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A NEW UPPER BOUND FOR $|\zeta(1 + it)|$

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Abstract

It is known that $\zeta(1 + it) \ll (\log t)^{2/3}$ when $t \gg 1$. This paper provides a new explicit estimate $|\zeta(1 + it)| \le \frac{3}{4}\log t$, for $t \ge 3$. This gives the best upper bound on $|\zeta(1 + it)|$ for $t \le 10^{2\cdot 10^5}$.

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1. Introduction

For $s = \sigma + it$ and $\sigma > 1$ one defines the Riemann zeta function to be $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. The zeta function can be continued analytically to the entire complex plane with the exception of the solitary point $s = 1$. For more properties on $\zeta(s)$ the reader is referred to [7, Ch. 2].

Mellin [5] (see also [7, Theorem 3.5]) was the first to show that

$$\zeta(1 + it) \ll \log t. \tag{1.1}$$

This was improved by Littlewood (see [7, Theorem 5.16]) to

$$\zeta(1 + it) \ll \frac{\log t}{\log\log t}. \tag{1.2}$$

This was improved in turn by several authors; the best known result (see [7, Equation (6.19.2)]) is

$$\zeta(1 + it) \ll (\log t)^{2/3}. \tag{1.3}$$

As usual, the Riemann hypothesis gives a stronger result, $\zeta(1 + it) \ll \log\log t$ when $t \gg 1$ (see [7, Section 14.18]).

As far as explicit results are concerned, Backlund [1] made (1.1) explicit by proving that

$$|\zeta(1 + it)| \le \log t, \tag{1.4}$$

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for $t \geq 50$. Ford [3] has made (1.3) explicit by proving that

$$|\zeta(1+it)| \leq 72.6(\log t)^{2/3}, \tag{1.5}$$

for $t \geq 3$. Ford's result is actually much more general: he obtains excellent bounds for $|\zeta(\sigma+it)|$ where σ is near 1. Should one be interested in a bound only on $\sigma = 1$, one can improve on (1.5) slightly. The integral inequality on [3, page 622], originally verified for $y \geq 0$, can now be evaluated at $y = 0$ only. This shows that $|\zeta(1+it)| \leq 62.6(\log t)^{2/3}$. Note that this improves on (1.4) when $t \geq 10^{10^5}$. Without a complete overhaul of Ford's paper it seems unlikely that his methods could furnish a bound superior to (1.4) when t is at all modest, say $t \leq 10^{100}$.

To the knowledge of the author there is no explicit bound of the form (1.2). One could follow the arguments of [7, Section 5.16] to produce such a bound, though this leads to a result that only improves on (1.4) when t is astronomically large. However, one can still use the ideas in [7, Section 5.16] to re-prove (1.1). Indeed, if one were lucky, as the author was, one may even be able to supersede (1.4). This fortune is summarised in the following theorem.

Theorem 1.1. *When* $t \geq 3$,

$$|\zeta(1+it)| \leq \tfrac{3}{4}\log t.$$

Good explicit bounds on $|\zeta(1+it)|$ enable one to bound the zeta function more effectively throughout the critical strip. Since, for $\sigma > 1$,

$$|\zeta(\sigma+it)| = \left|\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} = \zeta(\sigma),$$

one has a bound for the zeta function to the right of the line $\sigma = 1$. By the functional equation (see [7, Section 2.1]) this bounds the zeta function to the left of the line $\sigma = 0$. One may now apply the Phragmen–Lindelöf theorem to bound $\zeta(\sigma+it)$ in $-\eta \leq \sigma \leq 1+\eta$ for some fixed positive η. This leads to a bound of the type $\zeta(\sigma+it) \ll t^{(1+\eta-\sigma)/2}$. This bound throws away rather a lot of information since we know that $\zeta(1+it) \ll \log t$.

It is better to bound $\zeta(\sigma+it)$ for $-\eta \leq \sigma \leq 1+\eta$ by dividing this strip into three strips

$$\{s : -\eta \leq \sigma \leq 0\} \cup \{s : 0 \leq \sigma \leq 1\} \cup \{s : 1 \leq \sigma \leq 1+\eta\}$$

and applying the bound on $\zeta(1+it)$, and that of $\zeta(it)$, obtained from the functional equation, on each strip. Indeed, Theorem 1.1 has been used in [8] to improve the estimate on $\zeta(s)$ for $-\eta \leq \sigma \leq 1+\eta$.

Throughout this paper $\lfloor x \rfloor$ and $\{x\}$ denote respectively the integer part and the fractional part of x.

2. Backlund's result

To prove (1.4) consider $\sigma > 1$ and $t > 1$, and write $\zeta(s) - \sum_{n\leq N} n^{-s} = \sum_{N<n} n^{-s}$. Now invoke the following version of the Euler–Maclaurin summation formula—this can be found in [6, Theorem 2.19].

