J. Austral Math. Soc. (*Series A*) **27** (1979) 467–478

THE MOORE–PENROSE INVERSE OF PARTICULAR BORDERED MATRICES

FRANK J. HALL

(Received 21 December 1976; revised 23 June 1978)

Communicated by H. Lausch

Abstract

The Moore–Penrose inverse of a general bordered matrix is found under various conditions. The Moore–Penrose inverses obtained by Hall and Hartwig (1976) are shown to be special cases of these more general results.

Subject classification (*Amer. Math. Soc.* (*MOS*) *1970*): primary 15 A 09; secondary 15 A 21.

1. Introduction

In this paper we consider the general bordered matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

and find the Moore-Penrose inverse $M\dagger$ of M under various conditions. These conditions involve $MM\dagger$ and $M\dagger$–M being block diagonal, and the forms for $M\dagger$ obtained by Hall and Hartwig (1976) when $D = 0$ are special cases of the present results. We again make use of some of the techniques given by Ben-Israel and Greville (1974).

All matrices of this paper are over the complex field. If A is a complex matrix, $R(A)$ denotes the range of A, A^* the conjugate transpose of A, $N(A)$ the null space of A and $P_{N(A)}$ the orthogonal projection onto $N(A)$. The Moore–Penrose inverse $A\dagger$ of A is the unique matrix X which satisfies the Penrose equations:

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3)\ (AX)^* = AX,\quad (4)\ (XA)^* = XA.$$

https://doi.org/10.1017/S144678870001346X Published online by Cambridge University Press

In general, if a matrix X satisfies equations (i), (j) and (k), then X is called an (i,j,k)-inverse of A. For properties of these various inverses the reader can see Ben-Israel and Greville (1974).

2. Results

We first prove the following lemma.

LEMMA 2.1. *Suppose that*

$$\text{(i)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \text{and} \quad \text{(ii)}\ R(B^*) \cap R(D^*) = \{0\}.$$

Then

$$N(A) \cap N(C) = N((I - BB\dagger)A) \cap N((I - DD\dagger)C). \tag{2.1}$$

PROOF. Letting B and D have m columns, condition (ii) is equivalent to

$$\{R(B^*) \cap R(D^*)\}^{\perp} = C^m \quad \text{or} \quad N(B) + N(D) = C^m.$$

Using Lemma 2, Chapter 5, in Ben-Israel and Greville (1974), we then have

$$(P_{N(B)} + P_{N(D)})(P_{N(B)} + P_{N(D)})\dagger = P_{N(B)+N(D)} = I.$$

Hence, condition (ii) implies that

$$BP_{N(D)}(P_{N(B)} + P_{N(D)})\dagger = B. \tag{2.2}$$

Here and subsequently P_X, where X is some expression, is to be interpreted as P with subscript X.

Now, let $x \in N((I - BB\dagger)A) \cap N((I - DD\dagger)C)$. Then $Ax = BB\dagger Ax$, $Cx = DD\dagger Cx$, and from (2.2) it follows that

$$\begin{bmatrix} A \\ C \end{bmatrix} x = \begin{bmatrix} B \\ D \end{bmatrix} (D\dagger Cx + P_{N(D)}(P_{N(B)} + P_{N(D)})\dagger(B\dagger A - D\dagger C)x).$$

But then from condition (i) we have $Ax = 0$ and $Cx = 0$, and so

$$N((I - BB\dagger)A) \cap N((I - DD\dagger)C) \subseteq N(A) \cap N(C).$$

Clearly, the opposite inclusion is always the case, and (2.1) is now proved.

We now give one of the forms for $M\dagger$.

