Proceedings of the Edinburgh Mathematical Society (2003) **46**, 575–595 ©
DOI:10.1017/S0013091502000925 Printed in the United Kingdom

SIMILARITY AND THE POINT SPECTRUM OF SOME NON-SELFADJOINT JACOBI MATRICES

JAN JANAS[1], MARIA MALEJKI[2] AND YAROSLAV MYKYTYUK[3]

[1]*Institute of Mathematics, Polish Academy of Sciences, ul. Sw. Tomasza 30, 31-027 Kraków, Poland* (najanas@cyf-kr.edu.pl)

[2]*Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland* (malejki@uci.agh.edu.pl)

[3]*Department of Mathematics, Lvov State University, Lvov, Ukraine* (mykytyuk@email.lviv.ua)

(Received 26 September 2002)

Abstract In this paper spectral properties of non-selfadjoint Jacobi operators J which are compact perturbations of the operator $J_0 = S + \rho S^*$, where $\rho \in (0, 1)$ and S is the unilateral shift operator in ℓ^2, are studied. In the case where $J - J_0$ is in the trace class, Friedrichs's ideas are used to prove similarity of J to the rank one perturbation T of J_0, i.e. $T = J_0 + (\cdot, p)e_1$. Moreover, it is shown that the perturbation is of 'smooth type', i.e. $p \in \ell^2$ and

$$\overline{\lim_{n\to\infty}} |p(n)|^{1/n} \leqslant \rho^{1/2}.$$

When $J - J_0$ is not in the trace class, the Friedrichs method does not work and the transfer matrix approach is used. Finally, the point spectrum of a special class of Jacobi matrices (introduced by Atzmon and Sodin) is investigated.

Keywords: non-selfadjoint Jacobi matrix; one-dimensional perturbation; point spectrum; essential spectrum

2000 *Mathematics subject classification:* Primary 47B36
Secondary 47B37

1. Introduction

Let $\ell^2 = \ell^2(\mathbb{N})$ and $\{e_j\}_{j\in\mathbb{N}}$ be the standard orthonormal basis in ℓ^2 ($\mathbb{N} = \{1, 2, \dots\}$). Denote by S the unilateral shift operator in ℓ^2, i.e.

$$Se_j = e_{j+1}, \quad j \in \mathbb{N}.$$

In this paper we study spectral properties of non-selfadjoint Jacobi operators J which are compact perturbations of the operator

$$J_0 = S + \rho S^* \tag{1.1}$$

where $\rho \in (0, 1)$.

In the case where $J - J_0$ is in the trace class, Friedrichs's approach [**8**,**9**] is used to prove similarity of J to the rank one perturbation T of J_0, i.e. $T = J_0 + (\cdot, p)e_1$. Moreover, this perturbation is of 'smooth type' (see Theorem 2.1). In this way spectral analysis of J is somehow reduced to that of T. On the other hand, when $J - J_0$ is not in the trace class, the Friedrichs approach does not work and the idea of transfer matrix analysis is used. This idea has been already used in studies of the point spectrum of selfadjoint Jacobi operators [**12**].

A linear operator $A : \ell^2 \to \ell^2$ is called diagonal, with the diagonal $d(A) = a = \{a(j)\}_{j\in\mathbb{N}}$, if

$$Ae_j = a(j)e_j, \quad j \in \mathbb{N}.$$

Let $\mathcal{D}$ be the set of all diagonal operators. Define $\mathcal{D}_\mathrm{c} = \mathcal{D} \cap \mathcal{B}_\infty(\ell^2)$ and $\mathcal{D}_1 = \mathcal{D} \cap \mathcal{B}_1(\ell^2)$, where $\mathcal{B}_\infty(\ell^2)$ (respectively, $\mathcal{B}_1(\ell^2)$) is the set of all compact (respectively, trace class) operators in ℓ^2. We are going to study the eigenvalues of non-selfadjoint Jacobi matrices of the following form:

$$\mathcal{J} = SM_1 + M_2S^* + M_3,$$

where $M_1, M_2 \in \mathcal{D}$ and $M_3 \in \mathcal{D}_\mathrm{c}$, $d(M_1) = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \{0\}$, $d(M_2) = \{\beta_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \{0\}$, $\lim_{n\to\infty} \alpha_n = a_1$, $\lim_{n\to\infty} \beta_n = a_2$ and $a_1, a_2 \in \mathbb{C} \setminus \{0\}$.

Since operators $M_1 - a_1I$, $M_2 - a_2I$ and M_3 are compact, the essential spectrum $\sigma_\mathrm{ess}(\mathcal{J})$ of the operator $\mathcal{J}$ coincides with the essential spectrum of the operator

$$\mathcal{J}_0 = a_1S + a_2S^*.$$

In the case where $|a_1| = |a_2|$ the operator $\mathcal{J}$ has properties similar to those of a non-selfadjoint difference Schrödinger operator $S + S^* + Q$ ($Q \in \mathcal{D}_1$), which has been well studied in numerous works. In this paper we only consider the case $|a_1| \neq |a_2|$. We are going to answer the question: when is the point spectrum $\sigma_\mathrm{p}(\mathcal{J})$ empty or when does it consist of only a finite number of eigenvalues of $\mathcal{J}$? In what follows we will denote by $\sigma_\mathrm{p}^0(\mathcal{J})$ the set of all isolated eigenvalues of finite algebraic multiplicity. Additionally, we will denote by $\sigma_\mathrm{r}(\mathcal{J})$ the residual spectrum and by $\sigma_\mathrm{c}(\mathcal{J})$ the continuous spectrum of $\mathcal{J}$. Because $\mathcal{J}$ is a compact perturbation of $\mathcal{J}_0$, it is natural to look at the differences between spectra of these two operators. From general perturbation theory it follows that the set $\sigma(\mathcal{J}) \setminus \sigma(\mathcal{J}_0)$ consists of isolated eigenvalues of finite algebraic multiplicity. However, the condition $Q \in \mathcal{D}_1$ does not ensure finiteness of the point spectrum of the Schrödinger operator [**13**].

