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The Cubic Dirac Operator for Infinite-Dimensonal Lie Algebras

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Abstract. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be an infinite-dimensional graded Lie algebra, with $\dim \mathfrak{g}_i < \infty$, equipped with a non-degenerate symmetric bilinear form B of degree 0. The quantum Weil algebra $\widehat{\mathcal{W}}\mathfrak{g}$ is a completion of the tensor product of the enveloping and Clifford algebras of $\mathfrak{g}$. Provided that the Kac–Peterson class of $\mathfrak{g}$ vanishes, one can construct a cubic Dirac operator $\mathcal{D} \in \widehat{\mathcal{W}}(\mathfrak{g})$, whose square is a quadratic Casimir element. We show that this condition holds for symmetrizable Kac–Moody algebras. Extending Kostant's arguments, one obtains generalized Weyl–Kac character formulas for suitable "equal rank" Lie subalgebras of Kac–Moody algebras. These extend the formulas of G. Landweber for affine Lie algebras.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra equipped with a non-degenerate invariant symmetric bilinear form B. For $\xi \in \mathfrak{g}$, the corresponding generators of the enveloping algebra $U(\mathfrak{g})$ are denoted $s(\xi)$, while those of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ are denoted simply by ξ. The *quantum Weil algebra* [1] is the super algebra

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}),$$

with even generators $s(\xi)$ and odd generators ξ. Let $\mathcal{D} \in \mathcal{W}(\mathfrak{g})$ be the odd element written in terms of a basis e_a of $\mathfrak{g}$ as

$$\mathcal{D} = \sum_a s(e_a)e^a - \frac{1}{12}\sum_{abc} f_{abc}e^a e^b e^c,$$

where e^a is the B-dual basis and f_{abc} are the structure constants. The key property of this element is that its square lies in the center of $\mathcal{W}(\mathfrak{g})$:

$$\mathcal{D}^2 = \mathrm{Cas}_{\mathfrak{g}} + \frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}}),$$

where $\mathrm{Cas}_{\mathfrak{g}} = \sum_a s(e_a)s(e^a) \in U(\mathfrak{g})$ is the quadratic Casimir element. The element $\mathcal{D}$ is called the *cubic Dirac operator*, following Kostant [10]. More generally, Kostant introduced cubic Dirac operators $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ for pairs of a quadratic Lie algebra $\mathfrak{g}$ and a quadratic Lie subalgebra $\mathfrak{u}$. For $\mathfrak{g}$ semi-simple and $\mathfrak{u}$ an equal rank subalgebra, he used this to prove, among other things, generalizations of the Bott–Borel–Weil theorem and of the Weyl character formula (see also [2, 11]).

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In this article, we will consider generalizations of this theory to infinite-dimensional Lie algebras. We assume that $\mathfrak{g}$ is $\mathbb{Z}$-graded, with finite dimensional graded pieces $\mathfrak{g}_i$, and equipped with a non-degenerate invariant symmetric bilinear form B of degree 0. A priori, the formal expressions defining $\mathcal{D}$, $\mathrm{Cas}_\mathfrak{g}$ are undefined, since they involve infinite sums. It is possible to replace these expressions with "normal-ordered" sums, leading to well-defined elements $\mathcal{D}', \mathrm{Cas}'_\mathfrak{g}$ in suitable completion of $\mathcal{W}(\mathfrak{g})$. However, it is no longer true in general that $(\mathcal{D}')^2 - \mathrm{Cas}'_\mathfrak{g}$ is a constant, and in any case $\mathrm{Cas}'_\mathfrak{g}$ is not a central element. One may attempt to define elements $\mathcal{D}$, $\mathrm{Cas}_\mathfrak{g}$ having these properties by adding lower order correction terms to $\mathcal{D}', \mathrm{Cas}'_\mathfrak{g}$. Our main observation is that this is possible if and only if the *Kac–Peterson class* $[\psi_{KP}] \in H^2(\mathfrak{g})$ is zero. In fact, given $\rho \in \mathfrak{g}_0^*$ with $\psi_{KP} = \mathrm{d}\rho$, the elements $\mathcal{D} = \mathcal{D}' + \rho$ and $\mathrm{Cas}_\mathfrak{g} = \mathrm{Cas}'_\mathfrak{g} + 2\rho$ have the desired properties. These results are motivated by the work of Kostant-Sternberg [12], who had exhibited the Kac–Peterson class as an obstruction class in their BRST quantization scheme.

For symmetrizable Kac–Moody algebras, the existence of a corrected Casimir element $\mathrm{Cas}_\mathfrak{g}$ is a famous result of Kac [4]. In particular, $[\psi_{KP}] = 0$ in this case. As we will see, Kostant's theory carries over to the symmetrizable Kac–Moody case in a fairly straightforward manner. For suitable "regular" Kac–Moody subalgebras $\mathfrak{u} \subset \mathfrak{g}$, we thus obtain generalized Weyl–Kac character formulas as sums over multiplets of $\mathfrak{u}$-representations.

For *affine* Lie algebras or loop algebras, similar Dirac operators were described in Kac–Todorov [7] and Kazama-Suzuki [8], and more explicitly in Landweber [14] and Wassermann [19]. In fact, Wassermann uses this Dirac operator to give a proof of the Weyl-Kac character formula for affine Lie algebras, while Landweber proves generalized Weyl character formulas for "equal rank loop algebras". The cubic Dirac operator $\mathcal{D}$ for general symmetrizable Kac–Moody algebras is discussed very briefly in Kitchloo [9].

2 Completions

In this section we will define completions of the exterior and Clifford algebras of a graded quadratic vector space. We recall from [6] how the Kac–Peterson cocycle appears in this context.

2.1 Kac–Peterson Cocycle

Let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded vector space over $\mathbb{C}$, with finite-dimensional graded components. The (graded) dual space is the direct sum over the duals of V_i, with grading $(V^*)_i = (V_{-i})^*$. Given another graded vector space V' with $\dim V'_i < \infty$, we let $\mathrm{Hom}(V, V')$ be the direct sum over the spaces $\mathrm{Hom}(V, V')_i = \bigoplus_r \mathrm{Hom}(V_r, V'_{r+i})$ of finite rank maps of degree i. We let

$$\widehat{\mathrm{Hom}}(V, V')_i = \prod_r \mathrm{Hom}(V_r, V'_{r+i})$$

be the space of *all* linear maps $V \to V'$ of degree i, and $\widehat{\mathrm{Hom}}(V, V')$ their direct sum. If $V = V'$, we write $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ and $\widehat{\mathrm{End}}(V) = \widehat{\mathrm{Hom}}(V, V)$. Note that

$\widehat{\mathrm{End}}(V)$ is an algebra with unit I.

Define a splitting $V = V_- \oplus V_+$, where $V_+ = \bigoplus_{i>0} V_i$, $V_- = \bigoplus_{i\leq 0} V_i$. Denote by π_-, π_+ the projections to the two summands. The *Kac–Peterson cocycle* ([6]; see also [5, Exercise 7.28]) on $\widehat{\mathrm{End}}(V)$ is a Lie algebra cocycle given by the formula,

$$\psi_{KP}(A_1, A_2) = \tfrac{1}{2}\operatorname{tr}(A_1\pi_- A_2\pi_+) - \tfrac{1}{2}\operatorname{tr}(A_2\pi_- A_1\pi_+). \tag{2.1}$$

This is well defined since the compositions $\pi_- A_i \pi_+ \colon V \to V$ have finite rank. Observe that ψ_{KP} has degree 0; that is, (2.1) vanishes unless the degrees of A_1, A_2 add up to zero. On the Lie subalgebra $\mathrm{End}(V) \subset \widehat{\mathrm{End}}(V)$, the Kac–Peterson cocycle restricts to a coboundary:

$$\psi_{KP}(A_1, A_2) = \tfrac{1}{2}\operatorname{tr}(\pi_+[A_1, A_2]). \tag{2.2}$$

2.2 Completion of Symmetric and Exterior Algebras

Let $S(V)$ be the symmetric algebra of V, with $\mathbb{Z}$-grading defined by assigning degree i to generators in V_i. Let V^* be the graded dual as above. The pairing between $S(V)$ and $S(V^*)$ identifies $S(V)_i$ as a subspace of the space of linear maps $S(V^*)_{-i} \to \mathbb{K}$. We define a completion $\widehat{S}(V)_i$ as the space of all linear maps $S(V^*)_{-i} \to \mathbb{K}$. Equivalently,

$$\widehat{S}(V)_i = \prod_{r\geq 0} S(V_-)_{i-r} \otimes S(V_+)_r.$$

We let $\widehat{S}(V)$ be the direct sum over the $\widehat{S}(V)_i$. The multiplication map of $S(V)$ extends to the completion, making $\widehat{S}(V)$ into a $\mathbb{Z}$-graded algebra. For each $k \geq 0$ one similarly has a completion $\widehat{S}^k(V) \subset \widehat{S}(V)$ of each component $S^k(V)$. Then $\widehat{S}(V)_i$ is the direct product over all $\widehat{S}^k(V)_i$. The space $\widehat{S}^2(V^*)_0$ may be identified with the space of symmetric bilinear maps $B\colon V \times V \to \mathbb{C}$ of degree 0; that is, $B(V_i, V_j) = 0$ for $i + j \neq 0$.

In a similar fashion, one defines a completions $\widehat{\wedge}(V)_i$ as the spaces of all linear maps $\widehat{\wedge}(V^*)_{-i} \to \mathbb{K}$, or equivalently

$$\widehat{\wedge}(V)_i = \prod_{r\geq 0} \wedge(V_-)_{i-r} \otimes \wedge(V_+)_r.$$

We let $\widehat{\wedge}(V)$ be the $\mathbb{Z}$-graded super algebra given as the direct sum over all $\widehat{\wedge}(V)_i$. Again, one also has completions of the individual $\wedge^k(V)$. The space $\widehat{\wedge}^2(V^*)_0$ may be identified with the skew-symmetric bilinear maps $V \times V \to \mathbb{C}$ of degree 0. In particular:

$$\psi_{KP} \in \widehat{\wedge}^2\big(\widehat{\mathrm{End}}(V)^*\big)_0.$$

2.3 Clifford Algebras

Suppose B is a (possibly degenerate) symmetric bilinear form on $V = \bigoplus_i V_i$ of degree 0. Let $\mathrm{Cl}(V)$ be the corresponding Clifford algebra, *i.e.,* the super algebra with odd generators $v \in V$ and relations $vw + wv = 2\,B(v, w)$ for $v, w \in V$. The $\mathbb{Z}$-grading on V defines a $\mathbb{Z}$-grading on $\mathrm{Cl}(V)$ compatible with the algebra structure.

Using the restrictions of the bilinear form to $V_\pm$, we may similarly form the Clifford algebras $\mathrm{Cl}(V_\pm)$. These are $\mathbb{Z}$-graded subalgebras of $\mathrm{Cl}(V)$, and the multiplication map defines an isomorphism of super vector spaces, $\mathrm{Cl}(V) \cong \mathrm{Cl}(V_-) \otimes \mathrm{Cl}(V_+)$. Note that $\mathrm{Cl}(V_+) = \wedge(V_+)$, since B restricts to 0 on V_+.

We obtain a $\mathbb{Z}$-graded superalgebra $\widehat{\mathrm{Cl}}(V)$ as the direct sum over all

$$\widehat{\mathrm{Cl}}(V)_i = \prod_{r \geq 0} \mathrm{Cl}(V_-)_{i-r} \otimes \mathrm{Cl}(V_+)_r.$$

Let $q^0 \colon \wedge(V) \to \mathrm{Cl}(V)$ denote the standard quantization map for the Clifford algebra, defined by super symmetrization:

$$q^0(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)},$$

where $\mathfrak{S}_k$ is the permutation group on k elements, and $\mathrm{sign}(\sigma) = \pm 1$ is the parity of the permutation σ. The map q^0 is an isomorphism of super spaces preserving the $\mathbb{Z}$-gradings and taking $\wedge(V_\pm)$ to $\mathrm{Cl}(V_\pm)$. While q^0 itself does not extend to the completions, we obtain a well-defined *normal-ordered quantization map*

$$q \colon \widehat{\wedge}(V) \to \widehat{\mathrm{Cl}}(V)$$

by taking the direct sum over $i \in \mathbb{Z}$ and direct product over $r \geq 0$ of

$$q^0 \otimes q^0 \colon \wedge(V_-)_{i-r} \otimes \wedge(V_+)_r \to \mathrm{Cl}(V_-)_{i-r} \otimes \mathrm{Cl}(V_+)_r.$$

The quantization map is an isomorphism of $\mathbb{Z}$-graded super vector spaces, with the property that for $\lambda \in \widehat{\wedge}^k(V)$, $\mu \in \widehat{\wedge}^l(V)$,

$$q^{-1}\big(q(\lambda)q(\mu)\big) = \lambda \wedge \mu \mod \widehat{\wedge}^{k+l-2}(V).$$

Any element $v \in V$ defines an odd derivation ι_v, called a *contraction*, of the super algebra $\wedge(V)$, given on generators by $\iota_v(w) = B(v, w)$. The same formula also defines a derivation of the Clifford algebra, again denoted ι_v. In both cases, the contractions extend to the completions. The map $q \colon \widehat{\wedge}(V) \to \widehat{\mathrm{Cl}}(V)$ intertwines contractions:

$$q \circ \iota_v = \iota_v \circ q,$$

since $q^0 \circ \iota_v = \iota_v \circ q^0$ and since contractions preserve $\wedge(V_\pm)$ and $\mathrm{Cl}(V_\pm)$.