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THEOREM 2.2. *Under the assumptions that*

$$\text{(i)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)\cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right)=\{0\}\quad \textit{and}\quad \text{(ii)}\ R(B^*)\cap R(D^*)=\{0\},$$

the matrix

$$Y=\left[\begin{array}{c|c} \dfrac{Q(P+Q)^\dagger[(I-BB^\dagger)A]^\dagger}{HB^\dagger-KQ(P+Q)^\dagger[(I-BB^\dagger)A]^\dagger} & \dfrac{[(I-DD^\dagger)C]^\dagger-Q(P+Q)^\dagger[(I-DD^\dagger)C]^\dagger}{D^\dagger-HD^\dagger-K([(I-DD^\dagger)C]^\dagger-Q(P+Q)^\dagger[(I-DD^\dagger)C]^\dagger)} \end{array}\right]$$

is a $(1,2,4)$*-inverse for* M, *where*

$$P=P_{N((I-BB^\dagger)A)},\quad Q=P_{N((I-DD^\dagger)C)},\quad H=P_{N(D)}(P_{N(B)}+P_{N(D)})^\dagger$$

and

$$K=D^\dagger C+H(B^\dagger A-D^\dagger C).$$

If we further assume

$$\text{(iii)}\ R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right)\cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right)=\{0\},$$

then $Y=M^\dagger$.

PROOF. Since

$$[(I-BB^\dagger)A]^\dagger=[(I-BB^\dagger)A]^\dagger(I-BB^\dagger)$$

and

$$[(I-DD^\dagger)C]^\dagger=[(I-DD^\dagger)C]^\dagger(I-DD^\dagger)$$

we have by direct multiplication

$$YM=\left[\begin{array}{c|c} \dfrac{\begin{array}{c}[(I-DD^\dagger)C]^\dagger(I-DD^\dagger)C+Q(P+Q)^\dagger([(I-BB^\dagger)A]^\dagger\\ \times(I-BB^\dagger)A-[(I-DD^\dagger)C]^\dagger(I-DD^\dagger)C)\end{array}}{K-K\ ((1,1)\text{ position of } YM)} & \dfrac{0}{D^\dagger D+H(B^\dagger B-D^\dagger D)} \end{array}\right].$$

As in the proof of Theorem 6, Chapter 5, in Ben-Israel and Greville (1974), the $(1,1)$ and $(2,2)$ positions of YM become

$$I-P_{N((I-BB^\dagger)A)\cap N((I-DD^\dagger)C)}\quad\text{and}\quad I-P_{N(B)\cap N(D)},$$

respectively.

Now, assuming conditions (i)–(ii) we have (2.1) from the lemma, and hence

$$I-P_{N((I-BB\dagger)A)\cap N((I-DD\dagger)C)} = I-P_{N(A)\cap N(C)}. \tag{2.3}$$

It then follows that the (2, 1) position of YM is zero and that

$$YM = \left[\begin{array}{c|c} \dfrac{I-P_{N(A)\cap N(C)}}{0} & \dfrac{0}{I-P_{N(B)\cap N(D)}} \end{array}\right]. \tag{2.4}$$

Thus, $MYM = M$ and $(YM)^* = YM$.

As in the proof of the Theorem 6 in Ben-Israel and Greville (1974),

$$P_{N((I-BB\dagger)A)\cap N((I-DD\dagger)C)}\, Q(P+Q)\dagger = \tfrac{1}{2}P_{N((I-BB\dagger)A)\cap N((I-DD\dagger)C)}$$

and

$$P_{N(B)\cap N(D)}P_{N(D)}(P_{N(B)}+P_{N(D)})\dagger = \tfrac{1}{2}P_{N(B)\cap N(D)}.$$

Using (2.3)–(2.4) it can then be verified that $YMY = Y$.