Assume that $|a_1| > |a_2|$. Then there exists $\rho \in (0, 1)$, $b \in \mathbb{C}$, $|b| = 1$ and $c \in \mathbb{C}$ such that $ca_1 = b$, $ca_2 = \rho\bar{b}$. Because $\sigma(\mathcal{J}_0) = (1/c)\sigma(bS + (\rho bS)^*)$, the structure of the spectra are the same and we can consider $\mathcal{J}_0 = bS + (\rho bS)^*$ without loss of generality. Next notice that $\mathcal{J}_0$ given by the last formula is similar to $J_0 = S + \rho S^*$, because $|b| = 1$. Indeed, the diagonal operator given by $De_n = b^n e_n$ is bounded, boundedly invertible and $D^{-1}\mathcal{J}_0 D = J_0$.

Let

$$E = \left\{ \rho z + \frac{1}{z} \in \mathbb{C} : |z| = 1 \right\}$$

and

$$\Omega = \left\{\rho z + \frac{1}{z} \in \mathbb{C} : 1 < |z| < \rho^{-1}\right\}.$$

It is known that $\sigma(J_0) = \bar{\Omega} = \Omega \cup E$ and its essential spectrum $\sigma_{\rm ess}(J_0) = E$ (see [**6**]).

Therefore, we have a slightly simpler operator J:

$$J = SM_1 + M_2 S^* + M_3 = J_0 + B, \tag{1.2}$$

where $d(M_1) = \{\alpha_n\}_{n=1}^{\infty}$, $d(M_2) = \{\beta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \alpha_n = 1$, $\alpha_n, \beta_n \in \mathbb{C} \setminus \{0\}$, $\lim_{n\to\infty} \beta_n = \rho$, $M_3 \in \mathcal{D}_{\rm c}$, and B is a compact Jacobi operator, so that $\sigma(J) \cap (\mathbb{C} \setminus \bar{\Omega})$ consists of a discrete set of eigenvalues only.

It seems that there are not many papers devoted to spectral analysis of non-selfadjoint Jacobi matrices, but see [**1**], [**2**], [**3**], [**4**], [**5**] and [**11**]. Note that in the case $\alpha_n = \beta_n$, Jacobi operators have a close relation to the theory of formal orthogonal polynomials (in particular in the study of their asymptotics) and continued fractions. Conversely, formal orthogonal polynomials can be applied to spectral analysis of Jacobi operators (e.g. the characterization of the essential spectrum of J in terms of the asymptotic behaviour of formal orthogonal polynomials found by Beckermann in [**3**]). We also emphasize that although the Jacobi operators we study in this paper are compact perturbations of Toeplitz ones, the problem of similarity is rather subtle. This is clearly illustrated by the example given in § 6.

The paper is organized as follows. Section 2 contains the main similarity result. Its consequences are described in § 3. In turn, § 4 presents the transfer matrix method applied to analyse $\sigma_{\rm p}(J)$ lying on $\sigma_{\rm ess}(J)$. In § 5 the point spectrum of the special class of Jacobi matrices (introduced by Atzmon and Sodin in [**2**]) is studied. Finally, § 6 contains an example related to the main similarity result.

2. Similarity of perturbed Jacobi operators

Let I be the identity operator. Suppose that J is given by (1.2), where $M_1 - I$, $M_2 - \rho I$, M_3 all belong to $\mathcal{D}_1$. Notice that we can assume without loss of generality that $M_1 = I$. Indeed, if $M_2 - \rho I$, $M_3 \in \mathcal{D}_1$, $d(M_1) = \{\alpha_n - 1\} \in \ell^1$ and $\alpha_n \neq 0$ for every $n \in \mathbb{N}$, then $\Lambda J \Lambda^{-1} = S + (\rho I + R)S^* + Q$, where R, Q are still in $\mathcal{D}_1$ and Λ is a bounded and boundedly invertible in ℓ^2 diagonal operator with the diagonal

$$\lambda_n = \left(\prod_{k=1}^{n-1} \alpha_k\right)^{-1}, \quad \lambda_1 = 1.$$

Therefore, in what follows we shall study the operator $J : \ell^2 \to \ell^2$, which acts by the formula

$$J = S + (\rho I + R)S^* + Q, \tag{2.1}$$

where $\rho \in (0, 1)$, $R, Q \in \mathcal{D}_1$.

Let $(\cdot, \cdot)$ denote the scalar product in ℓ^2. The main result of this section is the following theorem.

Theorem 2.1. *Let the operator J be defined by (2.1) and $J_0 = S + \rho S^*$. Then J is similar to the operator*

$$T = J_0 + (\cdot, p)e_1, \tag{2.2}$$

where $p \in \ell^2$ and

$$\varlimsup_{n\to\infty} |p(n)|^{1/n} \leqslant \rho^{1/2}. \tag{2.3}$$

The proof of Theorem 2.1 is divided into several lemmas.

