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STRONG COMMUTATIVITY PRESERVING MAPS OF SEMIPRIME RINGS

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ABSTRACT. In this paper we characterize maps $f: R \longrightarrow R$ where R is semiprime, f is additive, and $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. It is shown that $f(x) = \lambda x + \xi(x)$ where $\lambda \in C$, $\lambda^2 = 1$, and $\xi: R \longrightarrow C$ is additive where C is the extended centroid of R.

1. **Introduction and preliminaries.** If R is a ring a map $f: R \longrightarrow R$ is *strong commutativity preserving* (SCP) on a set $S \subseteq R$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. It appears that this notion was first introduced by Bell and Mason in [3]. In [2] Bell and Daif studied non-trivial endomorphisms and derivations which are SCP on right ideals in prime or semiprime rings. In general they showed that the existence of such a map forces commutativity on a large part of the ring in question. In this note we study maps $f: R \longrightarrow R$ which are merely additive, but SCP on the entire semiprime ring R. Our main result states that such a map has the form $f(x) = \lambda x + \xi(x)$ where $\lambda \in C$, $\lambda^2 = 1$, and $\xi: R \longrightarrow C$ is an additive map from R to its extended centroid C.

In all that follows R will denote a semiprime ring, Q its Martindale ring of quotients, and C its extended centroid. If I is an ideal in R then $I^{\perp}$ will denote its annihilator.

We will need the following three results:

(A) [4, COROLLARY 3.2]. Suppose that $a, b \in R$ satisfy $axb = bxa$ for all $x \in R$. Then there exist idempotents $\epsilon_1, \epsilon_2, \epsilon_3 \in C$ such that $\epsilon_i \epsilon_j = 0$, $i \neq j$, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$, $\epsilon_1 a = 0$, $\epsilon_2 b = 0$, and $\epsilon_3 b = \lambda \epsilon_3 a$ for some invertible $\lambda \in C$.

(B) [4, THEOREM 4.1]. If $B: R \times R \longrightarrow R$ is a biderivation, then there exist an idempotent $\epsilon \in C$ and an element $\mu \in C$ such that $(1 - \epsilon)R \subseteq C$ and $\epsilon B(x, y) = \mu \epsilon [x, y]$ for all $x, y \in R$.

(C) [1 (ORIGINALLY), OR 4, COROLLARY 4.2]. If $f: R \longrightarrow R$ is an additive commuting map, then there exist $\lambda \in C$ and an additive map $\xi: R \longrightarrow C$ such that $f(x) = \lambda x + \xi(x)$ for all $x \in R$.

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2. **The main result.** We begin with a technical lemma.

LEMMA. *Let K be the ideal of R generated by all commutators in R. Suppose that $(\lambda_0\mu_0 - 1)K = 0$ for some $\mu_0, \lambda_0 \in C$. Then there exists an invertible element $\lambda \in C$ such that $(\lambda - \lambda_0)R \subseteq C$ and $(\lambda^{-1} - \mu_0)R \subseteq C$. Moreover, if $\lambda_0 = \mu_0$, then $\lambda = \lambda^{-1}$.*

PROOF. There exists an idempotent $\epsilon \in C$ such that $K^\perp = \epsilon Q \cap R$ (*cf.* [4]). Define $\lambda, \mu \in C$ by $\lambda = \lambda_0(1-\epsilon)+\epsilon$, $\mu = \mu_0(1-\epsilon)+\epsilon$. Whence $(\lambda\mu - 1) = (\lambda_0\mu_0 - 1)(1-\epsilon)$ which yields $(\lambda\mu - 1)(K \oplus K^\perp) = 0$ for $(\lambda_0\mu_0 - 1)K = 0$ and $(1-\epsilon)K^\perp = 0$. Since $K \oplus K^\perp$ is an essential ideal of R it follows that $\lambda\mu - 1 = 0$, that is, $\mu = \lambda^{-1}$. Clearly, $\lambda_0 = \mu_0$ implies $\lambda = \mu = \lambda^{-1}$.

We claim that $\epsilon R \subseteq C$. Indeed, there exists an essential ideal E such that $\epsilon E \subseteq R$ and hence $\epsilon E \subseteq R \cap \epsilon Q = K^\perp$, that is, $K\epsilon E = 0$ which gives $\epsilon K = 0$; thus, $[\epsilon R, R] = \epsilon[R,R] = 0$ which shows that $\epsilon R \subseteq C$. Therefore, as $\lambda - \lambda_0 = (1-\lambda_0)\epsilon$, we see that $(\lambda - \lambda_0)R \subseteq C$. Similarly, $(\lambda^{-1} - \mu_0)R = (1-\mu_0)\epsilon R \subseteq C$.