Let $\mathfrak{o}(V) \subset \mathrm{End}(V)$ and $\widehat{\mathfrak{o}}(V) \subset \widehat{\mathrm{End}}(V)$ denote the B-skew-symmetric endomorphisms. Let

$$\widehat{\wedge}^2(V) \to \widehat{\mathfrak{o}}(V), \quad \lambda \mapsto A_\lambda \tag{2.3}$$

be the map defined by $A_\lambda(v) = -2\iota_v\lambda$. The map (2.3) is $\widehat{\mathfrak{o}}(V)$-equivariant; that is, $A_{L_X\lambda} = [X, A_\lambda]$ for $X \in \widehat{\mathfrak{o}}(V)$.

Lemma 2.1 *For all* $\lambda \in \wedge^2(V)$, $q(\lambda) = q^0(\lambda) - \frac{1}{2}\,\mathrm{tr}(\pi_+ A_\lambda)$.

Proof It suffices to check for elements of the form $\lambda = u \wedge v$ for $u, v \in V$. We have $A_{u\wedge v}(w) = 2(B(v,w)u - B(u,w)v)$, hence $\mathrm{tr}(\pi_+ A_{u\wedge v}) = 2(B(\pi_+ u, v) - B(\pi_+ v, u))$. On the other hand, by considering the special cases that u, v are both in V_-, both in V_+, or $u \in V_-$, $v \in V_+$, we find

$$q(u \wedge v) = q^0(u \wedge v) + B(\pi_+ v, u) - B(\pi_+ u, v). \tag{2.4}$$ ■

The map q^0 is $\mathfrak{o}(V)$-equivariant. For the normal-ordered quantization map this is no longer the case.

Proposition 2.2 (Kac–Peterson [6]) *For all* $\lambda \in \widehat{\wedge}^2(V)$ *and* $X \in \widehat{\mathfrak{o}}(V)$, *one has*

$$L_X q(\lambda) = q(L_X \lambda) + \psi_{KP}(X, A_\lambda).$$

Proof It is enough to prove this for $X \in \mathfrak{o}(V)$ and $\lambda \in \wedge^2(V)$. Since q^0 intertwines Lie derivatives, Lemma 2.1 and (2.2) give

$$L_X q(\lambda) - q(L_X\lambda) = \tfrac{1}{2}\,\mathrm{tr}(\pi_+ A_{L_X\lambda}) = \tfrac{1}{2}\,\mathrm{tr}\big(\pi_+[X, A_\lambda]\big) = \psi_{KP}(X, A_\lambda).$$ ■

If B is non-degenerate, the map $\lambda \mapsto A_\lambda$ defines an isomorphism $\wedge^2(V) \to \mathfrak{o}(V)$. Let

$$\lambda\colon \mathfrak{o}(V) \to \wedge^2(V),\ A \mapsto \lambda(A)$$

be the inverse map. It extends to a map $\widehat{\mathfrak{o}}(V) \to \widehat{\wedge}^2(V)$ of the completions. In a basis e_a of V with B-dual basis e^a (*i.e.*, $B(e_a, e^b) = \delta_a^b$), one has

$$\lambda(A) = \frac{1}{4}\sum_a A(e_a) \wedge e^a.$$

If $A \in \mathfrak{o}(V)$, the elements $\gamma^0(A) = q^0(\lambda(A))$ are defined. As is well known, $[\gamma^0(A_1), \gamma^0(A_2)] = \gamma^0([A_1, A_2])$ for $A_i \in \mathfrak{o}(V)$, and $L_A = [\gamma^0(A), \cdot\,]$. If $A \in \widehat{\mathfrak{o}}(V)$, one still has $L_A = [\gamma'(A), \cdot\,]$ with $\gamma'(A) = q(\lambda(A))$, but the map γ' is no longer a Lie algebra homomorphism. Instead, Proposition 2.2 shows

$$\big[\gamma'(A_1), \gamma'(A_2)\big] = \gamma'\big([A_1, A_2]\big) + \psi_{KP}(A_1, A_2) \tag{2.5}$$

for $A_1, A_2 \in \widehat{\mathfrak{o}}(V)$.

3 Graded Lie Algebras

We will now specialize to the case that $V = \mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra. We show that in the quadratic case, the obstruction to defining a reasonable "Casimir operator" is precisely the Kac–Peterson class of $\mathfrak{g}$.

3.1 Kac–Peterson Cocycle of $\mathfrak{g}$

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a graded Lie algebra with $\dim \mathfrak{g}_i < \infty$. That is, we assume that the grading is compatible with the bracket: $[\mathfrak{g}_i, \mathfrak{g}_j]_\mathfrak{g} \subset \mathfrak{g}_{i+j}$. The map $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ defines a homomorphism of graded Lie algebras $\mathrm{ad}: \mathfrak{g} \to \widehat{\mathrm{End}}(\mathfrak{g})$. Recall that $\mathfrak{g}^* = \bigoplus_i (\mathfrak{g}^*)_i$ denotes the restricted dual where $(\mathfrak{g}^*)_i = (\mathfrak{g}_{-i})^*$. The algebra $\wedge(\mathfrak{g}^*)$ carries contraction operators and Lie derivatives ι_ξ, L_ξ for $\xi \in \mathfrak{g}$, given on generators by $\iota_\xi \mu = \langle \mu, \xi \rangle$ and $L_\xi \mu = (-\mathrm{ad}_\xi)^* \mu$. If $\dim \mathfrak{g} < \infty$, it also carries a differential d, given on generators by $\mathrm{d}\mu = 2\lambda(\mu)$, where $\lambda(\mu)$ is defined by $\iota_\xi \lambda(\mu) = \frac{1}{2} L_\xi \mu$. On generators,

$$(\mathrm{d}\mu)(\xi_1, \xi_2) = -\langle \mu, [\xi_1, \xi_2]_\mathfrak{g} \rangle.$$

In the infinite-dimensional case, $\lambda(\mu)$ and hence d are well defined on the completion $\widehat{\wedge}(\mathfrak{g}^*)$. The operators ι_ξ, L_ξ, d make $\widehat{\wedge}(\mathfrak{g}^*)$ into a $\mathfrak{g}$-differential algebra.

Define

$$\psi_{KP}(\xi_1, \xi_2) := \psi_{KP}(\mathrm{ad}_{\xi_1}, \mathrm{ad}_{\xi_2})$$

for $\xi_i \in \mathfrak{g}$. Thus $\psi_{KP} \in \widehat{\wedge}^2(\mathfrak{g}^*)_0$ is a degree 2 Lie algebra cocycle of $\mathfrak{g}$ called the *Kac–Peterson cocycle of* $\mathfrak{g}$. Its class $[\psi_{KP}] \in H^2(\mathfrak{g})$ will be called the Kac–Peterson class of the graded Lie algebra $\mathfrak{g}$. Note that d has $\mathbb{Z}$-degree 0, so that it restricts to a differential on each $\widehat{\wedge}(\mathfrak{g}^*)_i$. Hence, if ψ_{KP} admits a primitive in $\mathfrak{g}^*$, then it admits a primitive in $\mathfrak{g}_0^*$.

Example 3.1 ([6]) Suppose $\mathfrak{k}$ is a finite-dimensional Lie algebra, and let $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ be the loop algebra with its usual $\mathbb{Z}$-grading. Let $B^{\mathrm{Kil}}(x, y) = \mathrm{tr}_\mathfrak{k}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in \mathfrak{k}$ be the Killing form on $\mathfrak{k}$. One finds

$$\psi_{KP}(\xi, \zeta) = \mathrm{Res}\ B^{\mathrm{Kil}}\Big(\frac{\partial \xi}{\partial z}, \zeta\Big)$$

for $\xi, \zeta \in \mathfrak{k}[z, z^{-1}]$, where Res picks out the coefficient of z^{-1}. One may check that unless $B\mathrm{Kil} = 0$, the Kac–Peterson class $[\psi_{KP}]$ is non-zero.

Example 3.2 (Heisenberg algebra) Let $\mathfrak{g}$ be the Lie algebra having basis $K, e_1, f_1, e_2, f_2, \ldots$, where K is a central element and $[e_i, f_j]_\mathfrak{g} = \delta_{ij} K$. Define a grading on $\mathfrak{g}$ such that e_i has degree i and f_i has degree $-i$, while K has degree 0. One finds $\psi_{KP} = 0$.

Example 3.3 Suppose $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ and a system $\Delta^+ \subset \mathfrak{h}^*$ of positive roots. Let $\mathfrak{g}$ carry the principal grading, *i.e.*, $\mathfrak{g}_0 = \mathfrak{h}$, while $\mathfrak{g}_i, i \neq 0$ is the direct sum of root spaces for roots of height i. Using (2.2) one finds that $\psi_{KP} = \mathrm{d}\rho$, where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

3.2 Enveloping Algebras

The $\mathbb{Z}$-grading on $\mathfrak{g}$ defines a $\mathbb{Z}$-grading on the enveloping algebra $U(\mathfrak{g})$. Both $\mathfrak{g}_+ = \bigoplus_{i>0} \mathfrak{g}_i$ and $\mathfrak{g}_- = \bigoplus_{i\leq 0} \mathfrak{g}_i$ are graded Lie subalgebras, thus $U(\mathfrak{g}_\pm)$ are graded subalgebras of $U(\mathfrak{g})$. By the Poincaré–Birkhoff–Witt theorem, the multiplication map

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defines an isomorphism of vector spaces $U(\mathfrak{g}) = U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+)$. We define a completion $\widehat{U}(\mathfrak{g})$ as a direct sum over

$$\widehat{U}(\mathfrak{g})_i = \prod_{r \geq 0} U(\mathfrak{g}_-)_{i-r} \otimes U(\mathfrak{g}_+)_r.$$

The multiplication map extends to the completion, making $\widehat{U}(\mathfrak{g})$ into a graded algebra. Let $q^0 \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ be the isomorphism given by the standard (PBW) symmetrization map

$$q^0(\xi_1 \cdots \xi_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}.$$

This preserves $\mathbb{Z}$-degrees and takes $S(\mathfrak{g}_\pm)$ to $U(\mathfrak{g}_\pm)$. While the map itself does not extend to the completions, we define a normal-ordered symmetrization (quantization) map $q \colon \widehat{S}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ by taking the direct sum over i and the direct product over r of the maps

$$q^0 \otimes q^0 \colon S(\mathfrak{g}_-)_{i-r} \otimes S(\mathfrak{g}_+)_r \to U(\mathfrak{g}_-)_{i-r} \otimes U(\mathfrak{g}_+)_r.$$

Then q is an isomorphism of $\mathbb{Z}$-graded vector spaces. Let

$$S^2(\mathfrak{g}) \to \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}), \ p \mapsto A_p$$

be the linear map given for $p = uv, \ u, v \in \mathfrak{g}$ by

$$A_p(\mu) = \langle \mu, u \rangle v + \langle \mu, v \rangle u.$$

It extends to a $\mathfrak{g}$-equivariant linear map $\widehat{S}^2(\mathfrak{g}) \to \widehat{\operatorname{Hom}}(\mathfrak{g}^*, \mathfrak{g})$. Let

$$\mathrm{br} \colon \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}) \to \mathfrak{g}$$

be the linear map given by the identification $\operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}$ followed by the Lie bracket. In a basis e_a of $\mathfrak{g}$ with dual basis $e^a \in \mathfrak{g}^*$, $\mathrm{br}(A) = \sum_a [A(e^a), e_a]_\mathfrak{g}$. The counterpart to Lemma 2.1 reads as follows.

Lemma 3.4 *For* $p \in S^2(\mathfrak{g})$, $q(p) = q^0(p) - \frac{1}{2} \mathrm{br}(\pi_+ A_p)$.

