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SOME RESULTS IN THE CONNECTIVE K-THEORY OF LIE GROUPS

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ABSTRACT. In this paper we give a description of:
(1) the Hopf algebra structure of $k^*(G; L)$ when G is a compact, connected Lie group and L is a ring of type $Q(P)$ so that $H^*(G; L)$ is torsion free;
(2) the algebra structure of $k^*(G_2; L)$ for $L = \mathbf{Z}_2$ or $\mathbf{Z}$.

Introduction. In this paper we study the connective K-theory of compact connected Lie groups. We use mainly Borel's results in the ordinary cohomology of Lie groups, L. Hodgkin's paper [6] about their K-theory, the Atiyah-Hirzebruch spectral sequence [2] and L. Smith's exact sequence relating connective K-theory with integral cohomology [9].

In the first paragraph we give some results in the connective K-theory that will be used later. In paragraph 2 we work out the Atiyah-Hirzebruch spectral sequence converging to $k^*(X)$ (connective K-cohomology of a compact CW complex). In the other paragraphs, using the previous results, we obtain the Hopf algebra structure of $k^*(G; L)$, L a ring of type $Q(P)$ (it will be defined in Section 2) so that $H^*(G; L)$ is torsion free, and the algebra structure of $k^*(G_2; L)$, $L = \mathbf{Z}_2$ or $\mathbf{Z}$.

We work in the homotopy category of (compact when stated) CW complexes.

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1. **Preliminaries.** Let $K = (K_n, \sigma_n)_{n\in\mathbf{Z}}$ be the spectrum for K-theory. We recall that K is a periodic, ring Ω-spectrum and $K^*(pt) = \mathbf{Z}[t, t^{-1}]$, the Laurent polynomial ring generated by the class of the reduced Hopf bundle $t^{-1} \in K^{-2}(pt)$ and its inverse [10].

The spectrum $k = (k_n, \bar{\sigma}_n)_{n\in\mathbf{Z}}$ for connective K-theory is obtained from the spectrum K by making it connective. Let $j:k \to K$ be the associated map of spectra. We note that k is a commutative, associative, ring Ω-spectrum, j is a

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map of ring spectra and $k^*(pt) = \mathbf{Z}[t^{-1}]$. Also there is a map of ring spectra $\eta: k \to H\mathbf{Z}$ ($H\mathbf{Z}$ denotes the Eilenberg-Maclane spectrum with integer coefficients) so that it induces the homomorphism $\eta^*: k^*(pt) \to H^*(pt; \mathbf{Z})$ given by $\eta^i = 0$ if $i > 0$ and identity if $i = 0$ ([8] pp. 35-37).

We can consider $\mathbf{Z}_p$ coefficients, p prime. We define $\tilde{k}^i(X; \mathbf{Z}_p) = \tilde{k}^{i+2}(X \wedge M_p)$, where M_p is the space obtained by attaching a 2-cell e^2 to S^1 by a map of degree p. There is a universal coefficient formula – $\tilde{k}^i(X; \mathbf{Z}_p) = \tilde{k}^i(X) \otimes \mathbf{Z}_p \oplus \mathrm{Tor}(\tilde{k}^{i+1}(X); \mathbf{Z}_p)$ – and an associative multiplication on $\tilde{k}^*(X; \mathbf{Z}_p)$ since $\tilde{k}^*$ satisfies the sufficient conditions for their existence [1]. If L is a free abelian group we define $k^*(X; L) = k^*(X) \otimes L$.

We note that if X is a CW-complex and L is a free abelian group or $\mathbf{Z}_p$ then $k^*(X; L)$ is a $L[t^{-1}]$ algebra.

We will use the following generalization of L. Smith's theorem [9]:

1.1 THEOREM. *Let X be a CW complex. Then there is an exact sequence*

$$0 \to L \otimes k^*(X; L) \overset{\eta_L^*}{\to} H^*(X; L) \to \mathrm{Tor}_{1,*}^{L[t^{-1}]}(L; k^*(X; L)) \to 0,$$

where η_L^* is induced by $1 \otimes \eta^*: L \otimes k^*(X) \to L \otimes H^*(X; \mathbf{Z})$ if L is a free abelian group or η_L^* is $1 \otimes \eta^*: \mathbf{Z} \otimes k^*(X) \to \mathbf{Z} \otimes H^*(X; \mathbf{Z})$ "reduced mod p" ($p > 1$) *if $L = \mathbf{Z}_p$, the tensor products being taken over $L[t^{-1}]$.*