Definition 2.2. Let $\alpha \geqslant 1$. Denote by $\mathcal{A}_\alpha$ ($\mathcal{A}_{\alpha,1}$) the set of all operators $V \in \mathcal{B}(\ell^2)$ such that

$$V = \sum_{n=0}^{\infty} S^{*n} V_n, \quad V_n \in \mathcal{D} \quad (V_n \in \mathcal{D}_1), \tag{2.4}$$

$$|V|_\alpha := \sum_{n=0}^{\infty} \alpha^n \|V_n\| < \infty \quad \biggl(|V|_{\alpha,1} := \sum_{n=0}^{\infty} \alpha^n \|V_n\|_{\mathcal{B}_1} < \infty\biggr). \tag{2.5}$$

Let

$$\mathcal{A}_\alpha^0 = \{S^* V : V \in \mathcal{A}_\alpha\}.$$

The set $\mathcal{A}_\alpha^0$ coincides with the set of all operators from $\mathcal{A}_\alpha$ which have a strictly upper triangular matrix.

Remark 2.3. It is easy to see that operators $V \in \mathcal{A}_\alpha$ can be uniquely written in the form (2.4), and (2.5) determines the norm in $\mathcal{A}_\alpha$ ($\mathcal{A}_{\alpha,1}$). Obviously, $\mathcal{A}_\alpha$ ($\mathcal{A}_{\alpha,1}$) with the norm $|\cdot|_\alpha$ ($|\cdot|_{\alpha,1}$) is a Banach space.

Lemma 2.4. *$\mathcal{A}_\alpha$ is the Banach algebra, and $\mathcal{A}_\alpha^0$ is the two-sided closed ideal in $\mathcal{A}_\alpha$.*

Proof. It suffices to check that for arbitrary $V_1, V_2 \in \mathcal{A}_\alpha$

$$|V_1 \cdot V_2|_\alpha \leqslant |V_1|_\alpha \cdot |V_2|_\alpha.$$

Let

$$V_j = \sum_{k=0}^{\infty} S^{*k} V_{kj}, \quad j = 1, 2.$$

Then

$$V_1 V_2 = \sum_{r=0}^{\infty} S^{*r} \sum_{k+p=r} S^p V_{k1} S^{*p} V_{p2}.$$

Since

$$\alpha^r \biggl\| \sum_{k+p=r} S^p V_{k1} S^{*p} V_{p2} \biggr\| \leqslant \sum_{k+p=r} (\alpha^k \|V_{k1}\|)(\alpha^p \|V_{p2}\|),$$

then

$$|V_1 V_2|_\alpha \leqslant \sum_{r=0}^{\infty} \sum_{k+p=r} (\alpha^k \|V_{k1}\|)(\alpha^p \|V_{p2}\|) = |V_1|_\alpha \cdot |V_2|_\alpha.$$

The lemma is proved. □

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Denote by P_n ($n \in \mathbb{N}$) the projection in ℓ^2, which acts by

$$P_n x = \sum_{j=1}^{n} x(j) e_j, \quad x = \{x(j)\}_{j \in \mathbb{N}}. \tag{2.6}$$

Proposition 2.5. *Let $V \in \mathcal{A}^0_\alpha$ and*

$$\lim_{n\to\infty} |V - VP_n|_\alpha = 0.$$

Then the operator $I + V$ is invertible in the algebra $\mathcal{A}_\alpha$.

Proof. Choose $n \in \mathbb{N}$ such that $|V - VP_n|_\alpha < 1$. Since V is strictly upper triangular we have $I + V = [I + V(I - P_n)](I + VP_n)$. The first term in the product is invertible in $\mathcal{A}_\alpha$ because $|V - VP_n|_\alpha < 1$ and the second term is also invertible in $\mathcal{A}_\alpha$ as we notice that $(VP_n)^n = 0$. □

Denote by $\varGamma$ the linear operator acting from $\mathcal{A}_{1,1}$ into $\mathcal{A}_1$ by the formula

$$\varGamma(V) = \sum_{n=1}^{\infty} (S^*)^n V S^n.$$

By dint of simple calculations we obtain the following lemma.

Lemma 2.6.

(1) *If $A \in \mathcal{D}_1$ and $n \in \mathbb{N}$, then $\varGamma(A) \in \mathcal{B}_\infty(\ell^2)$ and*

$$\|\varGamma(A)\| \leqslant \|A\|_{\mathcal{B}_1}.$$

(2) *For $V \in \mathcal{A}_{1,1}$,*

$$\varGamma(VS^*) = S^*(V + \varGamma(V)). \tag{2.7}$$

(3) *The operator $\varGamma$ continuously maps $\mathcal{A}_{1,1}$ into $\mathcal{A}_1$ and*

$$|\varGamma(V)|_1 \leqslant |V|_{1,1}, \quad V \in \mathcal{A}_{1,1}. \tag{2.8}$$

(4) *For arbitrary $V \in \mathcal{A}_{1,1}$,*

$$(I - P_1)(\varGamma(V)S - S\varGamma(V) + VS) = 0. \tag{2.9}$$

(5) *If $V \in \mathcal{A}_1$, $A \in \mathcal{D}_1$ and $n \in \mathbb{N}$, then $VAS^{*n} \in \mathcal{A}_{1,1}$ and*

$$|VAS^{*n}|_{1,1} \leqslant |V|_1 \cdot \|A\|_{\mathcal{B}_1}. \tag{2.10}$$

Define the sequence $\{U_n\}_{n=0}^{\infty} \subset \mathcal{B}(\ell^2)$ by the following recurrence relation:

$$U_{n+1} = \rho S^* U_n S^* + \varGamma(U_n(QS^* + RS^{*2})), \quad n \geqslant 1, \tag{2.11}$$

with the initial conditions

$$U_0 = I, \qquad U_1 = \varGamma(QS^* + RS^{*2}). \tag{2.12}$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Lemma 2.7.

