, *Then,* c_σ *in* (5), *or in other words* $\chi(\sigma, 1)$ *in Propositoin* 2, *is given by:*

	conditions on σ		value of $c_\sigma = \chi(\sigma, 1)$
$2\mid c$ *and*			
	$c \neq 0,$	$d \equiv 1 \pmod 4$	$(c, d)\left(\frac{2c}{d}\right)$
	$c \neq 0,$	$d \equiv -1 \pmod 4$	$i(c, d)\left(\frac{2c}{d}\right)$
	$c = 0,$	$d = 1$	1
	$c = 0,$	$d = -1$	$-i$
$2\mid d$ *and*			
	$d \neq 0,$	$c \equiv 1 \pmod 4$	$\eta\left(\frac{2d}{c}\right)$
	$d \neq 0,$	$c \equiv -1 \pmod 4$	$\eta^{-1}\left(\frac{2d}{c}\right)$
	$d = 0,$	$c = 1$	η
	$d = 0,$	$c = -1$	η^{-1}

Proof. If $d \equiv 1 \pmod 4$ and $c \equiv 0 \pmod 4$, then $a \equiv 1 \pmod 4$, and the theorem follows at once from Proposition 1. So, we assume $d \equiv 1$, $c \equiv 2 \pmod 4$. Put $\tau' = \begin{pmatrix} 1 & -2 \\ & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix}$, $\rho = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$; then $\rho\tau = \tau'\rho$, and a $a(\rho, \tau) = 1$, $a(\tau', \rho) = 1$ by (11). Therefore, $\chi(\tau, 1) = 1$, where $(\tau, 1)$ stands for an element of $\tilde{\Gamma}$. Hence, under the additional assumption $c + 2d \neq 0$, Proposition 2 and the results for the case of $c \equiv 0 \pmod 4$ imply

$$\begin{aligned} c_\sigma &= \chi(\sigma, 1) = \chi(\sigma\tau, 1)a(\sigma, \tau) \\ &= (c + 2d, d)\left(\frac{2(c + 2d)}{d}\right)(c, 2)(-2c,\ c + 2d) \\ &= (-2cd,\ c + 2d)^{6)}\left(\frac{2c}{d}\right) = (c,\ 2d)\left(\frac{2c}{d}\right) \\ &= (c, d)\left(\frac{2c}{d}\right). \end{aligned}$$

If $c + 2d = 0$, then d must be 1. So,

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6) Apply here the formula $(a, b)(-a^{-1}b,\ a+b) = 1$ of Hilbert's symbol.

$$c_\sigma = (c, 2)(-2c, d) = 1 = (c, d)\left(\frac{2c}{d}\right).$$

Thus the theorem is verified for the case of $d \equiv 1 \pmod 4$.

Next we put $\tau = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$, $\rho = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ to have $\rho^2 = \tau$, $a(\rho, \rho) = -1$. Since (3) implies $\chi(\rho, 1) = \eta$, $\chi(\tau, 1)$ must be $-i$. Therefore, if $d \equiv -1 \pmod 4$ and $c \neq 0$, then

$$\begin{aligned} c_\sigma &= \chi(\sigma, 1) = \chi(\sigma\tau, 1)\,\chi(\tau, 1)^{-1}\,a(\sigma, \tau) \\ &= i(-c, -d)\left(\frac{-2c}{-d}\right)(-1, c)(c, -c) \\ &= i(-1, -d)(c, d)\left(\frac{-2c}{-d}\right) = i(c, d)\left(\frac{2c}{d}\right). \end{aligned}$$

The assertion for $c = 0$ is almost the same thing as $\chi(\tau, 1) = -i$. Thus the proof for the case of $2|c$ is finished.

If $2|d$ and $d \neq 0$, then $a(\sigma, \rho) = (-c, d)$ for $\rho = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

So,

$$c_\sigma = \chi(\sigma\rho, 1)\,\chi(\rho, 1)^{-1}(-c, d) = \eta^{-1}(-c, d)\,\chi(\sigma\rho, 1),$$

and our assertion reduces to former cases. If $d = 0$, then $c_\sigma = \eta^{-1}(-c, -c)\cdot \chi(\sigma\rho, 1)$, and the theorem is still valid.
This completes the proof.

§2. Remarks on the reciprocity law.

In a previous paper [4], the author has shown that the reciprocity law of the power residue symbol of an arbitrary degree in a totally imaginary number field is essentially equivalent with the multiplicativity of a function defined by means of the power residue symbol on an arithmetically defined discontinuous subgroup of $SL(2, \boldsymbol{C})$. For the rational number field, a corresponding result is stated in Proposition 2 of this paper using the quadratic residue symbol which is the only residue symbol of a number field with real conjugates. Proposition 2 shows that, whenever a number field has a real conjugate, $SL(2, \boldsymbol{R})$ is not enough to get a corresponding result to the theorem of [4] for the number field, but we must use the covering group $\tilde{G}$ which is not an algebraic group. Although Proposition 2 concerns only the rational number field, the situation is not completely different for the

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general case; we merely need such a theta function of several variables as is used in the integral representation of Dedeking's zeta function instead of the theata function in (1), to have a generalization of Proposition 2, i.e., a result like the theorem of [4].

As well as the theorem of [4] is proved by an elementary computation, it is possible to see the equivalence of Proposition 2 and the quadratic reciprocity law directly without any analytic function. For example, put $\sigma = \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}$, $(c \neq 0)$, $\tau = \begin{pmatrix} 1 & 2m \\ & 1 \end{pmatrix}$. Since then $a(\sigma, \tau) = 1$, the relation $\chi(\tilde{\sigma}\tilde{\tau}) = \chi(\tilde{\sigma})\chi(\tilde{\tau})$ together with Theorem 1 yields

$$(c, d)\left(\frac{c}{d}\right) = (c, d + 4cm)\left(\frac{c}{d + 4cm}\right),$$

which is a somewhat non-explicit formulation of the quadratic reciprocity. Conversely, assuming the quadratic recipricity, we can prove Proposition 2 by the method in [4]. The procedure becomes, however, rather complicated. In this manner one can any way understand the mechanism of the so-called analytic proof of the reciprocity law.

Proposition 2 gives various different forms, or formal generalizations, of the quadratic recipirocity law. Put for instance $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $\sigma \equiv \tau \equiv 1 \pmod 4$, $c \neq 0$, $c' \neq 0$, $ca' + dc' \neq 0$. Then, Proposition 2 entails

$$(c, d)\left(\frac{2c}{d}\right) \cdot (c', d')\left(\frac{2c'}{d'}\right)$$

$$= (ca' + dc',\ cb' + dd')\left(\frac{2(ca' + dc')}{cb' + dd'}\right) \cdot (c, c')(-cc', ca' + dc').$$

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Mathematical Institute,
Nagoya University.