Finally, using (2.2) we have by direct multiplication

$$MY = \left[\begin{array}{c|c} \dfrac{BB\dagger+(I-BB\dagger)AQ(P+Q)\dagger[(I-BB\dagger)A]\dagger}{0} & \dfrac{(I-BB\dagger)A[(I-DD\dagger)C]\dagger-(I-BB\dagger)AQ(P+Q)\dagger[(I-DD\dagger)C]\dagger}{DD\dagger+(I-DD\dagger)C[(I-DD\dagger)C]\dagger} \end{array}\right].$$

We now assume condition (iii), from which it follows that

$$R(A^*(I-BB\dagger))\cap R(C^*(I-DD\dagger)) = \{0\}.$$

Hence, as in the proof of Lemma 2.1,

$$(P+Q)(P+Q)\dagger = I, \tag{2.5}$$

and therefore $(I-BB\dagger)AQ(P+Q)\dagger = (I-BB\dagger)A$. Thus

$$MY = \left[\begin{array}{c|c} \dfrac{BB\dagger+(I-BB\dagger)A[(I-BB\dagger)A]\dagger}{0} & \dfrac{0}{DD\dagger+(I-DD\dagger)C[(I-DD\dagger)C]\dagger} \end{array}\right]$$

and $(MY)^* = MY$. The proof of the theorem is now complete.

It can be seen from the proof of the theorem that we need only assume (2.1) in order for Y to be a (1, 2, 4)-inverse for M. And, under this assumption YM is block diagonal. Consequently, from the results in Hall and Hartwig (1976) we have

$$N(A)\cap N(C) = N((I-BB\dagger)A)\cap N((I-DD\dagger)C) \Rightarrow R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)\cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\}$$

$\Leftrightarrow M\dagger M$ is block diagonal $\Leftrightarrow$ the blocks in the (1, 3)-inverses of M are independent of each other.

https://doi.org/10.1017/S144678870001346X Published online by Cambridge University Press

From Lemma 2.1 the first implication goes both ways if we assume

$$R(B^*)\cap R(D^*) = \{0\}.$$

In the same way, the blocks in the (1, 4)-inverses for M are independent of each other $\Leftrightarrow MM^\dagger$ is block diagonal

$$\Leftrightarrow R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right)\cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\} \Rightarrow R(A^*(I-BB^\dagger))\cap R(C^*(I-DD^\dagger)) = \{0\}.$$

If we assume $R(B^*)\cap R(D^*) = \{0\}$, the last implication goes both ways.

If $D = 0$ the conditions in Theorem 2.2 are the same as the conditions in Theorem 4.1 in Hall and Hartwig (1976). Furthermore, when $D = 0$,

$$P_{N(D)}(P_{N(B)}+P_{N(D)})^\dagger B^\dagger = I(P_{N(B)}+I)^\dagger B^\dagger = \tfrac{1}{2}(I+B^\dagger B)B^\dagger = B^\dagger,$$

and thus the matrix given in Theorem 4.1 in Hall and Hartwig is a special case of the matrix given in Theorem 2.2. It is also possible to give generalizations of the other forms in Hall and Hartwig.

In the particular case where $R(B^*)\subseteq N(D)$ we have

$$(P_{N(B)}+P_{N(D)})^\dagger P_{N(D)} B^\dagger = B^\dagger$$

from (2.2) and hence

$$(P_{N(B)}+P_{N(D)})^\dagger B^\dagger = B^\dagger.$$

But $R(B^*)\subseteq N(D) \Leftrightarrow R(D^*)\subseteq N(B)$ and so we also get

$$(P_{N(B)}+P_{N(D)})^\dagger D^\dagger = D^\dagger.$$

In this case the matrix Y of Theorem 2.2 simplifies and we have the following corollary.

COROLLARY 2.3. *Under the assumptions that*

$$\text{(i)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)\cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \text{and} \quad \text{(ii)}\ R(B^*)\subseteq N(D)$$

the matrix

$$Y_1 = \left[\begin{array}{c|c} \dfrac{Q(P+Q)^\dagger[(I-BB^\dagger)A]^\dagger}{B^\dagger-(D^\dagger C+B^\dagger A)Q(P+Q)^\dagger[(I-BB^\dagger)A]^\dagger} & \dfrac{[(I-DD^\dagger)C]^\dagger-Q(P+Q)^\dagger[(I-DD^\dagger)C]^\dagger}{D^\dagger-(D^\dagger C+B^\dagger A)([(I-DD^\dagger)C]^\dagger-Q(P+Q)^\dagger[(I-DD^\dagger)C]^\dagger)} \end{array}\right]$$