(1) *For each* $n \in \mathbb{N}$,

$$U_n = \sum_{j=n}^{2n} S^{*j} U_{nj}, \qquad U_{nj} \in \mathcal{D} \cap \mathcal{B}_\infty(\ell^2) \tag{2.13}$$

and

$$(I - P_1)[U_{n+1}S - SU_{n+1} + \rho(U_n S^* - S^* U_n) + U_n(Q + RS^*)] = 0. \tag{2.14}$$

(2)

$$\varlimsup_{n\to\infty} |U_n|_1^{1/n} \leqslant \rho. \tag{2.15}$$

Proof. Since $SS^* = I - P_1$, (2.14) follows from (2.9) and (2.11). Applying Lemma 2.6 (1), (2) and induction we obtain the representation (2.13).

Now let us check (2.15). Using (2.11) and (2.8), we have

$$|U_{n+1}|_1 \leqslant \rho |U_n|_1 + |U_n(QS^* + RS^{*2})|_{1,1}.$$

From (2.13) it follows that

$$U_n Q S^* = U_n Q_n S^*, \qquad U_n R S^{*2} = U_n R_n S^{*2},$$

where $Q_n = Q(I - P_n)$, $R_n = R(I - P_n)$ (see (2.6)). Therefore, by (2.10), we have

$$|U_n(QS^* + RS^{*2})|_{1,1} \leqslant C_n |U_n|_1,$$

where

$$C_n = \|Q_n\|_{\mathcal{B}_1} + \|R_n\|_{\mathcal{B}_1}.$$

Consequently,

$$|U_{n+1}|_1 \leqslant (\rho + C_n)|U_n|_1. \tag{2.16}$$

Since $Q, R \in \mathcal{B}_1(H)$, we have $\lim_{n\to\infty} C_n = 0$. Hence using (2.16), we get (2.15). The lemma is proved. □

Put

$$U = \sum_{n=0}^{\infty} U_n. \tag{2.17}$$

Lemma 2.8. *For each* $\alpha \in [1, \rho^{-1/2})$ *we have*

(1) $(U - I) \in \mathcal{A}_\alpha^0$;

(2) *the operator* U *is invertible in the algebra* $\mathcal{A}_\alpha$ *and*

$$(I - P_1)(UJU^{-1} - J_0) = 0. \tag{2.18}$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Proof. Let $1 \leqslant \alpha < \beta < \rho^{-1/2}$. Since $\beta^{-2} > \rho$, from (2.13) and (2.15) it follows that

$$\sum_{j=n}^{2n} \|U_{nj}\| = |U_n|_1 \leqslant c\beta^{-2n}, \quad n \in \mathbb{N},$$

where c is a positive constant. Therefore,

$$\|U_{nj}\| \leqslant c\beta^{-2n} \leqslant c\beta^{-j}, \quad n \in \mathbb{N}, \quad n \leqslant j \leqslant 2n. \tag{2.19}$$

Using (2.13) and (2.17) we obtain

$$U = I + \sum_{j=1}^{\infty} S^{*j} \sum_{j/2 \leqslant n \leqslant j} U_{nj}.$$

Hence, taking into account (2.19), we have

$$|U|_\alpha = 1 + \sum_{j=1}^{\infty} \alpha^j \Bigg\| \sum_{j/2 \leqslant n \leqslant j} U_{nj} \Bigg\| \leqslant 1 + c \sum_{j=1}^{\infty} j \left(\frac{\alpha}{\beta}\right)^j < \infty,$$

i.e. $(U - I) \in \mathcal{A}_\alpha^0$.

Let $V = U - I$. We claim that

$$\lim_{n \to \infty} |V - VP_n|_\alpha = 0.$$

Indeed, we have

$$|V - VP_n|_\alpha = \sum_{j=1}^{\infty} \alpha^j \Bigg\| \sum_{j/2 \leqslant k \leqslant j} U_{kj}(I - P_n) \Bigg\| \leqslant |U|_\alpha < +\infty.$$

Define $V_j = \sum_{j/2 \leqslant k \leqslant j} U_{kj}$. Observe that the sequence $F_n(j) := \|V_j(I - P_n)\|$ belongs to ℓ^1 (as a function of j) and $|F_n(j)| \leqslant \|V_j\|$, $j \in \mathbb{N}$. Since $\{\alpha^j \|V_j\|\} \in \ell^1$ and $\lim_{n \to \infty} F_n(j) = 0$ (due to compactness of V_j) the Lebesgue-dominated convergence theorem proves the claim. Therefore, according to Proposition 2.5 U is invertible in $\mathcal{A}_\alpha$. Equality (2.14) evidently implies that

$$(I - P_1)(UJ - J_0U) = 0,$$

and (2.18) follows. The lemma is proved. □

Proof of Theorem 2.1. Fix an arbitrary $\alpha \in [1, \rho^{-1/2})$. Applying Lemma 2.8 we have $(U - I) \in \mathcal{A}_\alpha^0$, $U^{-1} \in \mathcal{A}_\alpha$. Therefore, $(U^{-1} - I) \in \mathcal{A}_\alpha^0$ and

$$V = UJU^{-1} - J_0 = USU^{-1} - S + X,$$

where X belongs to $\mathcal{A}_\alpha$. But $U = I + S^*\tilde{V}$, for a certain $\tilde{V} \in \mathcal{A}_\alpha$, and so $USU^{-1} - S = (S^*\tilde{V}S - SS^*\tilde{V})U^{-1}$ also belongs to $\mathcal{A}_\alpha$. Let

