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ONE-REGULAR CUBIC GRAPHS OF ORDER A SMALL NUMBER TIMES A PRIME OR A PRIME SQUARE

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Abstract

A graph is *one-regular* if its automorphism group acts regularly on the set of its arcs. In this paper we show that there exists a one-regular cubic graph of order $2p$ or $2p^2$ where p is a prime if and only if 3 is a divisor of $p - 1$ and the graph has order greater than 25. All of those one-regular cubic graphs are Cayley graphs on dihedral groups and there is only one such graph for each fixed order. Surprisingly, it can be shown that there is no one-regular cubic graph of order $4p$ or $4p^2$.

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1. Introduction

Throughout this paper a graph means an undirected finite one, without loops or multiple edges. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its automorphism group, respectively. For further group- and graph-theoretic notation and terminology, we refer the reader to [12] and [13].

Let G be a finite group and S a subset of G such that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $X = \mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set

$$V(X) = G$$

and edge set

$$E(X) = \{(g, sg) \mid g \in G, s \in S\}.$$

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From the definition, Cay(G, S) is connected if and only if S generates the group G.

A permutation group G on a set Ω is said to be *semiregular* if for each $\alpha \in \Omega$, the stabilizer G_α of α in G is the identity group, and *regular* if it is semiregular and transitive. Let X be a graph. A subgroup G of Aut(X) is said to be *regular* and *one-regular* if it acts regularly on the vertex set and the arc set of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive*, *arc-transitive* and *one-regular* (or *arc-regular*) if Aut(X) is vertex-transitive, edge-transitive, arc-transitive and one-regular, respectively, and *half-transitive* if Aut(X) is vertex-transitive, edge-transitive, but not arc-transitive.

Clearly, a one-regular graph of regular valency must be connected and a graph of valency 2 is one-regular if and only if it is a cycle. Marušič [17] and Malnič *et al.* [15] constructed two different kinds of infinite families of one-regular graphs of valency 4, and Xu [24] gave a classification of one-regular circulant graphs of valency 4. One-regular cubic graphs have also received considerable attention. The first example of one-regular cubic graph was constructed by Frucht [9] with 432 vertices, and lots of work has been done on one-regular cubic graphs as part of a more general problem dealing with the investigation of a class of arc-transitive cubic graphs (see [4, 6, 20]). In 1997, Marušič and Xu [19] showed a way to construct a one-regular cubic graph Y from a half-transitive graph X of valency 4 with girth 3 by letting the triangles of X be the vertices in Y with two triangles being adjacent in Y when they share a common vertex in X. Thus, one can construct infinitely many one-regular cubic graphs from the infinite family of half-transitive graphs of valency 4 with girth 3 constructed by Alspach *et al.* in [1] and from another infinite family of half-transitive graphs constructed by Marušič and Nedela in [18]. Recently, Feng *et al.* [8] classified one-regular cubic Cayley graphs on abelian or dihedral groups. In this paper, we classify one-regular cubic graphs of order $2p$, $4p$, $2p^2$ or $4p^2$, where p is a prime. A one-regular cubic graph of order $2p$ or $2p^2$ is a Cayley graph on a dihedral group. Such a graph exists only when 3 is a divisor of $p - 1$ and the graph has order greater than 25, and it is unique for each fixed order. Thus there exists a unique one-regular cubic Cayley graph on the dihedral group of order 26, which is the least one-regular cubic graph by Conder and Dobcsányi [3]. Surprisingly, there is no one-regular cubic graph of order $4p$ or $4p^2$.