PROOF. We consider the cofibration of spectra

$$k \underset{m}{\to} k \underset{\eta}{\to} H\mathbf{Z},$$

where m is the morphism of spectra corresponding to multiplication by t^{-1} in k-cohomology. It induces for every CW-complex X the long exact sequence

$$\ldots \to k^i(X) \overset{m^*}{\to} k^{i-2}(X) \overset{\eta^*}{\to} H^{i-2}(X; \mathbf{Z}) \overset{\delta^*}{\to} k^{i+1}(X) \to \ldots \ (i \geqq 2),$$

that splits into short exact sequences:

$$0 \to \mathrm{coker}\ m^i \overset{\eta^*}{\to} H^{i-2}(X; Z) \overset{\delta^*}{\to} \ker m^{i+1} \to 0$$

It is clear that tensoring by L or taking $X \wedge M_p$ instead of X does not affect exactness. Then the result follows as in [9]. □

To simplify the notation we shall write η^* instead of η_L^*.

2. **Spectral sequences.** From now on we deal with compact spaces. Let X be a compact CW-complex. We are going to consider the following Atiyah-Hirzebruch spectral sequences: $(E_r^{**}(X), d_r)_{r \geqq 2}$ converging to $K^*(X)$, $('E_r^{**}(X), d_r)_{r \geqq 2}$ converging to $k^*(X)$. Let $F_p^m(X) = \ker[K^m(X) \to K^m(X^{p-1})]$ and $'F_p^m(X) = \ker[k^m(X) \to k^m(X^{p-1})]$ be the filtrations. The first spectral sequence is compatible with the Bott isomorphism.

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To simplify the notation we omit X when there will be no confusion about the space concerned.

We note that, since $K^q(pt) = 0 = k^q(pt)$ if q is odd and $k^q(pt) = 0$ if $q > 0$, then $E_r^{p,q} = 0 = {}'E_r^{p,q}$ for all $p \in \mathbf{Z}$, $r \geqq 2$, q an odd integer, ${}'E_r^{p,q} = 0$ if $q > 0$ and all the differentials of even degree are zero. Moreover, we have for all $i, n \in \mathbf{Z}$: $F_{n-1}^i = F_n^i$ and ${}'F_{n-1}^i = {}'F_n^i$ if $n - i$ is even; $F_n^i = F_{n+1}^i$ and ${}'F_n^i = {}'F_{n+1}^i$ if $n - i$ is odd; $m^*(F_n^i) = F_n^{i-2}$; ${}'F_n^n(X) = k^n(X)$.

2.1 Proposition. *Let X be a compact CW-complex. Then*:

(i) $j_s^{**}: {}'E_s^{p,q} \to E_s^{p,q}$ *is an isomorphism for* $q \leqq -\dim X + 1$;

(ii) *if* $d_r = 0$ *for* $r > s$ *then* $j^*|_{{}'F_n^m}$ *is an isomorphism onto* F_n^m *for all* $m \in \mathbf{Z}$, $n \geqq m + s - 1$.

Proof.

(i) One can easily show by induction on $r \geqq 2$ a more general result: $j_r^{**}: {}'E_r^{p,q} \to E_r^{p,q}$ is surjective if $-r + 3 \leqq q \leqq 0$ and an isomorphism if $q \leqq -r + 2$.

This proof can be done by diagram chasing:

$$\begin{array}{ccccc}
{}'E_s^{p-s,q+s-1} & \xrightarrow{d'_s} & {}'E_s^{p,q} & \xrightarrow{d'_s} & {}'E_s^{p+s,q-s+1} \\
\downarrow j_s^{**} & & \downarrow j_s^{**} & & \downarrow j_s^{**} \\
E_s^{p-s,q+s-1} & \xrightarrow{d_s} & E_s^{p,q} & \xrightarrow{d_s} & E_s^{p+s,q-s+1}
\end{array}$$

(ii) Now we consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}'F_{p+1}^m & \longrightarrow & {}'F_p^m & \longrightarrow & {}'E_\infty^{p,m-p} & \longrightarrow & 0 \\
 & & \downarrow j_{p+1}^* & & \downarrow j_p^* & & \downarrow j_\infty^{**} & & \\
0 & \longrightarrow & F_{p+1}^m & \longrightarrow & F_p^m & \longrightarrow & E_\infty^{p,m-p} & \longrightarrow & 0
\end{array}$$

Using (i), the 5-lemma and decreasing induction on p, supposing $m + s - 1 \leqq p \leqq \dim X$, we get the result. □

We need to consider $Q(P)$ coefficients, where P is a set of prime numbers and $Q(P)$ the quotient ring of $\mathbf{Z}$ with respect to the multiplicative subset generated by P. The spectral sequence for $k^*(-; Q(P)) = k^*(-) \otimes Q(P)$ is obtained from that one for $k^*(-)$ by tensoring by $Q(P)$. The idea of taking $Q(P)$ is "to kill" the p-torsion when suitable.