https://doi.org/10.1017/S144678870001346X Published online by Cambridge University Press

is a $(1,2,4)$*-inverse for* M *where* $P = P_{N((I-BB^\dagger)A)}$ *and* $Q = P_{N((I-DD^\dagger)C)}$. *If we further assume*

$$\text{(iii)}\quad R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\},$$

then $Y_1 = M^\dagger$.

When we replace the condition

$$R(B^*) \cap R(D^*) = \{0\}$$

by the condition

$$R(A^*) \cap R(C^*) = \{0\},$$

we obtain an analogous theorem, which we state without proof.

THEOREM 2.4. *Under the assumptions that*

$$\text{(i)}\quad R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \text{and} \quad \text{(ii)}\quad R(A^*) \cap R(C^*) = \{0\},$$

the matrix

$$W = \left[\begin{array}{c|c} HA^\dagger - KQ(P+Q)^\dagger[(I-AA^\dagger)B]^\dagger & C^\dagger - HC^\dagger - K([(I-CC^\dagger)D]^\dagger - Q(P+Q)^\dagger[(I-CC^\dagger)D]^\dagger) \\ \hline Q(P+Q)^\dagger[(I-AA^\dagger)B]^\dagger & [(I-CC^\dagger)D]^\dagger - Q(P+Q)^\dagger[(I-CC^\dagger)D]^\dagger \end{array}\right]$$

is a $(1,2,4)$*-inverse for* M, *where*

$$P = P_{N((I-AA^\dagger)B)}, \quad Q = P_{N((I-CC^\dagger)D)}, \quad H = P_{N(C)}(P_{N(A)} + P_{N(C)})^\dagger$$

and

$$K = C^\dagger D + H(A^\dagger B - C^\dagger D).$$

If we further assume

$$\text{(iii)}\quad R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\}$$

then $W = M^\dagger$.

As in Theorem 2.2 we have in this case

$$N(B) \cap N(D) = N((I-AA^\dagger)B) \cap N((I-CC^\dagger)D) \Rightarrow R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\}$$

and

$$R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\} \Rightarrow R(B^*(I-AA^\dagger)) \cap R(D^*(I-CC^\dagger)) = \{0\},$$

and these implications go both ways if we assume $R(A^*)\cap R(C^*) = \{0\}$.

In the particular case where $R(A^*)\subseteq N(C)$ we have the following simplification.

COROLLARY 2.5. *Under the assumptions that*

$$\text{(i)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \textit{and} \quad \text{(ii)}\ R(A^*) \subseteq N(C)$$

the matrix

$$W_1 = \left[\begin{array}{c|c} \dfrac{A^\dagger - (C^\dagger D + A^\dagger B)\, Q(P+Q)^\dagger[(I-AA^\dagger)\,B]^\dagger}{Q(P+Q)^\dagger[(I-AA^\dagger)\,B]^\dagger} & \dfrac{C^\dagger - (C^\dagger D + A^\dagger B)([(I-CC^\dagger)\,D]^\dagger - Q(P+Q)^\dagger[(I-CC^\dagger)\,D]^\dagger)}{[(I-CC^\dagger)\,D]^\dagger - Q(P+Q)^\dagger[(I-CC^\dagger)\,D]^\dagger} \end{array}\right]$$

is a $(1,2,4)$*-inverse for* M, *where* $P = P_{N((I-AA^\dagger)B)}$ *and* $Q = P_{N((I-CC^\dagger)D)}$. *If we further assume*

$$\text{(iii)}\ R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\},$$

then $W_1 = M^\dagger$.

We now combine the conditions of the above two theorems and obtain another simple form for $M^\dagger$.