We are now in a position to prove

THEOREM 1. *Let R be a semiprime ring with extended centroid C. Suppose that an additive map $f: R \longrightarrow R$ satisfies $[f(x), f(y)] = [x,y]$ for all $x, y \in R$. Then f is of the form $f(x) = \lambda x + \xi(x)$ where $\lambda \in C$, $\lambda^2 = 1$, and ξ is an additive map of R into C.*

PROOF. Our first goal is to prove that f is commuting. For $x, y \in R$ we have

$$\begin{aligned}\big[f(y^2), [y,x]\big] &= \big[f(y^2), [f(y), f(x)]\big]\\ &= \big[f(x), [f(y), f(y^2)]\big] + \big[f(y), [f(y^2), f(x)]\big]\\ &= \big[f(x), [y, y^2]\big] + \big[f(y), [y^2, x]\big]\\ &= \big[f(y), [y^2, x]\big].\end{aligned}$$

Thus,

$$\big[f(y^2), [y,x]\big] = \big[f(y), [y^2,x]\big] \quad \text{for all } x, y \in R. \tag{1}$$

In particular, $\big[f(y^2), [y,yx]\big] = \big[f(y), [y^2,yx]\big]$. But on the other hand,

$$\begin{aligned}\big[f(y^2), [y,yx]\big] &= \big[f(y^2), y[y,x]\big] = [f(y^2), y][y,x] + y\big[f(y^2), [y,x]\big],\\ \big[f(y), [y^2,yx]\big] &= \big[f(y), y[y^2,x]\big] = [f(y), y][y^2,x] + y\big[f(y), [y^2,x]\big].\end{aligned}$$

Comparing both results and using (1) we arrive at

$$[f(y^2), y][y,x] = [f(y), y][y^2,x] \quad \text{for all } x, y \in R.$$

Replacing x by xz and using $[y,xz] = [y,x]z + x[y,z]$, $[y^2,xz] = [y^2,x]z + x[y^2,z]$, we then get

$$[f(y^2), y]x[y,z] = [f(y), y]x[y^2,z] \quad \text{for all } x, y, z \in R.$$

Replacing y by $f(a)$ we thus obtain

$$\big[f\big(f(a)^2\big), f(a)\big]x[f(a), z] = \big[f\big(f(a)\big), f(a)\big]x[f(a)^2, z],$$

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which can be according to the initial assumption, written in the form

$$[f(a)^2, a]x[f(a), z] = [f(a), a]x[f(a)^2, z] \quad \text{for all } x, z, a \in R. \tag{2}$$

Now fix $a \in R$ and let us show that $[f(a), a] = 0$. As a special case of (2) we have

$$[f(a)^2, a]x[f(a), a] = [f(a), a]x[f(a)^2, a] \quad \text{for all } x \in R.$$

Applying (A) we see that there are mutually orthogonal idempotents $\epsilon_1, \epsilon_2, \epsilon_3 \in C$ with sum 1 such that $\epsilon_1[f(a), a] = 0$, $\epsilon_2[f(a)^2, a] = 0$, $\epsilon_3[f(a)^2, a] = \nu\epsilon_3[f(a), a]$ for some invertible $\nu \in C$. By (2) we thus obtain

$$\begin{aligned}
[f(a), a]x[f(a)^2, z] &= (\epsilon_1 + \epsilon_2 + \epsilon_3)[f(a), a]x[f(a)^2, z] \\
&= (\epsilon_2 + \epsilon_3)[f(a), a]x[f(a)^2, z] \\
&= (\epsilon_2 + \epsilon_3)[f(a)^2, a]x[f(a), z] \\
&= \epsilon_3[f(a)^2, a]x[f(a), z] \\
&= \nu\epsilon_3[f(a), a]x[f(a), z].
\end{aligned}$$

Setting $\mu = \nu\epsilon_3$ we thus have $[f(a), a]x[f(a)^2 - \mu f(a), z] = 0$ for all $x, z \in R$. That is, $[f(a)^2 - \mu f(a), R] \subseteq I$ where $I = \{q \in Q \mid [f(a), a]Rq = 0\}$. Of course, I is a right ideal of Q. Now, for any $z \in R$ we have

$$\begin{aligned}
\mu[a, z] - f(a)[a, z] - [a, z]f(a) &= \mu[f(a), f(z)] - f(a)[f(a), f(z)] - [f(a), f(z)]f(a) \\
&= [\mu f(a), f(z)] - [f(a)^2, f(z)] \\
&= [\mu f(a) - f(a)^2, f(z)],
\end{aligned}$$