We know that Cheng and Oxley [2] classified arc-transitive graphs of order $2p$. Among the graphs in their classification, there is a unique one-regular cubic graph for each prime $p \geq 13$ such that 3 is a divisor of $p - 1$. In this paper we show that a one-regular cubic graph of order twice an odd integer is a Cayley graph (Corollary 3.3), which implies that the unique one-regular cubic graph of a fixed order $2p$ in [2] must be a Cayley graph on a dihedral group. By using Corollary 3.3 we classify one-regular cubic graphs of order $2p^2$ and the same method can be used to classify one-regular cubic graphs of some orders, such as $6p$, $6p^2$, $2p^3$, $6p^3$. Note that it is easy to classify one-regular cubic graphs of order $3p$, $3p^2$, $5p$, $5p^2$, etc. since the valency 3 forces $p = 2$.

2. Preliminaries

We start with introducing five propositions for later applications in this paper. The first one has achieved a sort of folklore status, whereas the others are well known as group-theoretic results.

PROPOSITION 2.1. *A graph X is a Cayley graph if and only if* $\mathrm{Aut}(X)$ *contains a regular subgroup.*

PROPOSITION 2.2 ([23, Proposition 4.4]). *Any abelian transitive permutation group on a set is regular.*

PROPOSITION 2.3 ([23, Theorem 3.4]). *Let G be a permutation group on Ω and $\alpha \in \Omega$. Denote by α^G the orbit of α under G. Let p be a prime number and let p^m be a divisor of $|\alpha^G|$. Then p^m is also a divisor of $|\alpha^P|$ for any Sylow p-subgroup P of G.*

Let π be a nonempty set of primes and π' the set of primes which are not in π. A finite group G is called a *π-group*, if every prime factor of $|G|$ is in the set π. In this case, we also say that $|G|$ is a *π-number*.

Let G be a finite group. A π-subgroup H of G such that $|G : H|$ is a π'-number is called a *Hall π-subgroup* of G.

The following proposition is due to Hall [22].

PROPOSITION 2.4 ([22, Theorem 9.1.7]). *If G is a finite solvable group, then every π-subgroup is contained in a Hall π-subgroup of G. Moreover, all Hall π-subgroups of G are conjugate.*

Let p be a prime. A finite group G is called a *p-group* if it is a π-group for $\pi = \{p\}$.

PROPOSITION 2.5 ([13, Theorem 7.2]). *Let N be a nontrivial normal subgroup of a p-group G and $Z(G)$ the center of G. Then $N \cap Z(G) \neq 1$.*

The next two propositions give a classification of one-regular cubic Cayley graphs on abelian or dihedral groups.

PROPOSITION 2.6 ([8, Theorem 3.1]). *There is no one-regular cubic Cayley graph on an abelian group.*

PROPOSITION 2.7 ([8, Theorem 4.1]). *A cubic Cayley graph X on a dihedral group is one-regular if and only if X is isomorphic to* $\mathrm{Cay}(D_{2n}, \{a, ab, ab^{-k}\})$ *for $n \geq 13$, $3 \leq k < n$, and $k^2 + k + 1 \equiv 0 \pmod{n}$, where $D_{2n} = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1}\rangle$.*

By checking Conder and Dobcsányi's list [3] of arc-transitive cubic graphs up to 768 vertices, we have the following proposition.

PROPOSITION 2.8. *For any one-regular cubic graph X, $|V(X)| \geq 26$ and $|V(X)| \neq 4p$ or $4p^2$ for a prime $p \leq 13$.*

3. One-regular cubic graphs of order $2p$ or $2p^2$

In this section we classify one-regular cubic graphs of order $2p$ or $2p^2$, where p is a prime. Let $K_{3,3}$ be the bipartite graph of order 6. It is well-known that $\mathrm{Aut}(K_{3,3}) \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$ and so $\mathrm{Aut}(K_{3,3})$ has a normal Sylow 3-subgroup. From this, one may easily show the following lemma.