COROLLARY 2.6. *Under the assumptions that*

$$\text{(i)}\ R(A^*)\cap R(C^*) = \{0\}, \quad \text{(ii)}\ R(B^*)\cap R(D^*) = \{0\}$$

and

$$\text{(iii)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\},$$

$$M^\dagger = \left[\begin{array}{c|c} \dfrac{Q_1(P_1+Q_1)^\dagger[(I-BB^\dagger)\,A]^\dagger}{Q_2(P_2+Q_2)^\dagger[(I-AA^\dagger)\,B]^\dagger} & \dfrac{P_1(P_1+Q_1)^\dagger[(I-DD^\dagger)\,C]^\dagger}{P_2(P_2+Q_2)^\dagger[(I-CC^\dagger)\,D]^\dagger} \end{array}\right]$$

where

$$P_1 = P_{N((I-BB^\dagger)A)},\ Q_1 = P_{N((I-DD^\dagger)C)},\ P_2 = P_{N((I-AA^\dagger)B)}\ \textit{and}\ Q_2 = P_{N((I-CC^\dagger)D)}.$$

PROOF. It is easy to see that conditions (i)–(ii) imply that

$$R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\},$$

so that we have the conditions of both of the above theorems. Now, from (2.5) we obtain

$$\begin{aligned}&[(I-DD\dagger)C]\dagger - Q_1(P_1+Q_1)\dagger[(I-DD\dagger)C]\dagger \\ &\quad = (P_1+Q_1)(P_1+Q_1)\dagger[(I-DD\dagger)C]\dagger - Q_1(P_1+Q_1)\dagger[(I-DD\dagger)C]\dagger \\ &\quad = P_1(P_1+Q_1)\dagger[(I-DD\dagger)C]\dagger.\end{aligned}$$

Similarly, $(P_2+Q_2)(P_2+Q_2)\dagger = I$ and hence

$$[(I-CC\dagger)D]\dagger - Q_2(P_2+Q_2)\dagger[(I-CC\dagger)D]\dagger = P_2(P_2+Q_2)\dagger[(I-CC\dagger)D]\dagger.$$

Then, from the uniqueness of $M\dagger$, the result follows from the above theorems.

We will now present a form for $M\dagger$ where we assume $N(B) \subseteq N(D)$—a condition opposite to the condition $R(B^*) \cap R(D^*) = \{0\}$. We first establish the following lemma.

LEMMA 2.7. *Suppose that*

$$R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\}.$$

Then

(2.6) $$N(A) \cap N(C) = N((I-BB\dagger)A) \cap N(C-DB\dagger A).$$

PROOF. Let $x \in N((I-BB\dagger)A) \cap N(C-DB\dagger A)$. Then $Ax = BB\dagger Ax$, $Cx = DB\dagger Ax$, and hence

$$\begin{bmatrix} A \\ C \end{bmatrix} x = \begin{bmatrix} B \\ D \end{bmatrix} B\dagger Ax.$$

But then from our assumption it follows that $Ax = 0$ and $Cx = 0$; thus

$$N((I-BB\dagger)A) \cap N(C-DB\dagger A) \subseteq N(A) \cap N(C).$$

Clearly the opposite inclusion always holds and (2.6) is now proved.

THEOREM 2.8. *Under the assumptions that*

$$\text{(i)}\ R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \textit{and} \quad \text{(ii)}\ N(B) \subseteq N(D),$$