Proof It suffices to check for $p = uv$, where the formula reduces to (cf. (2.4))

$$q(uv) = q^0(uv) + \tfrac{1}{2}[u, \pi_+ v]_\mathfrak{g} + \tfrac{1}{2}[v, \pi_+ u]_\mathfrak{g}.$$

This is straightforward in each of the cases that u, v are both in $\mathfrak{g}_+$, both in $\mathfrak{g}_-$, or $u \in \mathfrak{g}_+, v \in \mathfrak{g}_-$. ■

In contrast to q^0, the map q is not $\mathfrak{g}$-equivariant. Similar to Proposition 2.2 we have the following proposition.

Proposition 3.5 *On* $\widehat{S}^2(\mathfrak{g})$,

$$L_\xi(q(p)) - q(L_\xi(p)) = \tfrac{1}{2} \mathrm{br}\left((\pi_+ \operatorname{ad}_\xi \pi_- - \pi_- \operatorname{ad}_\xi \pi_+) A_p \right).$$

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The right-hand side is well defined, because $\pi_- \operatorname{ad}_\xi \pi_+$ and $\pi_+ \operatorname{ad}_\xi \pi_-$ are in $\operatorname{Hom}(\mathfrak{g}, \mathfrak{g})$, and hence $(\pi_+ \operatorname{ad}_\xi \pi_- - \pi_- \operatorname{ad}_\xi \pi_+)A_p \in \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g})$.

Proof It suffices to verify this for $p \in S^2(\mathfrak{g})$, so that A_p has finite rank. Since $L_\xi q^0(p) - q^0(L_\xi p) = 0$, Lemma 3.4 gives

$$\begin{aligned} L_\xi q(p) - q(L_\xi p) &= -\tfrac{1}{2}\big(L_\xi \operatorname{br}(\pi_+ A_p) - \operatorname{br}(\pi_+ A_{L_\xi p})\big) \\ &= -\tfrac{1}{2} \operatorname{br}\big([L_\xi, \pi_+ A_p] - \pi_+[L_\xi, A_p]\big) \\ &= -\tfrac{1}{2} \operatorname{br}\big(L_\xi \pi_+ A_p - \pi_+ L_\xi A_p\big) \\ &= \tfrac{1}{2} \operatorname{br}\big((\pi_+ L_\xi \pi_- - \pi_- L_\xi \pi_+) A_p\big). \end{aligned}$$

■

3.3 Quadratic Lie Algebras

We assume that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ comes equipped with a non-degenerate ad-invariant symmetric bilinear form B of degree 0. Thus, $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for $i + j \neq 0$, while B defines a non-degenerate pairing between $\mathfrak{g}_i, \mathfrak{g}_{-i}$. We will often use B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. The examples we have in mind are the following:

(i) Let $\mathfrak{k}$ be a finite-dimensional Lie algebra with an invariant symmetric bilinear form $B_\mathfrak{k}$. Then B extends to an inner product on the loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$.

(ii) Let $\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$ be a graded Lie algebra, with finite-dimensional homogeneous components, and let $\mathfrak{l}^* = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i^*$ be its restricted dual, with grading $(\mathfrak{l}^*)_i = \mathfrak{l}^*_{-i}$. The semi-direct product $\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{l}^*$, with B given by the pairing, satisfies our assumptions. This case was studied by Kostant and Sternberg in [12].

(iii) Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a symmetrizable Kac–Moody Lie algebra with grading the principal grading (defined by the height of roots). Then $\mathfrak{g}$ carries a "standard" non-degenerate, invariant, symmetric, bilinear form; see [5]. We will return to the Kac–Moody case in Section 7.

Under the identification $\widehat{\wedge}^2(\mathfrak{g}) \cong \widehat{\mathfrak{o}}(\mathfrak{g})$, the Kac–Peterson cocycle ψ_{KP} corresponds to an element

$$\Psi_{KP} \in \widehat{\mathfrak{o}}(\mathfrak{g}), \quad \psi_{KP}(\xi, \zeta) = B(\Psi_{KP}(\xi), \zeta).$$

Since ψ_{KP} has $\mathbb{Z}$-degree 0, the transformation Ψ_{KP} preserves each $\mathfrak{g}_i$. Since ψ_{KP} is a cocycle, Ψ_{KP} is a derivation of the Lie bracket on $\mathfrak{g}$. Moreover, ψ_{KP} is a coboundary if and only if the derivation Ψ_{KP} is inner:

$$\psi_{KP} = \mathrm{d}\rho \;\Leftrightarrow\; \Psi_{KP} = [\rho^\sharp, \cdot\,]_\mathfrak{g}, \tag{3.1}$$

where $\rho^\sharp$ is the image of $\rho \in \mathfrak{g}_0^*$ under the isomorphism $B^\sharp \colon \mathfrak{g}^* \to \mathfrak{g}$.

Example 3.6 Let $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ with $\mathfrak{k}$ semi-simple and with bilinear form defined in terms of the Killing form on $\mathfrak{k}$ as $B(\xi, \zeta) = \operatorname{Res}(z^{-1} B^{\mathrm{Kil}}(\xi, \zeta))$ for $\xi, \zeta \in \mathfrak{k}[z, z^{-1}]$. Then Ψ_{KP} is the degree operator $\Psi_{KP}(\xi) = z\frac{\partial \xi}{\partial z}$.

3.4 Casimir Elements

Let $p \in \widehat{S}^2(\mathfrak{g})$ be the element $p = \sum_a e_a e^a \in \widehat{S}^2(\mathfrak{g})$, where e_a is a homogeneous basis of $\mathfrak{g}$, with B-dual basis e^a. The corresponding transformation $A_p \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$ is $2\,\mathrm{Id}_{\mathfrak{g}}$. We refer to $\mathrm{Cas}'_{\mathfrak{g}} = q(p) \in \widehat{U}(\mathfrak{g})$ as the *normal-ordered Casimir element*. It is not an element of the center, in general:

Theorem 3.7 *The normal-ordered Casimir element satisfies $L_\xi \mathrm{Cas}'_{\mathfrak{g}} = 2\Psi_{KP}(\xi)$ for all $\xi \in \mathfrak{g}$.*

Proof From the definition of br, one finds $B(\mathrm{br}(A), \zeta) = \mathrm{tr}(\mathrm{ad}_\zeta A)$ for all $A \in \mathrm{End}(\mathfrak{g})$ and $\zeta \in \mathfrak{g}$. Since $A_p = 2\,\mathrm{Id}_{\mathfrak{g}}$ and $L_\xi p = 0$, Proposition 3.5 therefore gives

$$\begin{aligned} B(L_\xi \mathrm{Cas}'_{\mathfrak{g}}, \zeta) &= B(\mathrm{br}(\pi_+ \mathrm{ad}_\xi \pi_- - \pi_- \mathrm{ad}_\xi \pi_+), \zeta) \\ &= \mathrm{tr}(\mathrm{ad}_\zeta \pi_+ \mathrm{ad}_\xi \pi_- - \mathrm{ad}_\zeta \pi_- \mathrm{ad}_\xi \pi_+) \\ &= 2\psi_{\mathrm{KP}}(\xi, \zeta) = 2B(\Psi_{\mathrm{KP}}(\xi), \zeta). \end{aligned}$$ ∎

The normal-ordered Casimir element $\mathrm{Cas}'_{\mathfrak{g}}$ admits a linear correction to a central element if and only if the Kac–Peterson class is zero.

Corollary 3.8 *For $\rho \in \mathfrak{g}_0^*$, $\mathrm{Cas}_{\mathfrak{g}} := \mathrm{Cas}'_{\mathfrak{g}} + 2\rho^\sharp$ lies in the center of $\widehat{U}(\mathfrak{g})$ if and only if $\psi_{KP} = d\rho$.*

Proof This is a direct consequence of Theorem 3.7, since $\psi_{KP} = \mathrm{d}\rho$ if and only if $L_\xi \rho^\sharp = -\Psi_{KP}(\xi)$, see Equation (3.1). ∎

Example 3.9 For a loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ with $\mathfrak{k}$ a semi-simple Lie algebra, the Kac–Peterson coycle of $\mathfrak{g}$ defines a non-trivial cohomology class. Hence it is impossible to make $\mathrm{Cas}'_{\mathfrak{g}}$ invariant by adding linear terms. On the other hand, for a symmetrizable Kac–Moody algebra $\mathfrak{g}$, a classical result of Kac shows that $\mathrm{Cas}'_{\mathfrak{g}}$ becomes invariant after a ρ-shift. Hence the Kac–Peterson class of such a $\mathfrak{g}$ is trivial. See Section 7.

3.5 The Structure Constants Tensor and its Quantization

Recall the definition of $\lambda\colon \widehat{\mathfrak{o}}(\mathfrak{g}) \to \widehat{\wedge}^2(\mathfrak{g})$. We will write $\lambda(\xi) = \lambda(\mathrm{ad}_\xi)$; that is, $\iota_\xi \lambda(\zeta) = \frac{1}{2}[\xi, \zeta]_{\mathfrak{g}}$. In a basis e_a of $\mathfrak{g}$, with B-dual basis e^a, we have $\lambda(\xi) = \frac{1}{4}\sum_a [\xi, e_a]_{\mathfrak{g}} \wedge e^a$.

Lemma 3.10 *There is a unique element $\phi \in \widehat{\wedge}^3(\mathfrak{g})_0$ with the property*

$$\iota_{\xi_1}\iota_{\xi_2}\iota_{\xi_3}\phi = \tfrac{1}{2}B\big([\xi_1, \xi_2]_{\mathfrak{g}}, \xi_3\big), \quad \xi_1, \xi_2, \xi_3 \in \mathfrak{g}.$$

Proof The right-hand side is a skew-symmetric trilinear form of degree 0 on $\mathfrak{g}$. Hence it defines an element of $\widehat{\wedge}^3(\mathfrak{g})$. ∎

Equivalently, $\iota_\xi \phi = 2\lambda(\xi),\ \ \xi \in \mathfrak{g}$. In a basis,

$$\phi = -\frac{1}{12}\sum_{abc} f_{abc} e^a \wedge e^b \wedge e^c,$$

where $f_{abc} = B([e_a, e_b]_{\mathfrak{g}}, e_c)$ are the structure constants.

From the definition, it is clear that ϕ is $\mathfrak{g}$-invariant. This need no longer be true of its normal-ordered quantization. Write

$$\gamma'(\xi) = q(\lambda(\xi)), \quad \phi'_{\mathrm{Cl}} = q(\phi),$$

so that $L_\xi = [\gamma'(\xi), \cdot)]$. Denote by $\psi^\sharp_{KP} \in \widehat{\wedge}^2(\mathfrak{g})$ the image of $\psi_{KP} \in \widehat{\wedge}^2(\mathfrak{g}^*)$ under the isomorphism $B^\sharp\colon \widehat{\wedge}(\mathfrak{g}^*) \to \widehat{\wedge}(\mathfrak{g})$.

Proposition 3.11 *The element $\phi'_{\mathrm{Cl}} \in \widehat{\mathrm{Cl}}(\mathfrak{g})$ satisfies $L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi)$, and its square is given by the formula*

$$(\phi'_{\mathrm{Cl}})^2 = q(\psi^\sharp_{KP}) + \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).$$

Here $\mathrm{Cas}_{\mathfrak{g}_0} \in U(\mathfrak{g}_0)$ is the quadratic Casimir element for $\mathfrak{g}_0$, and $\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0})$ is its trace in the adjoint representation.