LEMMA 3.1. *Let G be a vertex-transitive automorphism group of the graph $K_{3,3}$. If $|G| = 18$ then G has a regular subgroup of order 6 and its Sylow 3-subgroup contains a minimal normal subgroup of G isomorphic to $\mathbb{Z}_3$.*

LEMMA 3.2. *A solvable one-regular automorphism group of a connected cubic graph contains a regular subgroup.*

PROOF. Suppose to the contrary; let X be a counterexample of least order, that is, X is of the smallest order with the following properties: X is a connected cubic graph and its automorphism group $\mathrm{Aut}(X)$ contains a solvable one-regular subgroup G, which has no regular subgroup.

Let N be a minimal normal subgroup of G. Since G is solvable N is elementary abelian, say $N = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p = \mathbb{Z}_p^m$, for a prime p and a positive integer m. By Proposition 2.2, N cannot be transitive on $V(X)$. Denote by $\Sigma = \{B_0, B_1, \ldots, B_{l-1}\}$ the set of orbits of N on $V(X)$. Since $N \lhd G$, Σ is a complete block system of G. Consider the quotient graph $\overline{X}$ of X defined by $V(\overline{X}) = \Sigma$ and $(B_i, B_j) \in E(\overline{X})$ if and only if there exist $v_i \in B_i$ and $v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. If N has more than two orbits, Lorimer [14, Theorem 9] showed that $\overline{X}$ is a cubic graph and G/N is a solvable one-regular subgroup of $\mathrm{Aut}(\overline{X})$ (also see [21]). The minimality of X implies that G/N has a regular subgroup, say H/N on $V(\overline{X})$ and so H acts regularly on $V(X)$, a contradiction. Thus we may assume that N has only two orbits; $\Sigma = \{B_0, B_1\}$. Let K be the subgroup of G which fixes B_0 setwise and let $u \in B_0$.

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It follows that $G/K \cong \mathbb{Z}_2$ and the one-regularity of G implies $G_u \cong \mathbb{Z}_3$, where G_u is the stabilizer of u in G. We also denote by K_u and N_u the stabilizers of u in K and N, respectively. Then $G_u \le K$, $G_u = K_u$, and $K = NK_u = NG_u$. If N is not semiregular, $N_u \cong \mathbb{Z}_3$. Since N is abelian N_u fixes B_0 pointwise. This implies $X \cong K_{3,3}$, the complete bipartite graph of order 6, and consequently $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, which is impossible since N is not minimal in G by Lemma 3.1. If N is semiregular then $|K| = |N||G_u| = 3p^m$ and $|G| = 6p^m$. Now we consider three cases: $p = 2$, $p = 3$ or $p \ne 2, 3$.

Case I: $p = 2$. In this case $|G| = 2^{m+1} \cdot 3$ and $|V(X)| = 2^{m+1}$. By Proposition 2.3, each Sylow 2-subgroup of G is transitive on $V(X)$ and so is regular because both the Sylow 2-subgroup and the graph X have the same order 2^{m+1}. It is impossible.

Case II: $p \ne 2, 3$. In this case $|G| = 2 \cdot 3 \cdot p^m$ and $|V(X)| = 2p^m$. Let $\pi = \{2, p\}$. By Proposition 2.4, G has a Hall π-subgroup, say H. Then $|H| = 2p^m$. Since $G_u \cong \mathbb{Z}_3$ and $|H|$ has no divisor 3, we have $H_u = 1$, where H_u is the stabilizer of u in H. Thus H has an orbit of length $2p^m$ and so acts regularly on $V(X)$, a contradiction.

Case III: $p = 3$. In this case $|G| = 2 \cdot 3^{m+1}$ and $|V(X)| = 2 \cdot 3^m$. It is easy to see that K is the unique Sylow 3-subgroup of G. Therefore $Z(K) \ne 1$ (a nilpotent group has a non-trivial center) and $Z(K) \lhd\!\lhd K$, that is $Z(K)$ is a characteristic subgroup of K. Thus $Z(K) \lhd G$. By Proposition 2.5 we have $N \cap Z(K) \ne 1$, and since $N \lhd G$ and $Z(K) \lhd G$, $N \cap Z(K) \lhd G$. By the minimality of N, $N \cap Z(K) = N$, which forces $N \le Z(K)$. Let $u, v \in B_0$. Then $N \le Z(K)$ implies $K_u = K_v$. It follows that K_u fixes B_0 pointwise and so $X \cong K_{3,3}$. By Lemma 3.1 G has a regular subgroup, a contradiction. □