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the matrix

$$X = \begin{bmatrix} [(I-BB^\dagger)A]^\dagger - P(P+Q)^\dagger([(I-BB^\dagger)A]^\dagger + (C-DB^\dagger A)^\dagger DB^\dagger) & P(P+Q)^\dagger(C-DB^\dagger A)^\dagger \\ \hline B^\dagger - HB^\dagger - K([(I-BB^\dagger)A]^\dagger - P(P+Q)^\dagger([(I-BB^\dagger)A]^\dagger + (C-DB^\dagger A)^\dagger DB^\dagger)) & HD^\dagger - KP(P+Q)^\dagger(C-DB^\dagger A)^\dagger \end{bmatrix}$$

is a (1, 2, 4)*-inverse for* M, *where*

$$P = P_{N((I-BB^\dagger)A)},\ Q = P_{N(C-DB^\dagger A)},\quad H = P_{N(B)}(P_{N(B)} + P_{N(D)})^\dagger$$

and

$$K = B^\dagger A + H(D^\dagger C - B^\dagger A).$$

If we further assume

$$\text{(iii)}\ R(A^*(I-BB^\dagger)) \cap R((C-DB^\dagger A)^*) = \{0\}$$

and

$$\text{(iv)}\ (C-DB^\dagger A)(C-DB^\dagger A)^\dagger DB^\dagger = DB^\dagger,$$

then $X = M^\dagger$.

PROOF. The details of the proof are similar to the proofs of the above two theorems. Since $[(I-BB^\dagger)A]^\dagger = [(I-BB^\dagger)A]^\dagger(I-BB^\dagger)$ and $DB^\dagger B = D$ assuming condition (ii), we have by direct multiplication

$$XM = \begin{bmatrix} [(I-BB^\dagger)A]^\dagger(I-BB^\dagger)A + P(P+Q)^\dagger((C-DB^\dagger A)^\dagger (C-DB^\dagger A) - [(I-BB^\dagger)A]^\dagger(I-BB^\dagger)A) & 0 \\ \hline K - K\ ((1,1) \text{ position of } XM) & B^\dagger B + H(D^\dagger D - B^\dagger B) \end{bmatrix}.$$

As in the proof of Theorem 2.2, the (1, 1) and (2, 2) positions of XM become

$$I - P_{N((I-BB^\dagger)A)\cap N(C-DB^\dagger A)} \quad \text{and} \quad I - P_{N(B)\cap N(D)},$$

respectively.

Now, assuming condition (i) we have (2.6) from the lemma, and hence

$$(2.7) \qquad I - P_{N((I-BB^\dagger)A)\cap N(C-DB^\dagger A)} = I - P_{N(A)\cap N(C)}.$$

It then follows that the (2, 1) position of XM is zero and that

$$XM = \left[\begin{array}{c|c} \dfrac{I - P_{N(A)\cap N(C)}}{0} & \dfrac{0}{I - P_{N(B)\cap N(D)}} \end{array}\right]. \tag{2.8}$$

Thus, $MXM = M$ and $(XM)^* = XM$.

As in the proof of Theorem 2.2,

$$P_{N((I-BB\dagger)A)\cap N(C-DB\dagger A)}P(P+Q)\dagger = \tfrac{1}{2}P_{N((I-BB\dagger)A)\cap N(C-DB\dagger A)}$$

and

$$P_{N(B)\cap N(D)}P_{N(B)}(P_{N(B)}+P_{N(D)})\dagger = \tfrac{1}{2}P_{N(B)\cap N(D)}.$$

Using (2.7–2.8) it can then be verified that $XMX = X$.

Finally, using condition (ii) we have by direct multiplication

$$MX = \left[\begin{array}{c|c} \dfrac{BB\dagger + (I-BB\dagger)A[(I-BB\dagger)A]\dagger}{\begin{array}{r}(C-DB\dagger A)[(I-BB\dagger)A]\dagger - (C-DB\dagger A)P(P+Q)\dagger \\ \times([(I-BB\dagger)A]\dagger + (C-DB\dagger A)\dagger DB\dagger) + DB\dagger\end{array}} & \dfrac{0}{(C-DB\dagger A)P(P+Q)\dagger(C-DB\dagger A)\dagger} \end{array}\right]$$

We now assume condition (iii); as in the proof of Lemma 2.1, we then have $(P+Q)(P+Q)\dagger = I$ and so $(C-DB\dagger A)P(P+Q)\dagger = C-DB\dagger A$. Thus

$$MX = \left[\begin{array}{c|c} \dfrac{BB\dagger + (I-BB\dagger)A[(I-BB\dagger)A]\dagger}{DB\dagger - (C-DB\dagger A)(C-DB\dagger A)\dagger DB\dagger} & \dfrac{0}{(C-DB\dagger A)(C-DB\dagger A)\dagger} \end{array}\right].$$

Condition (iv) then guarantees that $(MX)^* = MX$. The proof of the theorem is now complete.

