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JEŚMANOWICZ' CONJECTURE ON PYTHAGOREAN TRIPLES

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Abstract

In 1956, Jeśmanowicz conjectured that, for any positive integers m and n with $m > n$, $\gcd(m, n) = 1$ and $2 \nmid m + n$, the Diophantine equation $(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. In this paper, we prove the conjecture if $4 \nmid mn$ and $y \geq 2$.

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1. Introduction

In 1956, Sierpiński [10] showed that the equation $3^x + 4^y = 5^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. Jeśmanowicz [2] proved that each of the equations $5^x + 12^y = 13^z$, $7^x + 24^y = 25^z$, $9^x + 40^y = 41^z$, $11^x + 60^y = 61^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$, and conjectured that, for any positive integers a, b, c with $a^2 + b^2 = c^2$ and $\gcd(a, b) = 1$, the equation

$$a^x + b^y = c^z \tag{1.1}$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

It is well known that, if a, b, c are positive integers with

$$a^2 + b^2 = c^2, \quad \gcd(a, b) = 1, \quad 2 \mid b,$$

then there exist two integers m, n with

$$m > n > 0, \quad \gcd(m, n) = 1, \quad m + n \equiv 1 \pmod 2$$

such that

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

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Now Equation (1.1) becomes

$$(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z. \tag{1.2}$$

Jeśmanowicz' conjecture has been proved for many special cases. In 1959, Lu [4] proved that Jeśmanowicz' conjecture is true for $n = 1$. In 1995, Le [3] showed that if $2 \parallel mn$ and $m^2 + n^2$ is a power of an odd prime, then Jeśmanowicz' conjecture is true. In 2013, Miyazaki [7] showed that if $a \equiv \pm 1 \pmod{b}$ or $c \equiv 1 \pmod{b}$, then Jeśmanowicz' conjecture is true. Since $m^2 + n^2 \equiv 1 \pmod{2mn}$ for $m = n + 1$, Jeśmanowicz' conjecture is true for $m = n + 1$. In the following, we always assume that

$$m > n + 1 > 1, \quad \gcd(m, n) = 1, \quad m + n \equiv 1 \pmod{2}. \tag{1.3}$$

In 2014, Terai [15] proved Jeśmanowicz' conjecture is true for $n = 2$. In 2015, Miyazaki and Terai [8] proved that Jeśmanowicz' conjecture is true if $m > 72n$, $n \equiv 2 \pmod{4}$ and n satisfies at least one of the following conditions:

(C1) $n/2$ is a power of an odd prime;
(C2) $n/2$ has no prime factors congruent to 1 modulo 8;
(C3) $n/2$ is a square.

For more results on the conjecture, see [1, 5, 9, 11–14].

In this paper, we obtain the following result.

Theorem 1.1. *Suppose that $4 \nmid mn$. Then the equation*

$$(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z, \quad y \geq 2,$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

In view of (1.3), it is clear that $4 \nmid mn$ if and only if either $m \equiv 2 \pmod{4}$ or $n \equiv 2 \pmod{4}$. This is equivalent to $c = m^2 + n^2 \equiv 5 \pmod{8}$.

2. Preliminary lemmas

Lemma 2.1 [6, Theorem 1.5]. *Let (x, y, z) be a positive integer solution of Equation (1.2). If x and z are even, then both $x/2$ and $z/2$ are odd.*

Lemma 2.2. *Let (x, y, z) be a positive integer solution of Equation (1.2). If $y \leq 2$ and x, z are even integers, then $x = y = z = 2$.*

Proof. Let $x = 2x_1$ and $z = 2z_1$. By (1.2),

$$\begin{aligned}(2mn)^y &= ((m^2 + n^2)^{z_1} + (m^2 - n^2)^{x_1})((m^2 + n^2)^{z_1} - (m^2 - n^2)^{x_1}) \\ &\geq (m^2 + n^2)^{z_1} + (m^2 - n^2)^{x_1} \\ &> (m^2 + n^2)^{z_1} > (2mn)^{z_1}.\end{aligned}$$

It follows from $y \leq 2$ that $z_1 = 1$ and $y = 2$. Thus $z = 2$. By (1.2), $x = 2$. □

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Lemma 2.3 [6, Lemma 2.1]. *Let* (x, y, z) *be a positive integer solution of Equation (1.2). Then* x *is even if one of the following holds:*

(1) *there exists a divisor* d *of* m *such that* $d \not\equiv 1 \pmod 4$*;*
(2) $n \equiv 2 \pmod 4$.