Assume that X is a one-regular cubic graph and let $A = \mathrm{Aut}(X)$. If X has order $2n$ with n an odd integer, then $|A| = 2 \cdot 3 \cdot n$. Since a group of order twice an odd integer is solvable, A is solvable. By Lemma 3.2 and Proposition 2.1 we have the following corollary.

COROLLARY 3.3. *A one-regular cubic graph of order twice an odd integer is a Cayley graph.*

REMARK. Fang *et al.* [7] proved that Lemma 3.2 is also true for a connected graph of any prime valency.

Let $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ be the cyclic group of order n written additively and let $\mathbb{Z}_n^*$ be the multiplication group of $\mathbb{Z}_n$ consisting of numbers in $\mathbb{Z}_n$ coprime to n. Then $\mathbb{Z}_{p^m}^* \cong \mathbb{Z}_{(p-1)p^{m-1}}$ for any odd prime p and any positive integer m. If 3 is a divisor of $p-1$ then $\mathbb{Z}_{p^m}^*$ has a unique subgroup of order 3. The proof of the following lemma is easy and we omit it.

LEMMA 3.4. *Let $p > 3$ be a prime and $n = p$ or p^2. Then there exists an integer $1 \le k < n$ such that $k^2 + k + 1 \equiv 0 \pmod{n}$ if and only if k is an element of order 3 in $\mathbb{Z}_n^*$.*

THEOREM 3.5. *Let $n = p$ or p^2 for a prime p. Then there exists a one-regular cubic graph X of order $2n$ if and only if 3 is a divisor of $p - 1$ and $|V(X)| \ge 26$. Furthermore, for each prime p with 3 being a divisor of $p - 1$ and $n \ge 13$, there exists a unique one-regular cubic graph X of order $2n$ and $X = \mathrm{Cay}(G, S)$, where $G = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1}\rangle$ is a dihedral group and $S = \{a, ab, ab^{-k}\}$ with k being an element of order 3 in $\mathbb{Z}_n^*$.*

PROOF. Let X be a one-regular cubic graph of order $2n$ where $n = p$ or p^2, and let $A = \mathrm{Aut}(X)$. By Proposition 2.8, $p > 3$ and by Corollary 3.3 X is a Cayley graph, say $X = \mathrm{Cay}(G, S)$, where G is a group of order $2n$. Thus, Proposition 2.6 implies that G is nonabelian. Let A_1 denote the stabilizer of 1 in A and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Then $A_1 \cong \mathbb{Z}_3$ and $\mathrm{Aut}(G, S) \le A_1$. Since X is connected, $\langle S\rangle = G$. We claim that G is dihedral. But, it is obvious for $|G| = 2p$ because G is nonabelian.

Assume that $|G| = 2p^2$. From an elementary group theory we know that up to isomorphism there are three nonabelian groups of order $2p^2$ defined as:

$$G_1(p) = \langle a, b \mid a^2 = b^{p^2} = 1, aba = b^{-1}\rangle;$$
$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^2 = [a, b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle;$$
$$G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1}\rangle.$$