$$V := \sum_{n=0}^{\infty} S^{*n} V_n, \quad V_n \in D.$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Then $\sum_{n=0}^{\infty} \alpha^n \|V_n\| = |V|_\alpha < \infty$. Put $p = V^* e_1$. Using (2.18) we get

$$UJU^{-1} = J_0 + (\cdot, p)e_1.$$

Since

$$Ve_m = \sum_{n=0}^{\infty} S^{*n} V_n e_m = \sum_{n<m} S^{*n} V_n e_m,$$
$$p(m) = (V^* e_1, e_m) = (e_1, Ve_m) = (e_m, V_{m-1} e_m),$$

we have

$$|p(m)| \leqslant \|V_{m-1}\| \leqslant \alpha^{-m+1} |V|_\alpha, \quad m \in \mathbb{N}.$$

Hence (2.3) follows. Theorem 2.1 is proved. □

3. Consequences of similarity

In this section Theorem 2.1 is used to describe the structure of $\sigma(J)$. In particular, we shall prove that $\sigma_{\mathrm{p}}^0(J)$ and $\sigma_{\mathrm{r}}(J) \cap E$ are finite. Similarity of J to a one-dimensional perturbation of J_0 allows us to present a rather complete description of the spectral picture of it.

Theorem 3.1. *Let $|a_1| = 1$, $|a_2| = \rho$, $\rho \in (0, 1)$, and $J = SM_1 + M_2 S^* + M_3$, where $M_1 - a_1 I$, $M_2 - a_2 I$, $M_3 \in \mathcal{D}_1$, then*

(1) $\bar{\Omega} \subset \sigma(J)$, $\sigma_{\mathrm{p}}(J) = \sigma_{\mathrm{p}}^0(J) \subset \mathbb{C} \setminus \bar{\Omega}$;

(2) $\Omega \subset \sigma_{\mathrm{r}}(J)$, $\sigma_{\mathrm{c}}(J) = E \setminus \sigma_{\mathrm{r}}(J)$;

(3) *the sets $\sigma_{\mathrm{p}}^0(J)$, $\sigma_{\mathrm{r}}(J) \cap E$ are finite.*

Proof. Because we can simplify the formula for J according to the remarks made above it suffices to prove the theorem for operators J of the form

$$J = S + (\rho I + R)S^* + Q,$$

where $\rho \in (0, 1)$, $Q, R \in D_1$.

According to Theorem 2.1 it is enough to study the spectrum of T, which is defined by (2.2). Denote by h the function

$$h(\xi) = \zeta^{-1} + \rho\zeta, \quad 0 < |\zeta| < \rho^{-1}.$$

By the above definitions it is easy to see that

$$\left.\begin{aligned} \Omega = \{h(\zeta) : 1 < |\zeta| < \rho^{-1}\}, \qquad E = \{h(\xi) : |\zeta| = 1\}, \\ \mathbb{C} \setminus \bar{\Omega} = \{h(\zeta) : 0 < |\zeta| < 1\} \end{aligned}\right\} \tag{3.1}$$

and

$$T - h(\zeta)I = (I - \rho\zeta S^*)(S - \zeta^{-1} I + (\cdot, p)e_1). \tag{3.2}$$

Let $M_\zeta = S - \zeta^{-1}I + (\cdot, p)e_1$, then

$$\ker M_\zeta = \{0\}, \quad 1 \leqslant |\zeta| < \rho^{-1}. \tag{3.3}$$

Indeed, if $x \in \ell^2$ and $M_\zeta x = 0$, then

$$(M_\zeta x, e_{j+1}) = ((S - \zeta^{-1}I)x, e_{j+1}) = 0, \quad j = 1, 2, \dots,$$

and, consequently,

$$x(j+1) = \zeta x(j), \quad j = 1, 2, \dots .$$

Since $x \in \ell^2$ and $|\zeta| \geqslant 1$, we have $x = 0$. Thus $\ker(T - \lambda) = \{0\}$ if $\lambda \in \Omega \cup E$.

Additionally, if $|\zeta| > 1$, then $S - \zeta^{-1}I$ is Fredholm and $\operatorname{ind}(S - \zeta^{-1}I) = -1$. Hence M_ζ is also Fredholm for $|\zeta| > 1$ and

$$\operatorname{ind} M_\zeta = -1.$$

Therefore, $T - h(\zeta)$ must be Fredholm as the product of invertible $I - \rho\zeta S^*$ and M_ζ, for $1 < |\zeta| < \rho^{-1}$. It follows that

$$\Omega \subset \sigma_{\mathrm{r}}(T).$$

On the other hand, if $0 < |\zeta| \leqslant 1$, then

$$T - h(\zeta)I = (I - \rho\zeta S^*)(S - \zeta^{-1}I + (\cdot, p)e_1) = (I - \rho\zeta S^*)(I + (\cdot, p_\zeta)e_1)(S - \zeta^{-1}I), \tag{3.4}$$

where

$$p_\zeta = -\sum_{n=0}^{\infty} \bar{\zeta}^{(n+1)} S^{*n} p. \tag{3.5}$$

The convergence of the series (3.5) follows from (2.3). To verify (3.4) it is enough to note that $S^* p_\zeta = p + (\bar{\zeta})^{-1} p_\zeta$, which is clear by definition of p_ζ. Observe that the operator