In particular, $mn \equiv 2 \pmod 4$ *implies that* x *is even.*

Lemma 2.4. *Let* (x, y, z) *be a positive integer solution of Equation (1.2) with* $y \geq 2$. *Suppose that* $mn \equiv 2 \pmod 4$. *Then* z *is even.*

Proof. Since $mn \equiv 2 \pmod 4$, it follows that

$$c = m^2 + n^2 \equiv 5 \pmod 8.$$

By Lemma 2.3, x is even. In view of $y \geq 2$, (1.2) and $4 \mid b$,

$$5^z \equiv c^z = a^x + b^y \equiv 1 \pmod 8.$$

It follows that z is even. □

3. Proof of Theorem 1.1

In this section, we assume that (x, y, z) is a positive integer solution of (1.2) with $y \geq 2$. Noting that $4 \nmid mn$, by Lemmas 2.3 and 2.4, $2 \mid x$ and $2 \mid z$. Let $u = m$ and $v = n$ if $n \equiv 2 \pmod 4$ and let $u = n$ and $v = m$ if $m \equiv 2 \pmod 4$. Then

$$u > 0, \quad v > 0, \quad \gcd(u, v) = 1, \quad u + v \equiv 1 \pmod 2, \quad v \equiv 2 \pmod 4.$$

It is clear that $u^2 + v^2 \equiv 5 \pmod 8$. Since $2 \mid x$, it follows from (1.2) that

$$(u^2 - v^2)^x + (2uv)^y = (u^2 + v^2)^z. \tag{3.1}$$

Let $x = 2x_1$ and $z = 2z_1$. By Lemma 2.1, $2 \nmid x_1$ and $2 \nmid z_1$. By Lemma 2.2, we may assume that $y \geq 3$. Now Equation (3.1) can be rewritten as

$$(2uv)^y = ((u^2 + v^2)^{z_1} + (u^2 - v^2)^{x_1})((u^2 + v^2)^{z_1} - (u^2 - v^2)^{x_1}). \tag{3.2}$$

If $u > v$, then

$$(u^2 + v^2)^{z_1} + (u^2 - v^2)^{x_1} > 0.$$

It follows from (3.2) that

$$(u^2 + v^2)^{z_1} - (u^2 - v^2)^{x_1} > 0.$$

If $u < v$, then, since $2 \nmid x_1$,

$$(u^2 + v^2)^{z_1} - (u^2 - v^2)^{x_1} > 0.$$

It follows from (3.2) that

$$(u^2 + v^2)^{z_1} + (u^2 - v^2)^{x_1} > 0.$$

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In both cases,

$$(u^2+v^2)^{z_1}+(u^2-v^2)^{x_1}>0,\quad (u^2+v^2)^{z_1}-(u^2-v^2)^{x_1}>0.$$

Noting that

$$((u^2+v^2)^{z_1}+(u^2-v^2)^{x_1},(u^2+v^2)^{z_1}-(u^2-v^2)^{x_1})=2$$

and

$$(u^2+v^2)^{z_1}-(u^2-v^2)^{x_1}\equiv 0 \pmod 4,$$

by (3.2), we see that

$$(u^2+v^2)^{z_1}+(u^2-v^2)^{x_1}=2(u_1v_1)^y, \tag{3.3}$$
$$(u^2+v^2)^{z_1}-(u^2-v^2)^{x_1}=2^{2y-1}(u_2v_2)^y, \tag{3.4}$$