Suppose to the contrary that $G \ne G_1(p)$. Let $x, y \in G_2(p)$ with $o(x) = 2$ and $o(y) = p$. It is easy to show that $\langle x, y\rangle$ has order $2p$, and hence $\langle x, y\rangle \ne G_2(p)$. Thus if $G = G_2(p)$ then S consists of three involutions of $G_2(p)$ since $G = G_2(p)$ has no element of order $2p$. Let $z \in G_2(p)$ be an element of order p which is not in $\langle y\rangle$. Then x, y and z satisfy the same relations as do c, a and b, and hence there is an automorphism of $G_2(p)$ mapping x, y and z to c, a and b, respectively. Thus we may assume that $S = \{c, ca, cb\}$ because $\langle S\rangle = G$, and since the automorphism of $G_2(p)$ induced by $c \to c$, $a \to b$ and $b \to a$, interchanges ca and cb, and fixes c, $|\mathrm{Aut}(G, S)|$ has a divisor 2. By $\mathrm{Aut}(G, S) \le A_1$, $|A_1|$ has a divisor 2, contrary to the fact that $A_1 \cong \mathbb{Z}_3$. If $G = G_3(p)$ then S consists of one involution, one element of order p or $2p$ and its inverse because all involutions of $G_3(p)$ can't generate $G_3(p)$. Since the automorphism group of $G_3(p)$ is transitive on the set of involutions of $G_3(p)$, we may assume that $S = \{c, a^ib^j, (a^ib^j)^{-1}\}$ or $\{c, ca^ib^j, (ca^ib^j)^{-1}\}$, where $a^i \ne 1$ and $b^j \ne 1$ since $\langle S\rangle = G$. The mapping $c \to c$, $a \to a^i$ and $b \to b^j$ induces an automorphism of $G_3(p)$, and so we may assume that $S = \{c, ab, a^{-1}b^{-1}\}$

or $\{c, cab, ca^{-1}b\}$. For $S = \{c, ab, a^{-1}b^{-1}\}$, X has a cycle of length p passing through 1 and ab but there exists no such cycle passing through 1 and c, contrary to the arc-transitivity of X. For $S = \{c, cab, ca^{-1}b\}$, let α be a permutation on $G = G_3(p)$ defined by $(a^i b^j c^k)^\alpha = a^{-i} b^j c^k$ where i, j and k are integers. Let $g \in G$ and denote by $N(g)$ the neighborhood of g in X. Now it is easy to check that $N((a^i b^j c^k)^\alpha) = (N(a^i b^j c^k))^\alpha$, implying that α is an automorphism of X. Since α fixes 1, we have $\alpha \in A_1$ and so $|A_1|$ has a divisor 2, a contradiction.

So far, we have proved that X is a Cayley graph on a dihedral group. By Lemma 3.4 and Proposition 2.7 we have $|V(X)| \geq 26$ and $X = \mathrm{Cay}(G, S)$, where $G = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1}\rangle$ and $S = \{a, ab, ab^{-k}\}$ with k being an element of order 3 in $\mathbb{Z}_n^*$. Note that $\mathbb{Z}_n^*$ has elements of order 3 if and only if 3 is a divisor of $p - 1$. To prove Theorem 3.5, we only need to prove the uniqueness of one-regular cubic graph of order $2n$ when $p - 1$ has a divisor 3 and $n \geq 13$. Since $\mathbb{Z}_n^*$ has only two elements of order 3, that is k and k^2, it suffices to prove that $\mathrm{Cay}(G, \{a, ab, ab^{-k}\}) \cong \mathrm{Cay}(G, \{a, ab, ab^{-k^2}\})$, which follows from the fact that the automorphism of G induced by $a \to a$ and $b \to b^{-k^2}$ maps $\{a, ab, ab^{-k}\}$ to $\{a, ab, ab^{-k^2}\}$. □

4. No one-regular cubic graphs of order $4p$ or $4p^2$

To show the non-existence of one-regular cubic graphs of order $4p$ or $4p^2$, we need to consider regular coverings of the complete graph K_4 of order 4.

A graph $\widetilde{X}$ is called a *covering* of X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The covering $\widetilde{X}$ is said to be *regular* (or K-*covering*) if there is a semiregular subgroup K of $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (in this paper all functions are composed from left to right). If the regular covering $\widetilde{X}$ is connected, then K is called a *covering transformation group*. The *fibre* of an edge or a vertex is its preimage under p. The graph $\widetilde{X}$ is called the *covering graph* and X is the *base graph*. The group of automorphisms of $\widetilde{X}$ which maps fibres to fibres is called the *fibre-preserving subgroup* of $\mathrm{Aut}(\widetilde{X})$.