Lemma 2.1 (Euler–Maclaurin summation). *Let k be a nonnegative integer and $f(x)$ be $k+1$ times differentiable on the interval $[a,b]$. Then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\,dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}(f^{(r)}(b) - f^{(r)}(a))B_{r+1} + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(x) f^{(k+1)}(x)\,dx,$$

where $B_j(x)$ is the jth periodic Bernoulli polynomial and $B_j = B_j(0)$.

Apply this to $f(n) = n^{-s}$, with $k=1$, $a=N$ and with b dispatched to infinity. Thus

$$\zeta(s) - \sum_{n\leq N-1} n^{-s} = \frac{N^{1-s}}{s-1} + \frac{1}{2N^s} + \frac{s}{12N^{s+1}} - \frac{s(s+1)}{2}\int_N^\infty \frac{\{x\}^2 - \{x\} + \frac{1}{6}}{x^{s+2}}\,dx, \quad (2.1)$$

where, since the right-hand side converges for $Re(s) > -1$, the equation remains valid when $s = 1+it$. Hence one can estimate the sum in (2.1) using

$$\sum_{n\leq N} \frac{1}{n} \leq \log N + \gamma + \frac{1}{N}, \quad (2.2)$$

which follows from partial summation, and in which γ denotes Euler's constant. Now if $N = \lfloor t/m \rfloor$, where m is a positive integer to be chosen later, (2.1) and (2.2) combine to show that

$$|\zeta(1+it)| - \log t \leq -\log m + \gamma + \frac{1}{t} + \frac{m}{2(t-m)} + \frac{m^2(1+t)(4+t)}{24(t-m)^2}. \quad (2.3)$$

The aim is to choose m and t_0 such that $t \geq t_0$ guarantees the right-hand side of (2.3) to be negative. It is easy to verify that when $m=3$, choosing $t = 49.385\ldots$ suffices. Thus (1.4) is true for all $t \geq 50$; a quick computation shows that (1.4) remains true for $t \geq 2.001\ldots$.

It seems impossible to improve upon (1.4) without a closer analysis of sums of the form $\sum_{a<n\leq 2a} n^{-it}$. Taking further terms in the Euler–Maclaurin expansion in (2.1) does not achieve an overall saving; choosing $N = \lfloor t^\alpha \rfloor$ for some $\alpha < 1$ in (2.2) means that the integral in (2.1) is no longer bounded when $t \to \infty$.

The next section aims to secure a good bound for $\sum_{a<n\leq 2a} n^{-it}$ for 'large' values of a. For 'small' values of a one may estimate the sum trivially. The inherent optimism is that, when combined, these two estimates give an improvement on (1.4).

3. Exponential sums: beyond Backlund

The following is an explicit version of [7, Theorem 5.9].

Lemma 3.1 (Cheng and Graham). *Assume that $f(x)$ is a real-valued function with two continuous derivatives when $x \in (a,c]$. If there exist two real numbers $V < W$*

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with $W > 1$ such that

$$\frac{1}{W} \le |f''(x)| \le \frac{1}{V}$$

for $x \in [a+1, c]$, then

$$\Big|\sum_{a<n\le c} e^{2\pi i f(n)}\Big| \le \frac{1}{5}\left(\frac{c-a}{V}+1\right)(8W^{1/2}+15).$$

Proof. See [2, Lemma 3]. □

Applying Lemma 3.1 to $f(x) = -(2\pi)^{-1} t \log x$ gives

$$\max_{a<c\le 2a}\Big|\sum_{a<n\le c} n^{-it}\Big| \le t^{1/2}\left(\frac{8}{5}\sqrt{\frac{2}{\pi}} + \frac{16\sqrt{2\pi}a}{5t} + \frac{3t^{1/2}}{2\pi a} + 3t^{-1/2}\right), \tag{3.1}$$

subject to $2\pi a^2 > t$. Imposing that $2\pi a^2 > t$ is to ensure that, in Lemma 3.1, $W > 1$; see (4.2). Now take $A_1 t^{1/2} < a \le \lfloor t/m \rfloor$ for some constant A_1 and positive integer m to be determined later. To ensure that this is a nonempty interval, see (4.2). If $t \ge t_0$ then (3.1) shows that

$$\max_{a<c\le 2a}\Big|\sum_{a<n\le c} n^{-it}\Big| \le A_2 t^{1/2},$$

and hence, by partial summation,

$$\Big|\sum_{a<n\le 2a} n^{-1-it}\Big| \le A_2 a^{-1} t^{1/2} \le \frac{A_2}{A_1}, \tag{3.2}$$

where

$$A_2 = \frac{8}{5}\sqrt{\frac{2}{\pi}} + \frac{16\sqrt{2\pi}}{5m} + \frac{3}{2\pi A_1} + 3t_0^{-1/2}.$$