2.2 Proposition. *Let X be a compact CW-complex, L a ring of type $Q(P)$* or $\mathbf{Z}_p$. *Then $x \in H^p(X; L)$ lies in the image of $\eta^*: k^*(X; L) \to H^*(X; L)$ if and only if x, considered as an element of ${}'E_2^{p,0}$, is an infinite cycle in the spectral sequence ${}'E_r^{**}$ converging to $k^*(X; L)$.*

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PROOF. It follows immediately from the morphism of spectral sequences for the cohomology theories $k^*(-; L)$ and $H^*(-; L)$ induced by the natural transformation $\eta^*:k^*(-; L) \to H^*(-; L)$. □

3. $k^*(G; L)$. Let G be a compact connected Lie group of rank r, dimension n. Borel proved [3] that $H^*(G; \mathbf{Q})$ is an exterior algebra over $\mathbf{Q}$ generated by elements of odd degree, $H^*(G; \mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_1, \ldots, x_r)$, $\sum_{j=1}^r$ degree $(x_j) = n$. Furthermore those elements are primitive, universally transgressive.

Hodgkin [6] proved that:

If $\pi_1(G)$ is torsion free, $K^*(G)$, graded by $\mathbf{Z}_2$, is (1) the exterior algebra over $\mathbf{Z}$ on the module of primitive elements of degree 1; (2) if G is semi-simple $K^*(G) = \Lambda_{\mathbf{Z}}(\beta(\rho_1), \ldots, \beta(\rho_r))$ where $\rho_1, \ldots, \rho_r$ are the "basic representations", $\beta:R(G) \to K^1(G)$ the homomorphism that takes a representation $\rho:G \to U(n)$ into the class $[i_n\rho]$ ($i_n:U(n) \to U$ is the standard inclusion), and those generators $\beta(\rho_1), \ldots, \beta(\rho_r)$ are primitive.

Using the above results we obtain the following theorem:

3.1 THEOREM. *Let L be a ring of type $Q(P)$ (P any set of prime numbers) such that $H^*(G; L)$ is torsion free. Then*:

(i) *$k^*(G; L) \approx \Lambda_{L[t^{-1}]}(y_1, \ldots, y_r)$ where y_j has odd degree i_j for all $1 \leqq j \leqq r$, $n = \sum_{j=1}^r i_j$*;

(ii) *the y_j can be chosen so that they are primitive in the Hopf algebra $k^*(G; L)$.*

PROOF.

(i) The spectral sequence converging to $k^*(G; L)$ is trivial and as $L[t^{-1}]$ modules $k^*(G; L) \approx H^*(G; L) \otimes L[t^{-1}]$. By 2.2 we can take generators $y_1, \ldots, y_r$ of the $L[t^{-1}]$ algebra $k^*(G; L)$ so that $\eta^*(y_j) = x_j$, $1 \leqq j \leqq r$, where $x_1, \ldots, x_r$ are the primitive, universally transgressive generators of $H^*(G; L)$. They are unique modulo Im m^*. Since every element in $K^1(G; L)$ has zero square and j^* is an injective ring homomorphism, $y_j^2 = 0$ if $1 \leqq j \leqq r$.

(ii) Now we take the universal G-bundle

$$G \to EG \underset{p}{\to} BG$$

and the induced exact sequences

$$\tilde{E}^m(G, L) \underset{\delta^*}{\overset{\approx}{\to}} E^{m+1}(EG, G; L) \underset{p^*}{\leftarrow} \tilde{E}^{m+1}(BG; L),$$

where E^* is one of the cohomology theories k^*, K^* or H^*.

Since the generators x_j are universally transgressive, the y_j in (i) can be taken in $\delta^{*-1}(p^*(k^*(BG; L)))$. But $\delta^{*-1}(p^*(\tilde{K}^0(BG; L)))$ is the module of primitive elements in the $\mathbf{Z}_2$ graded K-cohomology [6] and j^* is injective. Hence the y_j are primitive. □

3.2 REMARK. Let G be a simple connected Lie group such that $H^*(G, \mathbf{Z})$ is torsion free and suppose that $\rho_1, \ldots, \rho_r$ are the basic representations of G. If p is odd and greater than 3, the primitive generators $\beta(\rho_i) \in K^1(G)$ do not lie in $j^*(k^p(G))$, since on the one hand $x \in K^1(G)$ lies in $F_p(K^1(G))$ if and only if $\mathrm{ch}_j(x) = 0$ for $j < p$ [4] (ch_j denotes the j-component of the Chern character) and on the other hand $\mathrm{ch}_3(\beta(\rho_i)) = n_i x_3$, where $n_i \geqq 1$ and x_3 is a generator of $H^3(G; \mathbf{Z})$ [5].