Every edge of a graph X gives rise to a pair of opposite arcs. By e^{-1}, we mean the reverse arc to an arc e. Let K be a finite group and denote by $A(X)$ the arc-set of X. An *ordinary voltage assignment* (or, K-*voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in A(X)$. The values of ϕ are called *voltages*, and K is called the *voltage group*. The *ordinary derived graph* $X \times_\phi K$ derived from an ordinary voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = uv \in A(X)$ and $g \in K$. The first coordinate projection

$p_\phi : X \times_\phi K \to X$ is a regular covering since K is semiregular on $V(X \times_\phi K)$.

Let $p : \widetilde{X} \to X$ be a K-covering. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha} p = p\alpha$, we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the identity group. Gross and Tucker [11] showed that every K-covering of a graph X can be derived from a K-voltage assignment which assigns the identity voltage 1 to the arcs on an arbitrary fixed spanning tree of X.

Let $X \times_\phi K \to X$ be a connected K-covering, where $\phi = 1$ on the arcs of a spanning tree T of X. Such ϕ is called a *T-reduced voltage assignment*. Then the covering graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

The problem whether an automorphism α of X lifts can be grasped in terms of voltage as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Define the mapping $\overline{\alpha}$ from the set of voltages of fundamental closed walks based at a vertex v of the base graph X to the voltage group K as the following:

$$(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages of C and C^α, respectively. Note that if K is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree edges of X.

LEMMA 4.1 ([16]). *Let $X \times_\phi K \to X$ be a connected K-covering. Then an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

LEMMA 4.2. *Let $\widetilde{X}$ be a connected regular covering of the complete graph K_4, whose covering transformation group is cyclic or elementary abelian, and whose fibre-preserving subgroup is arc-transitive. Then $\widetilde{X}$ is not one-regular.*

PROOF. Let K be a cyclic or an elementary abelian group and let $\widetilde{X} = K_4 \times_\phi K$ be a connected regular covering of the graph K_4 satisfying the hypotheses in the theorem, where ϕ is a T-reduced K-voltage assignment with the spanning tree T as illustrated by dark lines in Figure 1. Identify the vertex set of K_4 with $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and we assign voltages z_1, z_2 and z_3 in K to the cotree arcs as shown in Figure 1.

Suppose to the contrary that the covering graph $K_4 \times_\phi K$ is one-regular. Since K_4 is not one-regular, we get $|K| > 1$, and since the fibre-preserving subgroup, say $\widetilde{L}$, acts arc-transitively on $K_4 \times_\phi K$ and $K_4 \times_\phi K$ is one-regular, we have $\mathrm{Aut}(\widetilde{X}) = \widetilde{L}$.

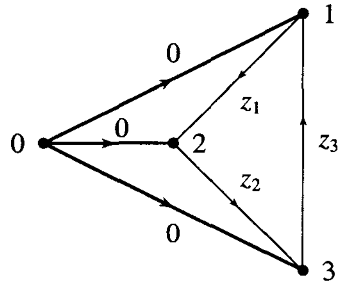

FIGURE 1. The complete graph K_4 with voltage assignment ϕ

Hence, the projection of $\widetilde{L}$, say L, acts regularly on the arc set of K_4. Then $|L| = 12$. Since $K_4 \times_\phi K$ is connected, $\{z_1, z_2, z_3\}$ generates the voltage group K, that is, $\langle z_1, z_2, z_3\rangle = K$. Noting that $\mathrm{Aut}(K_4) = S_4$ and $|L| = 12$, we have that $L = A_4$. Let $\alpha = (01)(23)$, $\beta = (123)$ and $\gamma = (12)$. Clearly, α, β and γ are automorphisms of K_4 and $\alpha, \beta \in L$.