$$N_\zeta = I + (\cdot, p_\zeta)e_1, \quad |\zeta| \leqslant 1,$$

is invertible in $\mathcal{B}(\ell^2)$ if

$$\varphi(\zeta) := 1 + (e_1, p_\zeta) \neq 0. \tag{3.6}$$

If $\varphi(\zeta) = 0$, then

$$\dim\ker N_\zeta = \operatorname{codim} \operatorname{Im} N_\zeta = 1.$$

Thus from (3.4) and (3.3), we obtain

$$\sigma_{\mathrm{p}}(T) = \{h(\zeta) : |\zeta| < 1,\ \varphi(\zeta) = 0\} \subset \mathbb{C} \setminus \bar{\Omega},$$
$$\sigma_{\mathrm{c}}(T) = \{h(\zeta) : |\zeta| = 1,\ \varphi(\zeta) \neq 0\} \subset E,$$
$$\sigma_{\mathrm{r}}(T) \cap E = \{h(\zeta) : |\zeta| = 1,\ \varphi(\zeta) = 0\}.$$

From (3.5) and (3.6) it follows that

$$\varphi(\zeta) = 1 - \sum_{n=1}^{\infty} \overline{p(n)} \zeta^n.$$

In view of (2.3), φ is analytic in the disc $|\zeta| < \rho^{-1/2}$. Therefore, the sets $\sigma_{\mathrm{p}}(T)$ and $\sigma_{\mathrm{r}}(T) \cap E$ are finite. Since $\sigma(J_0) = \bar{\Omega}$ and $(T - J_0) \in \mathcal{B}_\infty(\ell^2)$, we obtain

$$\sigma(T) \setminus \bar{\Omega} \subset \sigma_{\mathrm{p}}^0(T).$$

The above analysis and the similarity of T and J complete the proof. □

4. The transfer matrix approach

We do not have to assume that J is an ℓ^1-perturbation of J_0 to prove the relation $\sigma_{\mathrm{p}}(J) \cap \sigma_{\mathrm{ess}}(J) = \emptyset$. The transfer matrix point of view allows us to examine other sufficient conditions on entries of the Jacobi matrix J which guarantee the absence of eigenvalues of J on its essential spectrum.

In this section we follow the ideas from [**12**]. Consider

$$J = SM_1 + M_2 S^* + Q, \tag{4.1}$$

where $M_1 - I$, $M_2 - \rho I$, $Q \in \mathcal{D}_{\mathrm{c}}$ and $d(M_1) = \{\alpha_n\}_{n=1}^\infty$, $d(M_2) = \{\beta_n\}_{n=1}^\infty$, $d(Q) = \{q_n\}_{n=1}^\infty$. Then the equality $Jf = \lambda f$ is equivalent to the system of equations

$$\alpha_{n-1} f_{n-1} + q_n f_n + \beta_n f_{n+1} = \lambda f_n, \quad n = 1, 2, 3, \dots, \tag{4.2}$$

where $\alpha_0 = 0$.

Before we proceed to analyse when $\sigma_{\mathrm{p}}(J) \cap \sigma_{\mathrm{ess}}(J) = \emptyset$, observe that the relation $\Omega \cap \sigma_{\mathrm{p}}(J) = \emptyset$ is an easy consequence of the Perron Theorem. Indeed, if $\lambda \in \Omega$, i.e. $\lambda = \rho\zeta + 1/\zeta$, with $1 < |\zeta| < \rho^{-1}$, then $\lambda \notin \sigma_{\mathrm{p}}(J)$. Indeed, by the Poincaré Theorem (see [**7**, Theorem 8.10] or [**10**, Theorem 2.3.b]) every solution $\{f_n\}$ of (4.2) satisfies

$$\varlimsup_{n\to\infty} |f_n|^{1/n} = |z_+| \quad \text{or} \quad \varlimsup_{n\to\infty} |f_n|^{1/n} = |z_-|,$$

where $z_+ = 1/\rho\zeta$ and $z_- = \zeta$ are the roots of the equation

$$\rho t^2 - \lambda t + 1 = 0. \tag{4.3}$$

Since $|z_\pm| > 1$, there is no solution of (4.2) in ℓ^2.

If $\lambda \in E$, i.e. $\lambda = \rho\zeta + 1/\zeta$, where $|\zeta| = 1$, then again by the Perron theorem (see [**10**, Theorem 2.2]) there are solutions $\{f_n^-\}$ and $\{f_n^+\}$ of (4.2) such that $\lim_{n\to\infty} f_{n+1}^\pm / f_n^\pm = z_\pm$, where $z_\pm$ are the solutions of (4.3) given by the same formulae as above. Of course, $\{f_n^+\} \notin \ell^2$, but it is possible that $\{f_n^-\}$ belongs to ℓ^2, so we shall concentrate on this case.

For $\lambda \in E$ let us define the transfer matrix:

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\dfrac{\alpha_{n-1}}{\beta_n} & \dfrac{\lambda - q_n}{\beta_n} \end{pmatrix}. \tag{4.4}$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Using $B_n(\lambda)$ we can rewrite (4.2) as the system

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = B_n(\lambda)\cdots B_1(\lambda)\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \quad n = 1, 2, \dots. \tag{4.5}$$

Notice that

$$\lim_{n\to+\infty} B_n(\lambda) = B_\infty(\lambda) := \begin{pmatrix} 0 & 1 \\ -\dfrac{1}{\rho} & \dfrac{\lambda}{\rho} \end{pmatrix},$$

because of the assumed convergence of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{q_n\}$.