Proof The first formula follows from the second, since

$$L_\xi \phi'_{\mathrm{Cl}} = [\gamma'(\xi), \phi'_{\mathrm{Cl}}] = \iota_\xi (\phi'_{\mathrm{Cl}})^2.$$

Since

$$\iota_\xi(\phi'_{\mathrm{Cl}})^2 = [\gamma'(\xi), \phi'_{\mathrm{Cl}}] = L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi) = \iota_\xi q(\psi^\sharp_{KP}),$$

the difference $(\phi'_{\mathrm{Cl}})^2 - q(\psi^\sharp_{KP})$ is a constant. Let ϕ_r be the component of ϕ in $(\wedge\mathfrak{g}_-)_{-r} \otimes (\wedge\mathfrak{g}_+)_r$. The commutator of ϕ'_{Cl} with a term $q(\phi_r)$ for $r > 0$ is contained in the right ideal generated by $\mathfrak{g}_+$, and hence does not contribute to the constant. Hence the constant equals $q(\phi_0)^2$, where $\phi_0 \in \wedge^3\mathfrak{g}_0$ is the structure constants tensor of $\mathfrak{g}_0 \subset \mathfrak{g}$. By [1, 10] this constant is given by $\frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0})$. ∎

Corollary 3.12 *Suppose $\psi_{KP} = d\rho$ for some $\rho \in \mathfrak{g}_0^*$. Define elements of $\widehat{\mathrm{Cl}}(\mathfrak{g})$ by*

$$\phi_{\mathrm{Cl}} := \phi'_{\mathrm{Cl}} + \rho^\sharp, \ \ \gamma(\xi) = \gamma'(\xi) + \langle \rho, \xi \rangle,$$

for $\xi \in \mathfrak{g}$. The following commutator relations hold in $\widehat{\mathrm{Cl}}(\mathfrak{g})$:

$$[\xi, \zeta] = 2B(\xi, \zeta), \qquad [\gamma(\xi), \phi_{\mathrm{Cl}}] = 0,$$
$$[\xi, \phi_{\mathrm{Cl}}] = 2\gamma(\xi), \qquad [\gamma(\xi), \gamma(\zeta)] = \gamma([\xi, \zeta]_{\mathfrak{g}}),$$
$$[\gamma(\xi), \zeta] = [\xi, \zeta]_{\mathfrak{g}}, \qquad [\phi_{\mathrm{Cl}}, \phi_{\mathrm{Cl}}] = 2B(\rho^\sharp, \rho^\sharp) + \tfrac{1}{12}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).$$

Thus $\widehat{\mathrm{Cl}}(\mathfrak{g})$ becomes a $\mathfrak{g}$-differential algebra (see e.g., [16]*) with differential $d = [\phi_{\mathrm{Cl}}, \cdot\,]$, contractions $\iota_\xi = \frac{1}{2}[\xi, \cdot\,]$, and Lie derivatives $L_\xi = [\gamma(\xi), \cdot\,]$.*

Proof Observe first that $\lambda(\rho^\sharp) = -\psi_{KP}$, since

$$\iota_\zeta \iota_\xi \lambda(\rho^\sharp) = \iota_\zeta [\xi, \rho^\sharp]_{\mathfrak{g}} = B\big(\zeta, [\xi, \rho^\sharp]_{\mathfrak{g}}\big) = -\big\langle \rho, [\xi, \zeta]_{\mathfrak{g}} \big\rangle.$$

Consequently $[\rho^\sharp, \phi'_{\mathrm{Cl}}] = -q(\psi_{KP})$, which implies the formula for $[\phi_{\mathrm{Cl}}, \phi_{\mathrm{Cl}}]$. The other assertions are verified similarly. ∎

Still assuming $\psi_{KP} = \mathrm{d}\rho$, consider the algebra morphism

$$\gamma\colon U(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g}) \tag{3.2}$$

extending the Lie algebra homomorphism $\xi \mapsto \gamma(\xi)$.

Proposition 3.13 *The map* (3.2) *extends to an algebra morphism* $\gamma\colon \widehat{U}(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g})$.

Proof We claim that for all $i > 0$, $\gamma(\mathfrak{g}_i)$ is contained in

$$\coprod_{r\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r} \mathrm{Cl}(\mathfrak{g}_+)_{i+r} \subset \widehat{\mathrm{Cl}}(\mathfrak{g})_i \tag{3.3}$$

(*i.e.*, the components in $\mathrm{Cl}(\mathfrak{g}_+)$ have degree $\geq i$). Indeed, suppose $\xi \in \mathfrak{g}_i$ with $i > 0$. In particular, $\langle \rho, \xi \rangle = 0$. Let $e_a \in \mathfrak{g}$ be a basis consisting of homogeneous elements, and let e^a be the dual basis. Since $\langle \rho, \xi \rangle = 0$, and since $[\xi, e_a]_{\mathfrak{g}}$ Clifford commutes with e^a, we have

$$\gamma(\xi) = \frac{1}{2} \sum\nolimits_+ \big([\xi, e^a] e_a - e^a [\xi, e_a] \big) + \frac{1}{4} \sum\nolimits_0 [\xi, e_a] e^a,$$

where $\sum_+$ is a summation over indices with $e_a \in \mathfrak{g}_+$, and $\sum_0$ is a summation over indices with $e_a \in \mathfrak{g}_0$. The second and third terms in this expression are in (3.3) as are the summands $[\xi, e^a] e_a$ from the first sum for $e_a \in \mathfrak{g}_s$ with $s \geq i$. In the remaining case $s < i$ we have $[\xi, e^a] \in \mathfrak{g}_{i-s} \subset \mathfrak{g}_+$, and hence $[\xi, e^a] e_a \in \mathrm{Cl}(\mathfrak{g}_+)_i$. This proves the claim. By induction, one deduces that

$$\gamma\big(U(\mathfrak{g}_+)_i\big) \subset \prod_{r\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r} \mathrm{Cl}(\mathfrak{g}_+)_{i+r}.$$

Similarly, if $j \leq 0$,

$$\gamma\big(U(\mathfrak{g}_-)_j\big) \subset \prod_{r\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{j-r} \mathrm{Cl}(\mathfrak{g}_+)_r.$$

It follows that

$$\gamma\big(U(\mathfrak{g}_-)_{-r} U(\mathfrak{g}_+)_{i+r}\big) \subset \coprod_{m\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r-m} \mathrm{Cl}(\mathfrak{g}_+)_{i+r+m}.$$

Summing over all $r \geq 0$, one obtains a well-defined map $\widehat{U}(\mathfrak{g})_i \to \widehat{\mathrm{Cl}}(\mathfrak{g})_i$. ∎

4 Double Extension

For the loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ of a semisimple Lie algebra $\mathfrak{k}$, the Kac–Peterson class is non-trivial. On the other hand, the usual double extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra, hence its Kac–Peterson class is zero. In fact, one has a similar double extension in the general case, as we now explain.

We continue to work with the assumptions from the last sections; in particular $\mathfrak{g}$ carries an invariant, non-degenerate, symmetric, bilinear form B of degree 0. As noted above, the Kac–Peterson cocycle ψ_{KP} gives rise to a skew-symmetric derivation $\Psi_{KP} \in \widehat{\mathfrak{o}}(\mathfrak{g})$. By a general construction of Medina–Revoy [15], such a derivation can be used to define a double extension $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}\delta \oplus \mathbb{C}K$, with the following bracket. For $\xi, \xi_1, \xi_2 \in \mathfrak{g}$,

$$[\xi_1, \xi_2]_{\widetilde{\mathfrak{g}}} = [\xi_1, \xi_2]_{\mathfrak{g}} + \psi_{KP}(\xi_1, \xi_2)K, \qquad [\delta, \xi]_{\widetilde{\mathfrak{g}}} = \Psi_{KP}(\xi),$$
$$[\delta, K]_{\widetilde{\mathfrak{g}}} = 0, \qquad [\xi, K]_{\widetilde{\mathfrak{g}}} = 0.$$

The bilinear form B on $\mathfrak{g}$ extends to a non-degenerate invariant bilinear form on $\widetilde{\mathfrak{g}}$ in such a way that $\mathfrak{g}$ and $\mathbb{C}\delta \oplus \mathbb{C}K$ are orthogonal and

$$\widetilde{B}(\delta, K) = 1, \ \ \widetilde{B}(\delta, \delta) = \widetilde{B}(K, K) = 0.$$

Introduce the grading $\widetilde{\mathfrak{g}}_i = \mathfrak{g}_i$ for $i \neq 0$ and $\widetilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{C}\delta \oplus \mathbb{C}K$. The resulting splitting is

$$\widetilde{\mathfrak{g}}_- = \mathfrak{g}_- \oplus \mathbb{C}\delta \oplus \mathbb{C}K, \ \widetilde{\mathfrak{g}}_+ = \mathfrak{g}_+.$$

Let $\widetilde{\psi}_{KP}$ be the Kac–Peterson cocycle for this splitting, let $\widetilde{\Psi}_{KP}$ be the associated derivation, and denote by $\widetilde{\pi}_\pm \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}_\pm$ the projections along $\widetilde{\mathfrak{g}}_\mp$. The adjoint representation for $\widetilde{\mathfrak{g}}$ will be denoted $\widetilde{\mathrm{ad}}$.

Proposition 4.1 *The derivation $\widetilde{\Psi}_{KP}$ is inner: $\widetilde{\Psi}_{KP} = [\delta, \cdot\,]_{\widetilde{\mathfrak{g}}}$. Equivalently, $\widetilde{\psi}_{KP} = d\rho$, where $\rho = \widetilde{B}(\delta, \cdot\,)$.*

Proof The desired equation $\widetilde{\Psi}_{KP} = [\delta, \cdot\,]_{\widetilde{\mathfrak{g}}}$ means that

$$\widetilde{\Psi}_{KP}(\xi) = \Psi_{KP}(\xi), \quad \widetilde{\Psi}_{KP}(\delta) = 0, \quad \widetilde{\Psi}_{KP}(K) = 0.$$

Equivalently, we have to show that $\widetilde{\psi}_{KP}(\xi_1, \xi_2) = \psi_{KP}(\xi_1, \xi_2)$ for $\xi_1, \xi_2 \in \mathfrak{g}$, while both K, δ are in the kernel of $\widetilde{\psi}_{KP}$. The last claim follows from

$$\widetilde{\pi}_- \widetilde{\mathrm{ad}}_\delta \widetilde{\pi}_+ = 0 = \widetilde{\pi}_+ \widetilde{\mathrm{ad}}_\delta \widetilde{\pi}_-,$$

and similarly for ad_K, since ad_δ and ad_K preserve degrees. On the other hand, one checks that for $\xi_1, \xi_2 \in \mathfrak{g}$, the composition $\pi_+ \,\mathrm{ad}_{\xi_1}\, \pi_- \,\mathrm{ad}_{\xi_2}\, \pi_+ \colon \mathfrak{g}_+ \to \mathfrak{g}_+$ of operators on $\mathfrak{g}$ coincides with the composition $\widetilde{\pi}_+ \widetilde{\mathrm{ad}}_{\xi_1} \widetilde{\pi}_- \widetilde{\mathrm{ad}}_{\xi_2} \widetilde{\pi}_+ \colon \mathfrak{g}_+ \to \mathfrak{g}_+$ of operators on $\widetilde{\mathfrak{g}}$. Hence the Kac–Peterson coycles agree on elements of $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. ■

5 The Cubic Dirac Operator

We will define the cubic Dirac operator as an element of a completion of the quantum Weil algebra $\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. Following [1], we take the viewpoint that the commutator with $\mathcal{D}$ defines a differential, making $\widehat{\mathcal{W}}(\mathfrak{g})$ into a $\mathfrak{g}$-differential algebra.

5.1 Weil Algebra

We begin with an arbitrary $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}_i < \infty$. As usual $\mathfrak{g}^*$ denotes the restricted dual. Consider the tensor product $W(\mathfrak{g}^*) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ with grading

$$W^k(\mathfrak{g}^*) = \bigoplus_{2r+s=k} S^r(\mathfrak{g}^*) \otimes \wedge^s(\mathfrak{g}^*).$$

For $\mu \in \mathfrak{g}^*$ we denote by $s(\mu) = \mu \otimes 1$ the degree 2 generators and by $\mu = 1 \otimes \mu$ the degree 1 generators. Any $\xi \in \mathfrak{g}$ defines contraction operators ι_ξ; these are derivations of degree -1 given on generators by $\iota_\xi \mu = \mu(\xi)$, $\iota_\xi s(\mu) = 0$. The co-adjoint action on $\mathfrak{g}^*$ defines Lie derivatives $L_\xi = L_\xi^S \otimes 1 + 1 \otimes L_\xi^\wedge$. If $\dim(\mathfrak{g}) < \infty$, the algebra $W(\mathfrak{g})$ carries a *Weil differential* d^W, given on generators by[1]

$$\mathrm{d}^W \mu = 2(s(\mu) + \lambda(\mu)), \ \mathrm{d}^W s(\mu) = \sum\nolimits_a s(L_{e_a}\mu) e^a. \tag{5.1}$$

Here e_a is a basis of $\mathfrak{g}$ with dual basis $e^a \in \mathfrak{g}^*$.

In the general case, we need to pass to a completion in order for the differential to be defined. Define a second $\mathbb{Z}$-grading on $W(\mathfrak{g}^*)$ in such a way that the generators $s(\mu), \mu$ for $\mu \in (\mathfrak{g}^*)_i = (\mathfrak{g}_{-i})^*$ have degree i. Letting $\mathfrak{g}^*_+ = \bigoplus_{i>0} (\mathfrak{g}^*)_i$ and $\mathfrak{g}^*_- = \bigoplus_{i\le 0} (\mathfrak{g}^*)_i$ we define a completion $\widehat{W}(\mathfrak{g}^*)$ as the graded algebra with

$$\widehat{W}(\mathfrak{g}^*)_i = \prod_{r\ge 0} W(\mathfrak{g}^*_-)_{i-r} \otimes W(\mathfrak{g}^*_+)_r.$$

(Equivalently, $\widehat{W}(\mathfrak{g}^*)_i$ is the space of all linear maps $(S(\mathfrak{g}) \otimes \wedge(\mathfrak{g}))_{-i} \to \mathbb{K}$.) The Weil differential d^W is defined on generators by the formulas (5.1). Together with the natural extensions of ι_ξ, L_ξ, this makes $\widehat{W}(\mathfrak{g}^*)$ into a $\mathfrak{g}$-differential algebra.