which shows that

$$\mu[a, z] - f(a)[a, z] - [a, z]f(a) \in I \quad \text{for all } z \in R.$$

Replacing z by za it follows that

$$\mu[a, z]a - f(a)[a, z]a - [a, z]af(a) \in I.$$

On the other hand, since I is a right ideal, we have

$$\big(\mu[a, z] - f(a)[a, z] - [a, z]f(a)\big)a \in I.$$

Comparing the last two relations we get $[a, z][f(a), a] \in I$ for all $z \in R$. That is, $[f(a), a]R[a, z][f(a), a] = 0$ for every $z \in R$. Replacing z by $f(a)z$ and using $[a, f(a)z] = [a, f(a)]z + f(a)[a, z]$ it follows at once that $[f(a), a]R[a, f(a)]R[f(a), a] = 0$. Since R is semiprime it follows that $[f(a), a] = 0$. Thus we proved that f is commuting.

According to (C) we have $f(x) = \lambda_0 x + \xi_0(x)$, $x \in R$, where $\lambda_0 \in C$ and ξ_0 is an additive map of R into C. Therefore, the relation $[f(x), f(y)] = [x, y]$ can be rewritten as $(\lambda_0^2 - 1)[x, y] = 0$, which shows that $(\lambda_0^2 - 1)K = 0$. By the Lemma, there is $\lambda \in C$ such that $\lambda^2 = 1$ and $(\lambda - \lambda_0)R \subseteq C$. For any $x \in R$ we thus have

$$f(x) = \lambda_0 x + \xi_0(x) = \lambda x + (\lambda_0 - \lambda)x + \xi_0(x) = \lambda x + \xi(x),$$

where $\xi(x) = (\lambda_0 - \lambda)x + \xi_0(x) \in C$. This proves the theorem.

Assuming that f is onto, even a stronger result can be easily obtained:

THEOREM 2. *Let R be a semiprime ring with extended centroid C. Suppose that additive maps $f, g: R \longrightarrow R$ satisfy $[f(x), g(y)] = [x, y]$ for all $x, y \in R$. If f is onto, then there exists an invertible element $\lambda \in C$ and additive maps $\xi, \eta: R \longrightarrow C$ such that $g(x) = \lambda x + \xi(x)$, $f(x) = \lambda^{-1}x + \eta(x)$ for all $x \in R$.*

PROOF. Define a biadditive map $B: R \times R \longrightarrow R$ by $B(x, y) = [x, g(y)]$. Clearly, B is a derivation in the first argument. Pick $x_0 \in R$; as f is onto, we have $x_0 = f(x_1)$ for some $x_1 \in R$. Thus $B(x_0, y) = [f(x_1), g(y)] = [x_1, y]$. This shows that B is a derivation in the second argument, *i.e.*, B is a biderivation. By (B) there are $\epsilon, \mu \in C$, ϵ an idempotent, such that $(1-\epsilon)R \subseteq C$ and $\epsilon[x, g(y)] = \epsilon\mu[x, y]$ for all $x, y \in R$. Thus, $[R, \epsilon g(y) - \epsilon\mu y] = 0$ and so $\epsilon g(y) - \epsilon\mu y \in C$ for all $y \in R$. Whence $g(y) - \epsilon\mu y = \big(\epsilon g(y) - \epsilon\mu y\big) + (1-\epsilon)g(y) \in C$, and so $g(y) = \lambda_0 y + \xi_0(y)$ where $\lambda_0 = \epsilon\mu \in C$, $\xi_0(y) = g(y) - \epsilon\mu y \in C$. By the initial assumption it now follows that $[x, f(x)] = \big[f(x), g\big(f(x)\big)\big] = 0$, $x \in R$; that is, f is commuting. Therefore, f is of the form $f(x) = \mu_0 x + \eta_0(x)$, $\mu_0 \in C$, $\eta_0(x) \in C$. By $[f(x), g(y)] = [x, y]$ it now follows at once that $(\lambda_0\mu_0 - 1)K = 0$. By the Lemma there is an invertible $\lambda \in C$ such that $(\lambda - \lambda_0)R \subseteq C$, $(\lambda^{-1} - \mu_0)R \subseteq C$. Whence

$$f(x) = \mu_0 x + \eta_0(x) = \lambda^{-1}x + (\mu_0 - \lambda^{-1})x + \eta_0(x) = \lambda^{-1}x + \eta(x),$$
$$g(x) = \lambda_0 x + \xi_0(x) = \lambda x + (\lambda_0 - \lambda)x + \xi_0(x) = \lambda x + \xi(x),$$

where $\eta(x) = (\mu_0 - \lambda_1^{-1})x + \eta_0(x) \in C$, $\xi(x) = (\lambda_0 - \lambda)x + \xi_0(x) \in C$. The proof is completed.

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