By $i_1 i_2 \cdots i_s$, we denote a cycle which has vertex set $\{i_1, i_2, \ldots, i_s\}$, and edge set $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{s-1}, i_s), (i_s, i_1)\}$. There are three fundamental cycles 012, 023 and 031 in K_4, which are generated by the three cotree edges. Each cycle maps to a cycle of same length under the actions of α, β and γ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of K_4 and $\phi(C)$ denotes the voltage on the cycle C.

Consider the mapping $\overline{\alpha}$ from the set of voltages of the three fundamental cycles of K_4 to the voltage group K, defined by $\phi(C)^{\overline{\alpha}} = \phi(C^\alpha)$, where C ranges over all these three fundamental cycles. Similarly, one can define $\overline{\beta}$ and $\overline{\gamma}$. Since L lifts, by Lemma 4.1 $\overline{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of K, say α^* and β^*, respectively. However, $\overline{\gamma}$ can't be extended to an automorphism of K because of the one-regularity of $K_4 \times_\phi K$. From Table 1, $z_1^{\beta^*} = z_2$ and $z_2^{\beta^*} = z_3$. This implies that z_1, z_2 and z_3 have the same order. Now we consider the cases according to K being cyclic or elementary abelian.

Case I. $K = \mathbb{Z}_n$ $(n > 1)$. Since z_1, z_2 and z_3 have the same order and $\langle z_1, z_2, z_3\rangle = \mathbb{Z}_n$, each of them generates the cyclic group $\mathbb{Z}_n$. Thus we may assume that $z_1 = 1$. Let $1^{\beta^*} = k$. Then $(k, n) = 1$. By $z_1^{\beta^*} = z_2$, $z_2^{\beta^*} = z_3$ and $z_3^{\beta^*} = z_1$ (see Table 1), we have that $z_2 \equiv k \pmod n$, $z_3 \equiv k^2 \pmod n$ and $k^3 \equiv 1 \pmod n$. Let $1^{\alpha^*} = l$. Then $z_1^{\alpha^*} = z_3$ and $z_3^{\alpha^*} = z_1$ implies that $l \equiv k^2 \pmod n$ and $lk^2 \equiv 1 \pmod n$. From the latter equation and $k^3 \equiv 1 \pmod n$, we have $k \equiv l \pmod n$. Thus $l \equiv k^2 \pmod n$ implies that $k \equiv 1 \pmod n$ because $(k, n) = 1$. It follows that $z_1 \equiv z_2 \pmod n \equiv z_3 \pmod n \equiv 1 \pmod n$ and so $\overline{\gamma}$ can be extended to an automorphism of $\mathbb{Z}_n$ induced by $1 \mapsto -1$, a contradiction.

Case II. $K = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p = \mathbb{Z}_p^m$ (p prime, $m \geq 2$). By $\langle z_1, z_2, z_3\rangle = \mathbb{Z}_p^m$, we

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TABLE 1. Fundamental cycles and their images with corresponding voltages on K_4

C	$\phi(C)$	C^α	$\phi(C^\alpha)$	C^β	$\phi(C^\beta)$	C^γ	$\phi(C^\gamma)$
012	z_1	103	z_3	023	z_2	021	$-z_1$
023	z_2	132	$-z_1 - z_2 - z_3$	031	z_3	013	$-z_3$
031	z_3	120	z_1	012	z_1	032	$-z_2$

may assume that $K = \mathbb{Z}_p^2$ or $\mathbb{Z}_p^3$. If $K = \mathbb{Z}_p^3$, then z_1, z_2 and z_3 are linearly independent. Similarly, $-z_1$, $-z_2$ and $-z_3$ are also linearly independent. This implies that $\overline{\gamma}$ can be extended to an automorphism of $\mathbb{Z}_p^3$, a contradiction.