5.2 Quantum Weil Algebra

Suppose now that $\mathfrak{g}$ carries an invariant symmetric bilinear form B of degree 0. We use B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and will thus write $W(\mathfrak{g})$, $\widehat{W}(\mathfrak{g})$ and so on. The non-commutative *quantum Weil algebra* is the tensor product

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

It is a super algebra with even generators $s(\zeta) = \zeta \otimes 1$ and odd generators $\zeta = 1 \otimes \zeta$. Any $\xi \in \mathfrak{g}$ defines Lie derivatives $L_\xi = L_\xi^U \otimes 1 + 1 \otimes L_\xi^{Cl}$ and contraction operators

[1] The conventions for the differential follow [16, §6.11]. They are arranged to make the relation with the quantum Weil algebra appear most natural. One recovers the more standard conventions used in *e.g.,* [1, 3] by a simple rescaling of variables.

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ι_ξ, given as odd derivations with $\iota_\xi\zeta = B(\xi,\zeta)$, $\iota_\xi s(\zeta) = 0$. Super symmetrization defines an isomorphism

$$q^0\colon W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g}), \tag{5.2}$$

given simply as the tensor product of $q^0\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ and $q^0\colon \wedge(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$. Note that (5.2) intertwines the contractions and Lie derivatives. We define a completion $\widehat{\mathcal{W}}(\mathfrak{g})$ as the graded super algebra with

$$\widehat{\mathcal{W}}(\mathfrak{g})_i = \prod_{r\geq 0} \mathcal{W}(\mathfrak{g}_-)_{i-r} \otimes \mathcal{W}(\mathfrak{g}_+)_r.$$

The "normal-ordered" quantization map $q\colon \widehat{W}(\mathfrak{g}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ is defined by summing over all

$$q^0 \otimes q^0\colon W(\mathfrak{g}_-)_{i-r} \otimes W(\mathfrak{g}_+)_r \to \mathcal{W}(\mathfrak{g}_-)_{i-r} \otimes \mathcal{W}(\mathfrak{g}_+)_r.$$

It extends the quantization maps $q\colon \widehat{S}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ and $q\colon \widehat{\wedge}(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g})$.

5.3 The Element $q(D)$

If $\dim\mathfrak{g} < \infty$, one obtains a differential $\mathrm{d}^{\mathcal{W}}$ on $\mathcal{W}\mathfrak{g}$ as a derivation given on generators by formulas similar to (5.1):

$$\mathrm{d}^{\mathcal{W}}\zeta = 2(s(\zeta) + q_0(\lambda(\zeta))),\ \mathrm{d}^{\mathcal{W}}s(\zeta) = \sum\nolimits_a s(L_{e_a}\zeta)e^a;$$

see [1]. In fact, $\mathrm{d}^{\mathcal{W}} = [q^0(D),\cdot]$, where $D \in W^3(\mathfrak{g})$ is the element

$$D = \sum_a s(e_a)e^a + \phi$$

with $\phi \in \wedge^3\mathfrak{g} \subset W^3(\mathfrak{g})$ being the structure constants tensor. The fact that $\mathrm{d}^{\mathcal{W}}$ squares to zero means that $q^0(D)$ squares to a central element, and indeed one finds

$$q^0(D)^2 = \mathrm{Cas}_{\mathfrak{g}} + \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}}).$$

If $\dim\mathfrak{g} = \infty$, the element D is well defined as an element of the completion $\widehat{W}^3(\mathfrak{g})$, but $q^0(D)$ is ill-defined. On the other hand,

$$\mathcal{D}' = q(D) = \sum_a s(e_a)e^a + \phi'_{\mathrm{Cl}}$$

is defined but does not square to a central element.

Proposition 5.1 *The square of $\mathcal{D}' = q(D)$ is given by*

$$(\mathcal{D}')^2 = \mathrm{Cas}'_{\mathfrak{g}} + q(\psi^\sharp_{KP}) + \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).$$

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Proof We have $L_\xi \mathcal{D}' = L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi) = \iota_\xi q(\psi^\sharp_{KP})$ because $\sum_a s(e_a)e^a \in \widehat{\mathcal{W}}(\mathfrak{g})$ is $\mathfrak{g}$-invariant. Using that

$$\iota_\xi \mathcal{D}' = s(\xi) + \iota_\xi\big(q(\phi)\big) = s(\xi) + \gamma'(\xi)$$

are generators for the $\mathfrak{g}$-action on $\widehat{\mathcal{W}}(\mathfrak{g})$, we have

$$\iota_\xi((\mathcal{D}')^2 - q(\psi^\sharp_{KP})) = [\iota_\xi \mathcal{D}', \mathcal{D}'] - q(\psi^\sharp_{KP}) = 0.$$

This shows $(\mathcal{D}')^2 - q(\psi^\sharp_{KP}) \in \widehat{U}(\mathfrak{g}) \subset \widehat{\mathcal{W}}(\mathfrak{g})$. To find this element we calculate, denoting by $\cdots$ terms in the kernel of the projection $\widehat{\mathcal{W}}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$,

$$\begin{aligned}(\mathcal{D}')^2 &= \sum_{ab} s(e_a)s(e_b)e^a e^b + (\phi'_{\mathrm{Cl}})^2 + \cdots \\ &= \tfrac{1}{2}\sum_{ab} s(e_a)s(e_b)[e^a, e^b] + \frac{1}{24}\operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + \cdots \\ &= \mathrm{Cas}'_{\mathfrak{g}} + \frac{1}{24}\operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + \cdots\end{aligned}$$ ■

If the Kac–Peterson class is trivial, one obtains an element $\mathcal{D}$ with better properties.

Corollary 5.2 *Suppose that $\psi_{KP} = d\rho$ for some $\rho \in \mathfrak{g}_0^*$. Define*

$$\mathcal{D} = \mathcal{D}' + \rho^\sharp, \quad \gamma_{\mathcal{W}}(\xi) = s(\xi) + \gamma'_{\mathrm{Cl}}(\xi) + \langle \rho, \xi\rangle,$$

and put $\mathrm{Cas}_{\mathfrak{g}} = \mathrm{Cas}'_{\mathfrak{g}} + 2\rho^\sharp$ as before. Then

$$\mathcal{D}^2 = \mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \tfrac{1}{24}\operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + B(\rho^\sharp, \rho^\sharp).$$

One has the following commutator relations in $\widehat{\mathcal{W}}(\mathfrak{g})$,

$$\begin{aligned}[\mathcal{D}, \mathcal{D}] &= 2\,\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \tfrac{1}{12}\operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + 2B(\rho^\sharp, \rho^\sharp), \\ [\gamma_{\mathcal{W}}(\xi), \mathcal{D}] &= 0, \\ [\xi, \mathcal{D}] &= 2\gamma_{\mathcal{W}}(\xi), \\ [\gamma_{\mathcal{W}}(\xi), \gamma_{\mathcal{W}}(\zeta)] &= \gamma_{\mathcal{W}}([\xi, \zeta]_{\mathfrak{g}}), \\ [\gamma_{\mathcal{W}}(\xi), \zeta] &= [\xi, \zeta]_{\mathfrak{g}}, \\ [\xi, \zeta] &= 2B(\xi, \zeta).\end{aligned}$$

Thus $\widehat{\mathcal{W}}(\mathfrak{g})$ becomes a $\mathfrak{g}$-differential algebra with differential, Lie derivatives, and contractions given by

$$d^{\mathcal{W}} = [\mathcal{D}, \cdot\,], \quad L^{\mathcal{W}}_\xi = [\gamma_{\mathcal{W}}(\xi), \cdot\,], \quad \iota^{\mathcal{W}}_\xi = \tfrac{1}{2}[\xi, \cdot\,].$$

We will refer to $\mathcal{D} \in \widehat{\mathcal{W}}(\mathfrak{g})$ as the *cubic Dirac operator*, following Kostant [10].

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6 Relative Dirac Operators

In his paper [10], Kostant introduced more generally Dirac operators for any pair of a quadratic Lie algebra $\mathfrak{g}$ and a quadratic Lie subalgebra $\mathfrak{u}$. We now consider an extension of his results to infinite-dimensional graded Lie algebras.

Let $\mathfrak{g}, B$ be as in the last section, and suppose $\mathfrak{u} \subseteq \mathfrak{g}$ is a graded quadratic subalgebra. That is, $\mathfrak{u}_i \subseteq \mathfrak{g}_i$ for all i, and the non-degenerate symmetric bilinear form B on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{u}$. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{u}^{\perp}$. For any $\xi \in \mathfrak{u}$, the operator $\mathrm{ad}_\xi \in \widehat{\mathfrak{o}}(\mathfrak{g})$ breaks up as a sum $\mathrm{ad}_\xi = \mathrm{ad}^{\mathfrak{u}}_\xi + \mathrm{ad}^{\mathfrak{p}}_\xi$, $\xi \in \mathfrak{u}$ of operators $\mathrm{ad}^{\mathfrak{u}}_\xi \in \widehat{\mathfrak{o}}(\mathfrak{u})$, and $\mathrm{ad}^{\mathfrak{p}}_\xi \in \widehat{\mathfrak{o}}(\mathfrak{p})$. Accordingly,

$$\lambda(\xi) = \lambda_{\mathfrak{u}}(\xi) + \lambda_{\mathfrak{p}}(\xi), \;\; \xi \in \mathfrak{u}$$

with $\lambda_{\mathfrak{u}}(\xi) \in \widehat{\wedge}^2(\mathfrak{u})$ and $\lambda_{\mathfrak{p}}(\xi) \in \widehat{\wedge}^2(\mathfrak{p})$. Denote by $\gamma'_{\mathfrak{u}}(\xi),\ \gamma'_{\mathfrak{p}}(\xi)$ their images under $q\colon \widehat{W}(\mathfrak{g}) \to \widehat{\mathcal{W}}(\mathfrak{g})$. We have (cf. (2.5))

$$\big[\gamma'_{\mathfrak{p}}(\xi), \gamma'_{\mathfrak{p}}(\zeta)\big] = \gamma'_{\mathfrak{p}}\big([\xi,\zeta]\big) + \psi^{\mathfrak{p}}_{KP}(\xi,\zeta),$$

where $\psi^{\mathfrak{p}}_{KP}(\xi,\zeta) = \psi^{\mathfrak{p}}_{KP}(\mathrm{ad}^{\mathfrak{p}}_\xi, \mathrm{ad}^{\mathfrak{p}}_\zeta)$ defines a cocycle $\psi^{\mathfrak{p}}_{KP} \in \widehat{\wedge}^2(\mathfrak{u}^*)$. If $\psi^{\mathfrak{p}}_{KP} = \mathrm{d}\rho_{\mathfrak{p}}$ for some $\rho_{\mathfrak{p}} \in \mathfrak{u}_0^*$, then $\gamma_{\mathfrak{p}}(\xi) = \gamma'_{\mathfrak{p}}(\xi) + \langle \rho_{\mathfrak{p}}, \xi\rangle$ gives a Lie algebra homomorphism $\mathfrak{u} \to \widehat{\mathrm{Cl}}(\mathfrak{p})$ generating the adjoint action of $\mathfrak{u}$. One obtains an algebra homomorphism $j\colon \mathcal{W}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ given on generators by

$$j(\xi) = \xi, \quad j\big(s(\xi)\big) = s(\xi) + \gamma_{\mathfrak{p}}(\xi), \quad \xi \in \mathfrak{u}.$$

Proposition 6.1 *The homomorphism $\mathcal{W}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ extends to an algebra homomorphism for the completion: $j\colon \widehat{\mathcal{W}}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$. It intertwines Lie derivatives and contraction by elements $\xi \in \mathfrak{u}$.*

Proof The first part follows by an argument parallel to that for Proposition 3.13. The second part follows from

$$j \circ L_\xi = j \circ \big[s(\xi) + \gamma'_{\mathfrak{u}}(\xi),\,\cdot\,\big] = \big[s(\xi) + \gamma'_{\mathfrak{g}}(\xi),\,\cdot\,\big] \circ j = L_\xi \circ j,$$

and similarly $j \circ \iota_\xi = \frac{1}{2} j \circ [\xi,\,\cdot\,] = \frac{1}{2}[\xi,\,\cdot\,] \circ j = \iota_\xi \circ j$. ∎

Let $\mathcal{W}(\mathfrak{g},\mathfrak{u}) = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{u}}$ be the $\mathfrak{u}$-basic part of $\mathcal{W}(\mathfrak{g})$, *i.e.,* the subalgebra of elements annilated by all L_ξ and all ι_ξ for $\xi \in \mathfrak{u}$. Similarly let $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$ be the $\mathfrak{u}$-basic part of $\widehat{\mathcal{W}}(\mathfrak{g})$.