where

$$u=u_1u_2,\quad v=2v_1v_2,\quad (u_1,u_2)=1,\quad (v_1,v_2)=1.$$

By (3.3) and (3.4),

$$(u^2+v^2)^{z_1}=(u_1v_1)^y+2^{2y-2}(u_2v_2)^y. \tag{3.5}$$

In view of (3.5), $y\ge 3$ and $2\nmid z_1$,

$$(u_1v_1)^y\equiv (u^2+v^2)^{z_1}\equiv 5^{z_1}\equiv 5 \pmod 8.$$

So

$$2\nmid y,\quad u_1v_1\equiv 5 \pmod 8. \tag{3.6}$$

For any prime factor p of v_1, by (3.3),

$$u^{2z_1}+u^{2x_1}\equiv 0 \pmod p.$$

Thus

$$u^{2|z_1-x_1|}\equiv -1 \pmod p. \tag{3.7}$$

Since x_1 and z_1 are odd, it follows that $4\mid 2(z_1-x_1)$. By (3.7), the multiplicative order of u modulo p is divisible by 8 and so $8\mid p-1$. Hence $v_1\equiv 1 \pmod 8$ and by (3.6), $u_1\equiv 5 \pmod 8$.

By (3.3),

$$(u^2+v^2)^{z_1}\equiv 2(u_1v_1)^y \pmod{u+v}.$$

In the following, we use $(*/*)$ to denote the Jacobi symbol. Noting that y (see (3.6)) and z_1 are odd,

$$\left(\frac{u^2+v^2}{u+v}\right)^{z_1}=\left(\frac{2v^2}{u+v}\right)=\left(\frac{2}{u+v}\right),$$
$$\left(\frac{2}{u+v}\right)\left(\frac{u_1v_1}{u+v}\right)^{y}=\left(\frac{2}{u+v}\right)\left(\frac{u_1v_1}{u+v}\right).$$

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Hence
$$\left(\frac{u_1 v_1}{u+v}\right) = 1. \tag{3.8}$$
Since
$$u_1 \equiv 5 \pmod 8, \quad v_1 \equiv 1 \pmod 8,$$
it follows from (3.8) that
$$\left(\frac{u+v}{u_1}\right)\left(\frac{u+v}{v_1}\right) = 1,$$
that is,
$$\left(\frac{v}{u_1}\right)\left(\frac{u}{v_1}\right) = 1.$$
Hence
$$\left(\frac{2v_1 v_2}{u_1}\right)\left(\frac{u_1 u_2}{v_1}\right) = 1$$
and so
$$\left(\frac{2}{u_1}\right)\left(\frac{v_1}{u_1}\right)\left(\frac{v_2}{u_1}\right)\left(\frac{u_1}{v_1}\right)\left(\frac{u_2}{v_1}\right) = 1. \tag{3.9}$$
Since $u_1 \equiv 5 \pmod 8$, it follows that
$$\left(\frac{2}{u_1}\right) = -1, \quad \left(\frac{v_1}{u_1}\right)\left(\frac{u_1}{v_1}\right) = 1. \tag{3.10}$$
By (3.9) and (3.10),
$$\left(\frac{v_2}{u_1}\right) = -\left(\frac{u_2}{v_1}\right). \tag{3.11}$$
Since y is odd, it follows from (3.5) that
$$\left(\frac{u_2 v_2}{u_1}\right) = 1, \quad \left(\frac{u_1 v_1}{u_2}\right) = 1$$
and so
$$\left(\frac{u_2}{u_1}\right) = \left(\frac{v_2}{u_1}\right), \quad \left(\frac{u_1}{u_2}\right) = \left(\frac{v_1}{u_2}\right). \tag{3.12}$$
Noting that
$$u_1 \equiv 5 \pmod 8, \quad v_1 \equiv 1 \pmod 8,$$
by (3.11) and (3.12),
$$\left(\frac{v_2}{u_1}\right) = -\left(\frac{u_2}{v_1}\right) = -\left(\frac{v_1}{u_2}\right) = -\left(\frac{u_1}{u_2}\right) = -\left(\frac{u_2}{u_1}\right) = -\left(\frac{v_2}{u_1}\right),$$
a contradiction. This completes the proof of Theorem 1.1.

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