Now suppose that $K = \mathbb{Z}_p^2 = \langle a \rangle \times \langle b \rangle$. By $z_1^{\beta^*} = z_2$, $z_2^{\beta^*} = z_3$ and $z_3^{\beta^*} = z_1$, z_1 and z_2 must be linearly independent. We may assume that $z_1 = a$ and $z_2 = b$. Let $z_3 = ka + lb = z_2^{\beta^*}$. Then $z_3^{\beta^*} = z_1$ implies that $lk \equiv 1 \pmod p$ and $k + l^2 \equiv 0 \pmod p$, and by $lk \equiv 1 \pmod p$ we have $(l, p) = 1$. Since $z_1^{\alpha^*} = z_3$ and $z_2^{\alpha^*} = -z_1 - z_2 - z_3$ means that $a^{\alpha^*} = ka + lb$ and $b^{\alpha^*} = -(k+1)a - (l+1)b$, we may deduce that $a = (k^2 - lk - l)a + l(k - l - 1)b$ from $z_3^{\alpha^*} = z_1$, in which b has the coefficient $l(k - l - 1)$. Since $(l, p) = 1$, $k - l - 1 \equiv 0 \pmod p$. Noting that $lk \equiv 1 \pmod p$ and $k + l^2 \equiv 0 \pmod p$ we have $l^2 + l + 1 \equiv l^2 + l - 1 \pmod p \equiv 0 \pmod p$, implying that $p = 2$. This is impossible because the equation $l^2 + l + 1 \equiv 0 \pmod 2$ has no solution. □

THEOREM 4.3. *Let p be a prime. Then there is no one-regular cubic graph of order $4p$ or $4p^2$.*

PROOF. By Proposition 2.8 we may assume that $p \geq 17$ and suppose to the contrary that X is a one-regular cubic graph of order $4p$ or $4p^2$. Since X is connected, $A = \mathrm{Aut}(X)$ is transitive on $V(X)$. By the one-regularity of X, $|A| = 12p$ or $12p^2$. By [10, pp. 12–14], a non-abelian simple $\{2, 3, p\}$-group is one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$. By Conway *et al.* [5], the orders of these simple groups have divisor 8 or 9 except A_5. Since $|A|$ has no divisor 8 or 9 and $p \geq 17$, A is solvable. Let $\pi = \{2, p\}$. By Proposition 2.4, A has a Hall π-subgroup, say H.

We claim that A has a normal Sylow p-subgroup. Consider the conjugate action of A on the set of cosets of H in A. Then A/H_A is isomorphic to a subgroup of the symmetric group S_3 of degree 3, where H_A is the largest normal subgroup of A contained in H. Since $H_A \leq H$ and $|A : H| = 3$, we have $|A/H_A| = 3$ or 6. If $|A/H_A| = 3$ then $H_A = H$, and if $|A/H_A| = 6$ then $|H : H_A| = 2$. Thus $|H_A| = 2p$, $4p$, $2p^2$ or $4p^2$. By Sylow's theorem, the Sylow p-subgroup of H_A is normal in H_A and so is normal in A. Since Sylow p-subgroups of H_A are also Sylow p-subgroups of A, A has a normal Sylow p-subgroup.

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Let N be the normal Sylow p-subgroup of A. Since $|N|$ has no divisor 3, N acts semiregular on $V(X)$. It follows that N has four orbits. Recall that $\overline{X}$ is the quotient graph of X corresponding to the orbits of N, where $\overline{X}$ has the same definition as in the proof of Lemma 3.2. Then $\overline{X}$ is isomorphic to K_4, and hence X is a regular covering of K_4 with the covering transformation group N and with the fibre-preserving subgroup $\mathrm{Aut}(X)$. Since $|N| = p$ or p^2, N is cyclic or elementary abelian and by Lemma 4.2 X can't be one-regular, a contradiction. □

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