Proposition 6.2 *The subalgebra $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$ is the commutant of the range $j(\widehat{\mathcal{W}}(\mathfrak{u}))$.*

Proof Since $\iota_\xi = \frac{1}{2}[\xi,\,\cdot\,]$, an element of $\widehat{\mathcal{W}}(\mathfrak{g})$ commutes with the generators $j(\xi)$ for $\xi \in \mathfrak{u}$ precisely if it lies in the $\mathfrak{u}$-horizontal subspace given as the completion of $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. The elements $j(s(\xi)) = s(\xi) + \gamma'_{\mathfrak{p}}(\xi)$ generate the $\mathfrak{u}$-action on that subspace. Hence, an element of $\widehat{\mathcal{W}}(\mathfrak{g})$ commutes with all $j(\xi),\ j(s(\xi))$ if and only if it is $\mathfrak{u}$-basic. ∎

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We will now make the stronger assumption that the Kac–Peterson classes of both $\mathfrak{g}$ and $\mathfrak{u}$ are zero. Let $\rho \in \mathfrak{g}_0^*$, $\rho_\mathfrak{u} \in \mathfrak{u}_0^*$ be elements such that $\psi_{KP} = \mathrm{d}\rho$, $\psi_{KP}^\mathfrak{u} = \mathrm{d}\rho_\mathfrak{u}$, and take $\rho_\mathfrak{p} := \rho|_{\mathfrak{u}_0} - \rho_\mathfrak{u} \in \mathfrak{u}_0^*$ so that $\psi_{KP}^\mathfrak{p} = \mathrm{d}\rho_\mathfrak{p}$. Put

$$\gamma(\zeta) = \gamma'(\zeta) + \langle \rho, \zeta \rangle, \;\; \gamma_\mathfrak{u}(\xi) = \gamma_\mathfrak{u}'(\xi) + \langle \rho_u, \xi \rangle$$

for all $\zeta \in \mathfrak{g}$, $\xi \in \mathfrak{u}$, and let

$$\mathcal{D} = \mathcal{D}' + \rho^\sharp \in \widehat{\mathcal{W}}(\mathfrak{g}), \;\; \mathcal{D}_\mathfrak{u} = \mathcal{D}_u' + \rho_\mathfrak{u}^\sharp \in \widehat{\mathcal{W}}(\mathfrak{u})$$

be the cubic Dirac operators for $\mathfrak{g}, \mathfrak{u}$. The commutator with these elements defines differentials on the two Weil algebras.

Lemma 6.3 *The map $j\colon \widehat{\mathcal{W}}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ is a homomorphism of $\mathfrak{u}$-differential algebras.*

Proof It remains to show that the map j intertwines differentials. It suffices to check on generators. For $\xi \in \mathfrak{u}$,

$$j(\mathrm{d}\xi) = j\big(s_\mathfrak{u}(\xi) + \gamma_\mathfrak{u}(\xi)\big) = s(\xi) + \gamma_\mathfrak{p}(\xi) + \gamma_\mathfrak{u}(\xi) = s(\xi) + \gamma(\xi) = \mathrm{d}\, j(\xi),$$

and similarly $j(\mathrm{d}s_\mathfrak{u}(\xi)) = \mathrm{d}\, j(s_\mathfrak{u}(\xi))$. ■

We define the relative cubic Dirac operator $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ as a difference, $\mathcal{D}_{\mathfrak{g},\mathfrak{u}} = \mathcal{D} - j(\mathcal{D}_\mathfrak{u})$.

Proposition 6.4 *The element $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ lies in $\widehat{\mathcal{W}}(\mathfrak{g}, \mathfrak{u})$ and squares to an element of the center of $\widehat{\mathcal{W}}(\mathfrak{g}, \mathfrak{u})$. Explicitly,*

$$\mathcal{D}_{\mathfrak{g},\mathfrak{u}}^2 = \mathrm{Cas}_\mathfrak{g} - j(\mathrm{Cas}_\mathfrak{u}) + \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) - \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{u}_0}(\mathrm{Cas}_{\mathfrak{u}_0}) + B(\rho^\sharp, \rho^\sharp) - B(\rho_\mathfrak{u}^\sharp, \rho_\mathfrak{u}^\sharp).$$

Proof Using that j intertwines contractions ι_ξ, $\xi \in \mathfrak{u}$, we find

$$\begin{aligned} \iota_\xi \mathcal{D}_{\mathfrak{g},\mathfrak{u}} &= \iota_\xi \mathcal{D} - j(\iota_\xi \mathcal{D}_\mathfrak{u}) = s(\xi) + \gamma(\xi) - j\big(s_\mathfrak{u}(\xi) + \gamma_\mathfrak{u}(\xi)\big) \\ &= \gamma(\xi) - \gamma_\mathfrak{p}(\xi) - \gamma_\mathfrak{u}(\xi) = 0. \end{aligned}$$

Thus $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ is $\mathfrak{u}$-horizontal, and it is clearly $\mathfrak{u}$-invariant as well. Thus $\mathcal{D}_{\mathfrak{g},\mathfrak{u}} \in \widehat{\mathcal{W}}(\mathfrak{g}, \mathfrak{u})$. In particular, $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ commutes with $j(\mathcal{D}_\mathfrak{u})$. Consequently, $[\mathcal{D}, \mathcal{D}] = j([\mathcal{D}_\mathfrak{u}, \mathcal{D}_\mathfrak{u}]) + [\mathcal{D}_{\mathfrak{g},\mathfrak{u}}, \mathcal{D}_{\mathfrak{g},\mathfrak{u}}]$; that is, $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}^2 = \mathcal{D}^2 - j(\mathcal{D}_\mathfrak{u}^2)$. Now use Corollary 5.2. ■

7 Application to Kac–Moody Algebras

In [10], Kostant used the cubic Dirac operator $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ to prove generalized Weyl character formulas for any pair of a semi-simple Lie algebra $\mathfrak{g}$ and equal rank subalgebra $\mathfrak{u}$. In this section, we show that much of this theory carries over to symmetrizable Kac–Moody algebras with only minor adjustments.

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7.1 Notation and Basic Facts

Let us recall some notation and basic facts; our main references are the books by Kac [5] and Kumar [13].

Let $A = (a_{ij})_{1\le i,j\le l}$ be a generalized Cartan matrix, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Thus $\mathfrak{h}$ is a vector space of dimension $2l - \mathrm{rk}(A)$, and $\Pi = \{\alpha_1, \dots, \alpha_l\} \subset \mathfrak{h}^*$ (the set of simple roots) and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_l^\vee\} \subset \mathfrak{h}$ (the corresponding coroots) satisfy $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. The Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra generated by elements $h \in \mathfrak{h}$ and elements $e_j,\ f_j$ for $j = 1, \dots, l$, subject to relations

$$[h, e_i] = \langle \alpha_i, h\rangle e_i,\ [h, f_i] = -\langle \alpha_i, h\rangle f_i,\ [h, h'] = 0, \quad [e_i, f_j] = \delta_{ij}\alpha_i^\vee,$$
$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0, \quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j.$$

The non-zero weights $\alpha \in \mathfrak{h}^*$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ are called the roots; the corresponding root spaces are denoted $\mathfrak{g}_\alpha$. The set Δ of roots is contained in the lattice $Q = \bigoplus_{j=1}^l \mathbb{Z}\alpha_j \subset \mathfrak{h}^*$. Let $Q^+ = \bigoplus_{j=1}^l \mathbb{Z}_{\ge 0}\alpha_j$, and put $\Delta^+ = \Delta \cap Q^+$ and $\Delta^- = -\Delta^+$. One has $\Delta = \Delta^+ \cup \Delta^-$.

Let W be the Weyl group of $\mathfrak{g}$, *i.e.,* the group of transformations of $\mathfrak{h}$ generated by the simple reflections $\xi \mapsto \xi - \langle \alpha_j, \xi\rangle \alpha_j^\vee$. The dual action of W as a reflection group on $\mathfrak{h}^*$ preserves Δ. Let Δ^{re} be the set of real roots, *i.e.,* roots that are W-conjugate to roots in Π, and let Δ^{im} be its complement, the imaginary roots. For $\alpha \in \Delta^{\mathrm{re}}$ one has $\dim \mathfrak{g}_\alpha = 1$.

The length $l(w)$ of a Weyl group element may be characterized as the cardinality of the set $\Delta_w^+ = \Delta^+ \cap w\Delta^-$ of positive roots that become negative under w^{-1} ([13, Lemma 1.3.14]). We remark that $\Delta_w^+ \subset \Delta^{\mathrm{re}}$ ([5, §5.2]).

Fix a real subspace $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ containing $\Pi^\vee$. Let $C \subset \mathfrak{h}_\mathbb{R}$ be the *dominant chamber*, and let X be the *Tits cone* ([5, §3.12]). Thus C is the set of all $\xi \in \mathfrak{h}_\mathbb{R}$ such that $\langle \alpha, \xi\rangle \ge 0$ for all $\alpha \in \Pi$, while X is characterized by the property that $\langle \alpha, \xi \rangle < 0$ for at most finitely many $\alpha \in \Delta$. The W-action preserves X, and C is a fundamental domain in the sense that every W-orbit in X intersects C in a unique point.

For any $\mu = \sum_{j=1}^l k_j\alpha_j \in Q$ one defines $\mathrm{ht}(\mu) = \sum_{j=1}^l k_j$. The *principal grading* on $\mathfrak{g}$ is defined by letting $\mathfrak{g}_i$ for $i \neq 0$ be the direct sum of root spaces $\mathfrak{g}_\alpha$ with $\mathrm{ht}(\alpha) = i$, and $\mathfrak{g}_0 = \mathfrak{h}$. Letting $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$, it follows that $\mathfrak{g}_+ = \mathfrak{n}_+$ and $\mathfrak{g}_- = \mathfrak{n}_- \oplus \mathfrak{h}$.

7.2 The Kac–Peterson Cocycle

Suppose from now on that A is *symmetrizable*; that is, there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_1, \dots, \epsilon_l)$ such that $D^{-1}A$ is symmetric. In this case, $\mathfrak{g}$ carries a non-degenerate symmetric invariant bilinear form B with the property $B(\alpha_j^\vee, \xi) = \epsilon_j \langle \alpha_j, \xi\rangle,\ \xi \in \mathfrak{h}$ ([5, §2.2]). One refers to B as a *standard* bilinear form. Choose $\rho \in \mathfrak{h}^*$ with $\langle \rho, \alpha_j^\vee\rangle = 1$ for $j = 1, \dots, l$.

Proposition 7.1 *The Kac–Peterson cocycle of the symmetrizable Kac–Moody algebra $\mathfrak{g}$ is exact. In fact, $\psi_{KP} = d\rho$.*

Proof Use B to define $\mathrm{Cas}'_\mathfrak{g}$. As shown by Kac in [5, Theorem 2.6], the operator $\mathrm{Cas}_\mathfrak{g} := \mathrm{Cas}'_\mathfrak{g} + 2\rho^\sharp$ is $\mathfrak{g}$-invariant. By Corollary 3.8 this is equivalent to $\psi_{KP} = \mathrm{d}\rho$. ∎

7.3 Regular Subalgebras

We now introduce a suitable class of "equal rank" subalgebras. Following Morita and Naito ([17, 18]), consider a linearly independent subset $\Pi_{\mathfrak{u}} \subset \Delta^{\mathrm{re},+}$ with the property that the difference of any two elements in Π_u is not a root. We denote by $\mathfrak{u} \subset \mathfrak{g}$ the Lie subalgebra generated by $\mathfrak{h}$ together with the root spaces $\mathfrak{g}_{\pm\beta}$ for $\beta \in \Pi_{\mathfrak{u}}$. Let $\mathfrak{p} = \mathfrak{u}^{\perp}$, so that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$.

Examples 7.2 (a) If $\Pi_{\mathfrak{u}} = \emptyset$, one obtains $\mathfrak{u} = \mathfrak{h}$. (b) Suppose $\mathfrak{g}$ is an affine Kac–Moody algebra, *i.e.,* the double extension of a loop algebra $\mathfrak{k}[z, z^{-1}]$ of a semi-simple Lie algebra $\mathfrak{k}$. Let $\mathfrak{l} \subset \mathfrak{k}$ be an equal rank subalgebra of $\mathfrak{k}$. Let $\Pi_{\mathfrak{l}} \subset \Delta_{\mathfrak{k}}^{+}$ be the simple roots of $\mathfrak{l}$, and let $\Pi_{\mathfrak{u}} \subset \Delta^{+}$ be the corresponding affine roots. Then $\mathfrak{u} = \mathfrak{l}[z, z^{-1}]$. This is the setting considered in Landweber's paper [14].