4. **Calculation of $k^*(G_2; \mathbf{Z}_2)$ and $k^*(G_2)$.** We now prove two theorems about the exceptional Lie group G_2.

4.1 THEOREM. *The $\mathbf{Z}_2[t^{-1}]$ algebra $k^*(G_2; \mathbf{Z}_2)$ is generated by $y_i \in k^i(G_2; \mathbf{Z}_2)$ $i = 5, 6, 9$ with $t^{-1}y_6 = 0$, $y_6 y_9 = 0$, $y_i^2 = 0$.*

PROOF. $H^*(G_2; \mathbf{Z}_2)$ is a $\mathbf{Z}_2$-algebra with a simple system of generators x_3, x_5, x_6, degree $x_i = i$ [3]. Let $\{E_r^{**}, d_r\}$ be the spectral sequence converging to $K^*(G_2; \mathbf{Z}_2)$. The only non-zero differential is $d_3 = Sq^1Sq^2 + Sq^2Sq^1$ ([6], III, Proposition 1.2). Therefore, $d_3x_3 = x_6$, $d_3(x_3x_5) = x_5x_6$ and d_3 is zero otherwise. By 2.1 this result holds for the spectral sequence converging to $k^*(G_2; \mathbf{Z}_2)$. Also all the extension exact sequence split. Thus $k^i(G_2; \mathbf{Z}_2)$ is equal to: 0, if $i > 14$ or $i = 13$; $\mathbf{Z}_2$, if $i = 14, 12, 11, 10, 9, 8, 7, 4, 2$; and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, otherwise.

The $\mathbf{Z}_2[t^{-1}]$ module structure can be obtained by using:

(i) The short exact sequences

$$0 \to \mathrm{coker}\, m^{i+2} \xrightarrow{\eta^*} H^i(X; Z_2) \to \ker m^{i+3} \to 0$$

(ii) If $a \in k^*(X; L)$ projects to $\bar{a} \in E_\infty^{**}$ and $t^{-1}\bar{a} \neq 0$ then $t^{-1}a \neq 0$.

By 1.1 we can take elements $\bar{y}_j \in k^*(G_2; \mathbf{Z}_2)/\mathrm{Im}\, m^*$, degree $\bar{y}_j = j$, $j \in \{5, 6, 9, 11, 14\}$, such that $\eta^*(\bar{y}_j) = x_j$ for $j = 5, 6$, $\eta^*(\bar{y}_9) = x_3x_6$, $\eta^*(\bar{y}_{11}) = x_5x_6$ and $\eta^*(\bar{y}_{14}) = x_3x_5x_6$. Furthermore those elements are unique. We take a representative y_j of each class $\bar{y}_j$, choosing y_6 so that $t^{-1}y_6 = 0$.

Let y_0 denote the algebra unit of $k^0(G_2; \mathbf{Z}_2)$. Then: y_j, $t^{-i}y_k$ form a $\mathbf{Z}_2$ basis of $k^j(G_2; \mathbf{Z}_2)$ for $j \in \{14, 11, 9, 6, 5, 0\}$, where $i \geqq 1$, $-2i + k = j$ and $k \in \{0, 5, 9, 14\}$. Moreover, $t^{-1}y_6 = 0 = t^{-1}y_{11}$.

Now the algebra structure can be easily obtained. We just observe that η^* is a ring homomorphism, η^i is injective for $i = 14, 11$, all the elements of $K^1(G_2; \mathbf{Z}_2)$ have zero squares and $j^*: k^{10} \to K^{10}$ is injective. □

4.2 THEOREM. *The $\mathbf{Z}[t^{-1}]$ algebra $k^*(G_2)$ is generated by $z_i \in k^i(G_2)$, $i \in \{3, 6, 9, 11, 14\}$ so that*

$$2z_6 = t^{-1}z_6 = z_3z_6 = 0,\ t^{-1}z_{11} = 2z_9,\ z_3z_9 = t^{-1}z_{14},\ 2z_{14} = z_3z_{11},\ z_i^2 = 0$$

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for all i and $z_i z_j = 0$ *for* $i + j > 14$.

PROOF. $H^*(G_2; \mathbf{Z})$ is an algebra generated by h_3, h_{11} of degree 3, 11 respectively, subjected to the relations: $2h_3^2 = h_3^4 = h_{11}^2 = h_3^2 h_{11} = 0$ [3]. Using 4.1 and the universal coefficient theorem we get the $\mathbf{Z}$-module structure of $k^*(G_2)$.

The same technique as in 4.1 applies here to obtain the $\mathbf{Z}[t^{-1}]$ module and algebra structure. □

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