Since z_+ and z_- are the eigenvalues of $B_\infty(\lambda)$ and $z_+ \neq z_-$, we have

$$W\begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix} = B_\infty(\lambda)W,$$

where

$$W = \begin{pmatrix} 1 & 1 \\ z_+ & z_- \end{pmatrix}$$

is invertible and

$$W^{-1} = \frac{1}{z_+ - z_-}\begin{pmatrix} -z_- & 1 \\ z_+ & -1 \end{pmatrix}.$$

Put

$$\mathcal{B}_n = W^{-1}B_n(\lambda)W \tag{4.6}$$

and define

$$\epsilon_n = \frac{\alpha_{n-1}}{\beta_n} - \frac{1}{\rho}, \tag{4.7}$$

$$\delta_n = \frac{1}{\rho} - \frac{1}{\beta_n}. \tag{4.8}$$

Define

$$A_n^+ = \frac{1}{z_+ - z_-}\left(\epsilon_n + \delta_n z_+\lambda + \frac{q_n}{\beta_n}z_+\right) \tag{4.9}$$

and

$$A_n^- = \frac{1}{z_+ - z_-}\left(\epsilon_n + \delta_n z_-\lambda + \frac{q_n}{\beta_n}z_-\right). \tag{4.10}$$

Then $\lim_{n\to\infty} A_n^+ = \lim_{n\to\infty} A_n^- = 0$. After some calculations, we have

$$\mathcal{B}_n = \begin{pmatrix} z_+ - A_n^+ & -A_n^- \\ A_n^+ & z_- + A_n^- \end{pmatrix}. \tag{4.11}$$

Define

$$C_n = |z_+ - A_n^+|^2 + |A_n^-|^2 - |z_+|^2, \qquad D_n = |z_- + A_n^-|^2 + |A_n^+|^2 - 1$$

and

$$E_n = \bar{z}_+ A_n^+ - z_- \bar{A}_n^- - (|A_n^+|^2 + |A_n^-|^2).$$

It is trivial that $\lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n = \lim_{n\to\infty} E_n = 0$ and

$$B_n B_n^* = \begin{pmatrix} |z_+|^2 + C_n & \bar{E}_n \\ E_n & 1 + D_n \end{pmatrix}.$$

The eigenvalues of $B_n B_n^*$ are given by

$$w_n^{\pm} = \frac{|z_+|^2 + 1 + C_n + D_n}{2} \pm \sqrt{\left(\frac{|z_+|^2 - 1 + C_n - D_n}{2}\right)^2 + |E_n|^2}. \tag{4.12}$$

Hence $\|B_n v\|^2 \geqslant w_n^- \|v\|^2$ for every $v \in \mathbb{C}^2$.

From (4.5) we obtain

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = W B_n \cdots B_2 W^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad n = 2, 3, \dots,$$

so that

$$\left\| \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} \right\|^2 \geqslant \text{const.} w_n^- \cdots w_2^- .$$

But if $\lambda \in \sigma_{\mathrm{p}}(J)$, then there exists $f \in \ell^2$ such that (4.2) is fulfilled; therefore, for all $p > 0$,

$$\sum_{n=p}^{\infty} \prod_{k=p}^{n} w_k^- < +\infty.$$

Using (4.12) and the obvious inequality $|z_+|^2 + C_n \geqslant 1 + D_n$ (valid for sufficiently large n) we have the following estimation from below for w_n^-:

$$w_n^- \geqslant 1 + D_n - |E_n|.$$

Due to the definitions of D_n and E_n we have (remember that $|z_-| = 1$)

$$D_n \geqslant -2|A_n^-| + |A_n^+|^2 + |A_n^-|^2 \tag{4.13}$$

and

$$|E_n| \leqslant |z_+|\,|A_n^+| + |A_n^-| + |A_n^+|^2 + |A_n^-|^2. \tag{4.14}$$

Thus $D_n - |E_n| \geqslant -3|A_n^-| - |z_+|\,|A_n^+|$. Again by the definitions of A_n^+ and A_n^- one can easily check that

$$|A_n^+| \leqslant \frac{1}{|z_+ - z_-|} \left(|\epsilon_n| + |\delta_n|\,|z_+|\,|\lambda| + \left|\frac{q_n}{\beta_n}\right| |z_+| \right) \tag{4.15}$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

and, similarly,

$$|A_n^-| \leqslant \frac{1}{|z_+ - z_-|}\left(|\epsilon_n| + |\delta_n|\,|\lambda| + \left|\frac{q_n}{\beta_n}\right|\right), \tag{4.16}$$

so, finally,

$$w_n^- \geqslant 1 - \left(C_1(\lambda)|\epsilon_n| + C_2(\lambda)|\delta_n| + C_3(\lambda)\left|\frac{q_n}{\beta_n}\right|\right)$$

for n sufficiently large and the constants $C_k(\lambda)$ are given by the formulae:

$$\begin{aligned} C_1(\lambda) &= \frac{|z_+| + 3}{|z_+ - z_-|}, \\ C_2(\lambda) &= |\lambda|\frac{|z_+|^2 + 3}{|z_+ - z_-|}, \\ C_3(\lambda) &= \frac{|z_+|^2 + 3}{|z_+ - z_-|}. \end{aligned}$$

The above reasoning has proved the following theorem.

Theorem 4.1. *If J is given by (4.1), $\lambda \in E$, and*

$$\sum_{n=p}^{\infty}\prod_{k=p}^{n}\left(1 - \left(C_1(\lambda)|\epsilon_k| + C_2(\lambda)|\delta_k| + C_3(\lambda)\left|\frac{q_k}{\beta_k}\right|\right)\right) = +\infty$$

for some $p \in \mathbb{N}$, then $\lambda \notin \sigma_{\mathrm{p}}(J)$.