It was shown in [17, 18] that $\mathfrak{u}$ is a direct sum (as Lie algebras) of a symmetrizable Kac–Moody algebra $\widetilde{\mathfrak{u}}$ with a subalgebra of $\mathfrak{h}$.[2] Furthermore, the standard bilinear form B on $\mathfrak{g}$ restricts to a standard bilinear form on $\widetilde{\mathfrak{u}}$.

For any root $\alpha \in \Delta$ put $n_{\mathfrak{u}}(\alpha) = \dim \mathfrak{u}_{\alpha}$ and $n_{\mathfrak{p}}(\alpha) = \dim(\mathfrak{p}_{\alpha})$. Thus $n(\alpha) = n_{\mathfrak{u}}(\alpha) + n_{\mathfrak{p}}(\alpha)$ is the multiplicity of α in $\mathfrak{g}$. Let $\Delta_{\mathfrak{u}}$ (resp. $\Delta_{\mathfrak{p}}$) be the set of roots such that $n_{\mathfrak{u}}(\alpha) > 0$ (resp. $n_{\mathfrak{p}}(\alpha) > 0$). Thus $\Delta_{\mathfrak{u}}$ is the set of roots of $\mathfrak{u}$. Let $W_{\mathfrak{u}} \subset W$ be the Weyl group of $\mathfrak{u}$ (generated by reflections for elements of $\Pi_{\mathfrak{u}}$), and define a subset

$$W_{\mathfrak{p}} = \{w \in W \mid w^{-1}\Delta_{\mathfrak{u}}^{+} \subset \Delta^{+}\}.$$

Lemma 7.3 *We have $w \in W_{\mathfrak{p}} \Leftrightarrow \Delta_{w}^{+} \subset \Delta_{\mathfrak{p}}$. Every $w \in W$ can be uniquely written as a product $w = w_1 w_2$ with $w_1 \in W_{\mathfrak{u}}$ and $w_2 \in W_{\mathfrak{p}}$.*

Proof By definition, $w \in W_{\mathfrak{p}}$ if and only if the intersection $\Delta_{\mathfrak{u}}^{+} \cap w\Delta^{-} = \Delta_{\mathfrak{u}} \cap \Delta_{w}^{+}$ is empty. Since Δ_{w}^{+} consists of real roots, this means $\Delta_{w}^{+} \subset \Delta_{\mathfrak{p}}$. For the second claim, let $C_{\mathfrak{u}} \subset X_{\mathfrak{u}}$ be the chamber and Tits cone for $\mathfrak{u}$. One has $w \in W_{\mathfrak{p}}$ if and only if $w^{-1}\Delta_{\mathfrak{u}}^{+} \subset \Delta^{+}$, if and only if $wC \subset C_{\mathfrak{u}}$. Let $w \in W$ be given. Then $wC \subset X \subset X_{\mathfrak{u}}$ is contained in a unique chamber of $\mathfrak{u}$. Hence there is a unique $w_1 \in W_{\mathfrak{u}}$ such that $wC \subset w_1 C_{\mathfrak{u}}$. Equivalently, $w_2 := w_1^{-1} w \in W_{\mathfrak{p}}$. ■

We have a decomposition $\mathfrak{p} = \mathfrak{p}_{+} \oplus \mathfrak{p}_{-}$, where $\mathfrak{p}_{\pm} = \mathfrak{p} \cap \mathfrak{n}_{\pm}$. The splitting defines a spinor module $\mathsf{S}_{\mathfrak{p}} = \wedge \mathfrak{p}_{-}$ over $\mathrm{Cl}(\mathfrak{p})$, where the elements of $\mathfrak{p}_{+}$ act by contraction and those of $\mathfrak{p}_{-}$ by exterior multiplication. The Clifford action on this module extends to the completion $\widehat{\mathrm{Cl}}(\mathsf{S}_{\mathfrak{p}})$.

Fix $\rho_{\mathfrak{u}} \in \mathfrak{h}^{*}$ with $\langle \rho_{\mathfrak{u}}, \beta^{\vee} \rangle = 1$ for all $\beta \in \Pi_{\mathfrak{u}}$. Let $\rho_{\mathfrak{p}} = \rho|_{\mathfrak{u}} - \rho_{\mathfrak{u}}$ defining a Lie algebra homomorphism $\gamma_{\mathfrak{p}} = \gamma_{\mathfrak{p}}' + \rho_{\mathfrak{p}} \colon \mathfrak{u} \to \widehat{\mathrm{Cl}}(\mathfrak{p})$. By composition with the spinor action one obtains an integrable $\mathfrak{u}$-representation $\pi_{\mathsf{S}} \colon \mathfrak{u} \to \mathrm{End}(\mathsf{S}_{\mathfrak{p}})$.

Proposition 7.4 *The restriction of π_{S} to $\mathfrak{h} \subset \mathfrak{u}$ differs from the adjoint representation of $\mathfrak{h}$ by a $\rho_{\mathfrak{p}}$-shift:*

$$\pi_{\mathsf{S}}(\xi) = \langle \rho_{\mathfrak{p}}, \xi \rangle + \mathrm{ad}(\xi), \quad \xi \in \mathfrak{h}.$$

[2] In fact, Naito [18] constructs an explicit subspace $\widetilde{\mathfrak{h}} \subset \mathfrak{h}$ such that the Lie algebra $\widetilde{\mathfrak{g}}$ generated by $\widetilde{\mathfrak{h}}$ and the $\mathfrak{g}_{\pm\beta}$, $\beta \in \Pi_{\mathfrak{u}}$ is a Kac–Moody algebra. He also considers subsets $\Pi_{\mathfrak{u}}$ that do not necessarily consist of real roots, and finds that the resulting $\widetilde{u}$ is a symmetrizable *generalized* Kac–Moody algebra.

Hence, the weights for the action of $\mathfrak{h}$ on $\mathrm{S}_\mathfrak{p}$ are of the form $\rho_\mathfrak{p} - \sum_{\alpha\in\Delta_\mathfrak{p}^+} k_\alpha \alpha$, where $0 \leq k_\alpha \leq n_\mathfrak{p}(\alpha)$. The parity of the corresponding weight space is $\sum_\alpha k_\alpha \mod 2$. For all $w \in W_\mathfrak{p}$, the element $w\rho - \rho_\mathfrak{u}$ is a weight of $\mathrm{S}_\mathfrak{p}$, of multiplicity 1. The parity of the weight space $\mathrm{S}_\mathfrak{p}$ equals $l(w) \mod 2$.

Proof For each $\alpha \in \Delta_\mathfrak{p}^+$, fix a basis $e_\alpha^{(s)}$, $s = 1, \dots, n_\mathfrak{p}(\alpha)$ of $\mathfrak{p}_\alpha$, and let $e_{-\alpha}^{(s)}$ be the B-dual basis of $\mathfrak{p}_{-\alpha}$. By definition, we have $\gamma_\mathfrak{p}(\xi) = \langle \rho_\mathfrak{p}, \xi\rangle + \gamma'_\mathfrak{p}(\xi)$ with

$$\gamma'_\mathfrak{p}(\xi) = -\tfrac{1}{2} \sum_{\alpha\in\Delta_\mathfrak{p}^+} \sum_{s=1}^{n_\mathfrak{p}(\alpha)} \langle \alpha, \xi\rangle \, e_{-\alpha}^{(s)} e_\alpha^{(s)}.$$

The action of $\gamma'_\mathfrak{p}(\xi)$ on the spinor module is just the adjoint action of ξ. This proves the first assertion. It is now straightforward to read off the weights of the action on $\mathrm{S}_\mathfrak{p}$. For all $w \in W$ one has $\rho - w\rho = \sum_{\alpha\in\Delta_w^+} \alpha$ (cf. [13, Corollary 1.3.22]). If $w \in W_\mathfrak{p}$, so that $\Delta_w^+ \subset \Delta_\mathfrak{p}^+$, it follows that $w\rho - \rho_\mathfrak{u} = w\rho - \rho + \rho_\mathfrak{p} = \rho_\mathfrak{p} - \sum_{\alpha\in\Delta_w^+} \alpha$ is a weight of $\mathrm{S}_\mathfrak{p}$. We now use $\mathrm{S}_{\mathfrak{h}^\perp} = \mathrm{S}_\mathfrak{p} \otimes \mathrm{S}_{\mathfrak{u}\cap\mathfrak{h}^\perp}$ as modules over $\mathrm{Cl}(\mathfrak{h}^\perp) = \mathrm{Cl}(\mathfrak{p}) \otimes \mathrm{Cl}(\mathfrak{u}\cap\mathfrak{h}^\perp)$. Hence, the tensor product with a generator of the line $(\mathrm{S}_{\mathfrak{u}\cap\mathfrak{h}^\perp})_{\rho_\mathfrak{u}}$ defines an isomorphism of the weight space $(\mathrm{S}_\mathfrak{p})_{w\rho-\rho_\mathfrak{u}}$ with $(\mathrm{S}_{\mathfrak{h}^\perp})_{w\rho}$, but the latter is 1-dimensional, and its parity is given by $l(w) \mod 2$ (cf. [13, Lemma 3.2.6]). ■

7.4 Action of the Cubic Dirac Operator

The subalgebra $\mathfrak{u}$ inherits a $\mathbb{Z}$-grading from $\mathfrak{g}$ with $\mathfrak{u}_i$ the direct sum of root spaces $\mathfrak{u}_\alpha$ for $\alpha = \sum_r k_r\beta_r$ and $i = \sum_r k_r m_r$. It is thus the *grading of type* $m = (m_1, \dots, m_r)$ [5, §1.5] with $m_r = \mathrm{ht}(\beta_r)$. Let $\widehat{\mathcal{W}}(\mathfrak{u})$ be the completion of the quantum Weil algebra for this grading. (It is just the same as the completion defined by the principal grading of $\mathfrak{u}$).

Let $P \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{g}$, and let $P^+ \subset P$ be the dominant weights. Thus $\mu \in P$ if and only if $\langle \mu, \alpha_j^\vee\rangle \in \mathbb{Z}$ for $j = 1, \dots, l$, and $\mu \in P^+$ if these pairings are all non-negative. For any $\mu \in P^+$ let $L(\mu)$ be the irreducible integrable representation of $\mathfrak{g}$ of highest weight μ. By [5, §11.4], $L(\mu)$ carries a unique (up to scalar) Hermitian form for which the elements of the real form of $\mathfrak{g}$ are represented as skew-adjoint operators. The weights ν of $L(\mu)$ satisfy $\mu - \nu \in Q^+$, hence there is a $\mathbb{Z}$-grading on $L(\mu)$ such that elements of $L(\mu)_\nu$ have degree $j = -\mathrm{ht}(\mu - \nu)$. The $\mathfrak{g}$-action is compatible with the gradings; *i.e.*, the action map $\mathfrak{g} \otimes L(\mu) \to L(\mu)$ preserves gradings. The spinor module $\mathrm{S}_\mathfrak{p} = \wedge \mathfrak{p}_-$ carries the $\mathbb{Z}$-grading defined by the $\mathbb{Z}$-grading on $\mathfrak{p}_-$, and the module action $\mathrm{Cl}(\mathfrak{p}) \otimes \mathrm{S}_\mathfrak{p} \to \mathrm{S}_\mathfrak{p}$ preserves gradings. The action of $\mathcal{W}(\mathfrak{g}, \mathfrak{u})$ on the graded vector space $L(\mu) \otimes \mathrm{S}_\mathfrak{p}$ extends to an action of the completion $\widehat{\mathcal{W}}(\mathfrak{g}, \mathfrak{u})$. We denote by $\mathcal{D}_{L(\mu)} \in \widehat{\mathrm{End}}(L(\mu) \otimes \mathrm{S}_\mathfrak{p})$ the image of $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ under this representation. Then $\mathcal{D}_{L(\mu)}$ is an odd, skew-adjoint operator.

Since $\mathcal{D}_{L(\mu)}$ commutes with the diagonal action of $\mathfrak{u}$ on $L(\mu) \otimes \mathrm{S}_\mathfrak{p}$, its kernel $\ker(\mathcal{D}_{L(\mu)})$ is a $\mathbb{Z}_2$-graded $\mathfrak{u}$-representation.

Let $P_\mathfrak{u}^+ \subset P_\mathfrak{u} \subset \mathfrak{h}^*$ be the set of dominant weights for $\mathfrak{u}$. For any $\nu \in P_\mathfrak{u}^+$, let $M(\nu)$ be the corresponding irreducible highest weight representation of $\mathfrak{u}$. Parallel to [10, Theorem 4.24] we have the following theorem.

Theorem 7.5 *The kernel of the operator $\mathcal{D}_{L(\mu)}$ is a direct sum*

$$\ker(D_{L(\mu)}) = \bigoplus_{w\in W_{\mathfrak{p}}} M\big(w(\mu+\rho)-\rho_{\mathfrak{u}}\big).$$

Here the even (resp. odd) part of the kernel is the sum over the $w \in W_{\mathfrak{p}}$ such that $l(w)$ is even (resp. odd).