One may now apply (3.2) to each of the sums on the right-hand side of

$$\Big|\sum_{A_1 t^{1/2} < n \le (t/m)} \frac{1}{n^{1+it}}\Big| = \sum_{\frac{1}{2}(t/m)<n\le(t/m)} + \sum_{\frac{1}{4}(t/m)<n\le\frac{1}{2}(t/m)} + \cdots .$$

There are at most

$$\frac{\frac{1}{2}\log t - \log(mA_1) + \log 2}{\log 2} \tag{3.3}$$

such sums. This gives an upper bound for $\sum n^{-1-it}$ when $n > A_1 t^{1/2}$. When $n \le A_1 t^{1/2}$ one may use (2.2) to estimate the sum trivially.

4. Proof of Theorem 1.1

In $\zeta(s) - \sum_{n\leq N} n^{-s} = \sum_{N<n} n^{-s}$ use Lemma 2.1 and expand to k terms. Choosing $N-1 = \lfloor t/m \rfloor$, recalling (3.2) and (3.3), and estimating all complex terms trivially gives

$$|\zeta(1+it)| \leq \log t \left(\frac{1}{2} + \frac{A_2}{2A_1 \log 2}\right) + \frac{A_2(\log 2 - \log(mA_1))}{A_1 \log 2} + \log A_1 + \gamma$$
$$+ \frac{1}{A_1 t_0^{1/2}} + \frac{m}{2t} + \frac{1}{t} + \sum_{r=1}^{k} \frac{|B_{r+1}|}{(r+1)!}(1+t)\cdots(r+t)\left(\frac{m}{t}\right)^{r+1} \tag{4.1}$$
$$+ \frac{(1+t)\cdots(k+1+t)}{(k+1)\cdot(k+1)!} \max|B_{k+1}(x)| \left(\frac{m}{t}\right)^{k+1}.$$

Note that each term in the r-sum in (4.1) is $O_{m,k}(t^{-1})$. This is cheap relative to the last term which is $O_{m,k}(1)$. Thus one can take k somewhat large to reduce the burden of the final term. For a given t_0, when $t \geq t_0$ one can optimise (4.1) over k, m and A_1 subject to

$$A_1 > \frac{1}{\sqrt{2\pi}}, \quad mA_1 \leq t_0^{1/2}. \tag{4.2}$$

One finds that, when $k = 14, m = 6, A_1 = 23$ then $|\zeta(1+it)| \leq 0.749818\ldots$, for all $t \geq 10^8$. A numerical check on *Mathematica* suffices to extend the result to all $t \geq 2.391\ldots$, whence Theorem 1.1 follows.

4.1. Improvements. Lemma 3.1 is unable to furnish a value less than $\frac{1}{2}$ in Theorem 1.1. On the other hand, by verifying that $|\zeta(1+it)| < \frac{1}{2}\log t$ for t larger than 10^8 one will improve slightly on Theorem 1.1.

One could also take an analogue of Lemma 3.1 that incorporates higher derivatives. Such a result, giving explicit bounds on exponential sums of a function involving k derivatives, is given in [4, Proposition 8.2]. It is unclear how much could be gained from pursuing this idea.

References

[1] R. J. Backlund, 'Sur les zéros de la fonction $\zeta(s)$ de Riemann', *C. R. Math. Acad. Sci. Paris* **158** (1914), 1979–1982.

[2] Y. F. Cheng and S. W. Graham, 'Explicit estimates for the Riemann zeta function', *Rocky Mountain J. Math.* **34**(4) (2004), 1261–1280.

[3] K. Ford, 'Vinogradov's integral and bounds for the Riemann zeta function', *Proc. Lond. Math. Soc.* **85**(3) (2002), 565–633.

[4] A. Granville and O. Ramaré, 'Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients', *Mathematika* **43**(1) (1996), 73–197.

[5] Hj. Mellin, 'Eine Formel für den Logarithmus transcendenter Funktionen von endlichem Geschlecht', *Acta Soc. Sci. Fenn.* **24**(4) (1902), 1–50.

[6] M. Ram Murty, *Problems in Analytic Number Theory*, Graduate Texts in Mathematics, 206 (Springer, New York, 2008).

https://doi.org/10.1017/S0004972713000415 Published online by Cambridge University Press

[7] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd edn. (Oxford University Press, Oxford, 1986).
[8] T. S. Trudgian, 'An improved upper bound for the argument of the Riemann zeta-function on the critical line II'. Preprint.

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