It can be seen that the first matrix given after Theorem 4.1 in Hall and Hartwig (1976) is a special case of the matrix given in Theorem 2.8.

If we consider the condition $N(A) \subseteq N(C)$ we have the following analogous theorem, which we state without proof.

THEOREM 2.9. *Under the assumptions that*

$$\text{(i) } R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \quad \textit{and} \quad \text{(ii) } N(A) \subseteq N(C),$$

the matrix

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$$Z=\left[\begin{array}{c|c} A^\dagger-HA^\dagger-K([(I-AA^\dagger)B]^\dagger-P(P+Q)^\dagger((D-CA^\dagger B)^\dagger CA^\dagger \\ \qquad +[(I-AA^\dagger)B]^\dagger)) & HC^\dagger-KP(P+Q)^\dagger(D-CA^\dagger B)^\dagger \\ \hline [(I-AA^\dagger)B]^\dagger-P(P+Q)^\dagger((D-CA^\dagger B)^\dagger CA^\dagger+[(I-AA^\dagger)B]^\dagger) & P(P+Q)^\dagger(D-CA^\dagger B)^\dagger \end{array}\right]$$

is a $(1,2,4)$*-inverse for* M, *where*

$$P=P_{N((I-AA^\dagger)B)},\quad Q=P_{N(D-CA^\dagger B)},\quad H=P_{N(A)}(P_{N(A)}+P_{N(C)})^\dagger$$

and

$$K=A^\dagger B+H(C^\dagger D-A^\dagger B).$$

If we further assume

$$\text{(iii)}\quad R(B^*(I-AA^\dagger))\cap R((D-CA^\dagger B)^*)=\{0\}$$

and

$$\text{(iv)}\quad (D-CA^\dagger B)(D-CA^\dagger B)^\dagger CA^\dagger=CA^\dagger,$$

then $Z=M^\dagger$.

For the previous two theorems we have

$$R\left(\begin{bmatrix}A\\C\end{bmatrix}\right)\cap R\left(\begin{bmatrix}B\\D\end{bmatrix}\right)=\{0\}\Rightarrow N((I-BB^\dagger)A)\cap N(C-DB^\dagger A)=N(A)\cap N(C)$$

and

$$R\left(\begin{bmatrix}A\\C\end{bmatrix}\right)\cap R\left(\begin{bmatrix}B\\D\end{bmatrix}\right)=\{0\}\Rightarrow N((I-AA^\dagger)B)\cap N(D-CA^\dagger B)=N(B)\cap N(D),$$

and the two implications go both ways if we assume $N(B)\subseteq N(D)$ and $N(A)\subseteq N(C)$, respectively. Furthermore, under the assumptions of these two theorems, both $M^\dagger M$ and $MM^\dagger$ are again block diagonal in each case.

There are analogous forms for $M^\dagger$ when we assume $N(D)\subseteq N(B)$ and $N(C)\subseteq N(A)$, instead of $N(B)\subseteq N(D)$ and $N(A)\subseteq N(C)$, respectively.

We should note that various other forms for the Moore–Penrose inverse of bordered matrices have also been given in Burns *et al.* (1974), Hung and Markham (1975a, 1975b) and Hartwig (1976), using techniques different than those in the present paper.

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Department of Mathematics
Pembroke State University
Pembroke, North Carolina 28372
U.S.A.

Author's current address:
Department of Mathematics
Georgia State University
Atlanta, Georgia 30303
U.S.A.