Proof Given an integrable $\mathfrak{u}$-representation and any $\mathfrak{u}$-dominant weight $\nu \in P_{\mathfrak{u}}^+$, let the subscript $[\nu]$ denote the corresponding isotypical subspace. We are interested in $\ker(D_{L(\mu)})_{[\nu]}$. Since $\mathcal{D}_{L(\mu)}$ is skew-adjoint, its kernel coincides with that of its square:

$$\ker(\mathcal{D}_{L(\mu)}) = \ker(\mathcal{D}^2_{L(\mu)}).$$

The action of $\mathrm{Cas}_{\mathfrak{g}}$ on $L(\mu)$ is as a scalar $B(\mu+\rho,\mu+\rho)-B(\rho,\rho)$, and similarly for the action of $\mathrm{Cas}_{\mathfrak{u}}$ on $M(\nu)$. Hence,

$$\mathcal{D}^2_{L(\mu)} = B(\mu+\rho,\mu+\rho) - j(\mathrm{Cas}_{\mathfrak{u}}) - B(\rho_{\mathfrak{u}},\rho_{\mathfrak{u}})$$

acts on $(L(\mu)\otimes \mathrm{S}_{\mathfrak{p}})_{[\nu]}$ as a scalar, $B(\mu+\rho,\mu+\rho)-B(\nu+\rho_{\mathfrak{u}},\nu+\rho_{\mathfrak{u}})$. This shows that

$$\ker(\mathcal{D}_{L(\mu)})_{[\nu]} = {\textstyle\bigoplus_{\nu}'}(L(\mu)\otimes \mathrm{S}_{\mathfrak{p}})_{[\nu]},$$

where the sum $\bigoplus_{\nu}'$ is over all $\nu \in \Delta_{\mathfrak{u}}$ satisfying $B(\mu+\rho,\mu+\rho) = B(\nu+\rho_{\mathfrak{u}},\nu+\rho_{\mathfrak{u}})$. We want to identify this sum as a sum over $W_{\mathfrak{p}}$.

Suppose ν is any weight with $(L(\mu)\otimes\mathrm{S}_{\mathfrak{p}})_{\nu} \neq 0$. We will show that

$$B(\nu+\rho_{\mathfrak{u}},\ \nu+\rho_{\mathfrak{u}}) \le B(\mu+\rho,\ \mu+\rho).$$

By [5, Prop. 11.4(b)], an element $\nu \in P_{\mathfrak{u}}$ for which equality holds is automatically in $P_{\mathfrak{u}}^+$, and the multiplicity of $M(\nu)$ in $L(\mu)\otimes\mathrm{S}_{\mathfrak{p}}$ is then equal to the dimension of the highest weight space $(L(\mu)\otimes\mathrm{S}_{\mathfrak{p}})_{\nu}$. Write $\nu=\nu_1+\nu_2$, where $L(\mu)_{\nu_1}$ and $(\mathrm{S}_{\mathfrak{p}})_{\nu_2}$ are non-zero. By our description of the set of weights of $\mathrm{S}_{\mathfrak{p}}$, the element $\nu_2+\rho_{\mathfrak{u}}$ is among the weights of the $\mathfrak{g}$-representation $L(\rho)$, and in particular lies in the dual Tits cone $X^\vee$ of $\mathfrak{g}$. Since the Tits cone is convex, and $\nu_1 \in X^\vee$, it follows that $\nu_1+(\nu_2+\rho_{\mathfrak{u}}) = \nu+\rho_{\mathfrak{u}} \in X^\vee$. Consequently, there exists $w\in W$ such that $w^{-1}(\nu+\rho_{\mathfrak{u}}) \in C^\vee \subset \mathfrak{h}^*$. Since $\nu_2+\rho_{\mathfrak{u}}$ is a weight of $L(\rho)$, so is its image under w^{-1}. Hence, $\kappa_2 = \rho - w^{-1}(\nu_2+\rho_{\mathfrak{u}}) \in Q^+$. On the other hand, since $w^{-1}\nu_1$ is a weight of $L(\mu)$, we also have $\kappa_1 = \mu - w^{-1}\nu_1 \in Q^+$. Adding, we obtain $\mu+\rho = \kappa + w^{-1}(\nu+\rho_{\mathfrak{u}})$ with $\kappa = \kappa_1+\kappa_2 \in Q^+$. Since the pairing of κ with $w^{-1}(\nu+\rho_{\mathfrak{u}}) \in C^\vee$ is non-negative, the inequality $B(\mu+\rho,\mu+\rho) \ge B(\nu+\rho_{\mathfrak{u}},\nu+\rho_{\mathfrak{u}})$ follows. Equality holds if and only if $\kappa=0$, *i.e.,* $\kappa_1=0$ and $\kappa_2=0$, *i.e.,* $\nu_2 = w\rho-\rho_{\mathfrak{u}}$ and $\nu_1 = w\mu$. The $\mathfrak{h}$-weight spaces $(\mathrm{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{u}}}$ and $L(\mu)_{w\mu}$ are 1-dimensional, hence so is their tensor product, $(L(\mu)\otimes\mathrm{S}_{\mathfrak{p}})_{\nu}$. It follows that ν appears with multiplicity 1.

This shows that $M(\nu)$ appears in $\ker(D_{L(\mu)})$ if and only if it can be written in the form $\nu = w(\mu+\rho)-\rho_{\mathfrak{u}}$ for some $w\in W_{\mathfrak{p}}$, and in this case it appears with multiplicity 1. Note finally that w with this property is unique, since $\mu+\rho$ is regular. The parity of the ν-isotypical component follows, since $(\mathrm{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{u}}}$ has parity equal to that of $l(w)$. ■

The weights $\nu = w(\mu+\rho) - \rho_{\mathfrak{u}},\ w \in W_{\mathfrak{p}}$ are referred to as the *multiplet* corresponding to μ. Note that for given μ, the value of the quadratic Casimir $\mathrm{Cas}_{\mathfrak{u}}$ on the representations $M(w(\mu+\rho)-\rho_{\mathfrak{u}})$ is given by the constant value $B(\mu+\rho,\mu+\rho) - B(\rho_{\mathfrak{u}},\rho_{\mathfrak{u}})$, independent of w.

7.5 Characters

For any weight $\nu \in \mathfrak{h}^*$, we write $\mathrm{e}(\nu)$ for the corresponding formal exponential. We will regard the spinor module as a super representation, using the usual $\mathbb{Z}_2$-grading of the exterior algebra. The even and odd parts are denoted $\mathrm{S}_{\mathfrak{p}}^{\bar{0}}$ and $\mathrm{S}_{\mathfrak{p}}^{\bar{1}}$, and its formal character is denoted

$$\mathrm{ch}(\mathrm{S}_{\mathfrak{p}}) = \sum_{\nu} \big(\dim(\mathrm{S}_{\mathfrak{p}}^{\bar{0}})_\nu - \dim(\mathrm{S}_{\mathfrak{p}}^{\bar{1}})_\nu\big)\ \mathrm{e}(\nu).$$

Here $(\mathrm{S}_{\mathfrak{p}}^{\bar{0}})_\nu$ and $(\mathrm{S}_{\mathfrak{p}}^{\bar{1}})_\nu$ are the $\mathfrak{h}$ weight spaces, and $\mathrm{e}(\nu)$ is the formal character defined by ν (cf. [5, §10.2]).

Proposition 7.6 *The super character of the spin representation of* $\mathfrak{u}$ *on* $\mathfrak{p}$ *is given by the formula*

$$\mathrm{ch}(\mathrm{S}_{\mathfrak{p}}) = \mathrm{e}(\rho_{\mathfrak{p}}) \prod_{\alpha\in\Delta_{\mathfrak{p}}^+} \big(1-\mathrm{e}(-\alpha)\big)^{n_{\mathfrak{p}}(\alpha)}.$$

Proof For each root space $\mathfrak{p}_{-\alpha}$, the character of the adjoint action of $\mathfrak{h}$ on $\wedge\mathfrak{p}_{-\alpha}$ equals $(1-\mathrm{e}(-\alpha))^{n_{\mathfrak{p}}(\alpha)}$. The character of the adjoint action on $\wedge\mathfrak{p}_- = \bigotimes_{\alpha\in\Delta_{\mathfrak{p}}^+} \wedge\mathfrak{p}_{-\alpha}$ is the product of the characters on $\wedge\mathfrak{p}_{-\alpha}$. By Proposition 7.4 the action of $\mathfrak{h}$ as a subalgebra of $\mathfrak{u}$ differs from the adjoint action by a $\rho_{\mathfrak{p}}$-shift accounting for an extra factor $\mathrm{e}(\rho_{\mathfrak{p}})$. ■

Consider $L(\mu)\otimes \mathrm{S}_{\mathfrak{p}}$ as a super representation of $\mathfrak{u}$. Its formal super character is

$$\mathrm{ch}\big(L(\mu)\otimes\mathrm{S}_{\mathfrak{p}}\big) = \mathrm{ch}\big(L(\mu)\big)\,\mathrm{ch}(\mathrm{S}_{\mathfrak{p}}).$$

On the other hand, since $D_{L(\mu)}$ is an odd skew-adjoint operator on this space, this coincides with

$$\mathrm{ch}\big(\ker(D_{L(\mu)})\big) = \sum_{w\in\mathfrak{p}} (-1)^{l(w)} \mathrm{ch}\Big(M\big(w(\mu+\rho)-\rho_{\mathfrak{u}}\big)\Big).$$

This gives the generalized Weyl-Kac character formula,

$$\mathrm{ch}\big(L(\mu)\big) = \frac{\sum_{w\in W_{\mathfrak{p}}} (-1)^{l(w)} \mathrm{ch}\big(M(w(\mu+\rho)-\rho_{\mathfrak{u}})\big)}{\mathrm{e}(\rho_{\mathfrak{p}})\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}(1-\mathrm{e}(-\alpha))^{n_{\mathfrak{p}}(\alpha)}},$$

valid for quadratic subalgebras $\mathfrak{u}\subset\mathfrak{g}$ of the form considered above. For $\mathfrak{u}=\mathfrak{h}$ one recovers the usual Weyl–Kac character formula [5, §10.4] for symmetrizable Kac–Moody algebras. Note that the Weyl–Kac character formula also holds for the non-symmetrizable case; see Kumar [13, Chapter 3.2]. We do not know how to treat this general case using cubic Dirac operators.

Example 7.7 As a concrete example, consider the Kac–Moody algebra of hyperbolic type, associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ (cf. [5, Exercise 5.28]). The Weyl group W is generated by the reflections r_1, r_2 corresponding to α_1, α_2. The set P^+ of dominant weights is generated by $\varpi_1 = -\frac{1}{5}(2\alpha_1 + 3\alpha_2)$ and $\varpi_2 = -\frac{1}{5}(2\alpha_2 + 3\alpha_1)$. One has $\rho = \varpi_1 + \varpi_2 = -(\alpha_1 + \alpha_2)$.

Put $\Pi_\mathfrak{u} = \{\beta_1, \beta_2\}$ with $\beta_1 = \alpha_1$, $\beta_2 = r_2(\alpha_1) = \alpha_1 + 3\alpha_2$. Since $\beta_2 - \beta_1 = 3\alpha_2$ is not a root, $\Pi_\mathfrak{u}$ is the set of simple roots for a Kac–Moody Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$. One finds that $\rho_\mathfrak{u} = \varpi_1$, and the fundamental $\mathfrak{u}$-weights spanning $P^+_\mathfrak{u}$ are $\tau_1 = \varpi_1 - \frac{1}{3}\varpi_2$ and $\tau_2 = \frac{1}{3}\varpi_2$.

The Weyl group $W_\mathfrak{u}$ is generated by the reflections defined by β_1, β_2, *i.e.,* by r_1 and $r_2 r_1 r_2$. A general element of $W_\mathfrak{u}$ is thus a word in r_1, r_2, with an even number of r_2's. One has $W_\mathfrak{p} = \{1, r_2\}$, giving duplets of $\mathfrak{u}$-representations. Write weights $\mu \in P^+$ in the form $\mu = k_1\varpi_1 + k_2\varpi_2$. Then the corresponding duplet is given by the weights

$$\mu + \rho - \rho_\mathfrak{u} = k_1\varpi_1 + (k_2 + 1)\varpi_2 = k_1\tau_1 + (k_1 + 3k_2 + 3)\tau_2,$$

$$r_2(\mu + \rho) - \rho_\mathfrak{u} = (k_1 + 3(k_2 + 1))\varpi_1 - (k_2 + 1)\varpi_2 = (k_1 + 3k_2 + 3)\tau_1 + k_2\tau_2.$$

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