Corollary 4.2. *Let J be as in (4.1) and $\lambda \in E$.*
If

$$\sum_{n=1}^{\infty}\exp\left(-q\sum_{k=1}^{n}\left(C_1(\lambda)|\epsilon_k| + C_2(\lambda)|\delta_k| + C_3(\lambda)\left|\frac{q_k}{\beta_k}\right|\right)\right) = +\infty$$

for some $q > 1$, then $\lambda \notin \sigma_{\mathrm{p}}(J)$.

Proof. Let us fix $\lambda \in \mathbb{C}$ and let $\lambda = (1/\zeta) + \rho\zeta$. Because $\lambda \in E$ we can assume $|\zeta| = 1$, $z_- = \zeta$, $z_+ = 1/\rho\zeta$.

Using Theorem 4.1 it is enough to check that

$$\sum_{n=p}^{\infty}\prod_{k=p}^{n}\left(1 - \left(C_1(\lambda)|\epsilon_k| + C_2(\lambda)|\delta_k| + C_3(\lambda)\left|\frac{q_k}{\beta_k}\right|\right)\right) = +\infty$$

for some $p \in \mathbb{N}$.

The simple inequality $1 - x \geqslant \mathrm{e}^{-qx}$ is valid for any number $q > 1$ and $x \in [0, x_0]$ for sufficiently small x_0, and if there exists $p \in \mathbb{N}$ such that

$$C_1(\lambda)|\epsilon_k| + C_2(\lambda)|\delta_k| + C_3(\lambda)\left|\frac{q_k}{\beta_k}\right| \leqslant x_0$$

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

for $k \geqslant p$, then

$$\begin{aligned}\sum_{n=p}^{\infty}\prod_{k=p}^{n}&\biggl(1-\biggl(C_1(\lambda)|\epsilon_k|+C_2(\lambda)|\delta_k|+C_3(\lambda)\biggl|\frac{q_k}{\beta_k}\biggr|\biggr)\biggr)\\ &=\sum_{n=p}^{\infty}\exp\biggl(\sum_{k=p}^{n}\ln\biggl(1-\biggl(C_1(\lambda)|\epsilon_k|+C_2(\lambda)|\delta_k|+C_3(\lambda)\biggl|\frac{q_k}{\beta_k}\biggr|\biggr)\biggr)\biggr)\\ &\geqslant\sum_{n=p}^{\infty}\exp\biggl(-q\sum_{k=p}^{n}\biggl(C_1(\lambda)|\epsilon_k|+C_2(\lambda)|\delta_k|+C_3(\lambda)\biggl|\frac{q_k}{\beta_k}\biggr|\biggr)\biggr)\\ &=M\sum_{n=p}^{\infty}\exp\biggl(-q\sum_{k=1}^{n}\biggl(C_1(\lambda)|\epsilon_k|+C_2(\lambda)|\delta_k|+C_3(\lambda)\biggl|\frac{q_k}{\beta_k}\biggr|\biggr)\biggr)=+\infty.\end{aligned}$$

This completes the proof. □

As a simple immediate consequence of Theorem 4.1 we have the following corollary.

Corollary 4.3. *If* $\{\beta_n-\rho\}_{n\in\mathbb{N}}\in\ell^1$, $\{\alpha_n-1\}_{n\in\mathbb{N}}\in\ell^1$ *and* $\{q_n\}_{n\in\mathbb{N}}\in\ell^1$, *then* $E\cap\sigma_{\mathrm{p}}(J)=\emptyset$.

Surely the assumptions made in Corollary 4.3 are too strong; for example, one can easily obtain the following corollary.

Corollary 4.4. *If*

$$\biggl\{\exp\biggl(-\sum_{k=1}^{n}|\delta_n|\biggr)\biggr\}_{n=1}^{\infty}\notin l^p\quad\text{for any } p\in(1,+\infty)\tag{4.17}$$

and $\{\epsilon_n\}_{n=1}^{\infty},\{q_n\}_{n=1}^{\infty}\in\ell^1$, *then* $E\cap\sigma_{\mathrm{p}}(J)=\emptyset$.

Notice that if $\{\beta_n-\rho\}_{n\in\mathbb{N}}\in\ell^1$, then (4.17) is satisfied, but the condition (4.17) can be satisfied by sequences $\{\beta_n\}$ for which $\{\beta_n-\rho\}$ is not necessarily in ℓ^1.

One can formulate other variants of Theorem 4.1. The next result is based on more exact estimations of w_n^- from below.

Theorem 4.5. *Assume that* $\{\epsilon_n\}_{n=1}^{\infty}\in\ell^2$, $\{\delta_n\}_{n=1}^{\infty}\in\ell^2$. *Then* $E\cap\sigma_{\mathrm{p}}(J)=\emptyset$ *provided that*

(a) $\operatorname{Re}\epsilon_n\geqslant 0$, $\operatorname{Re}\delta_n\geqslant 0$ *for large* $n\in\mathbb{N}$ *and both* $\{\operatorname{Im}\epsilon_n+2\operatorname{Im}\delta_n\}_{n=1}^{\infty}$ *and* $\{q_n\}_{n=1}^{\infty}$ *are summable; or*

(b)
$$\sup_N\biggl|\sum_{n=1}^{N}\delta_n\biggr|<+\infty,\qquad\sup_N\biggl|\sum_{n=1}^{N}\epsilon_n\biggr|<+\infty,\qquad\sup_N\biggl|\sum_{n=1}^{N}\biggl(\frac{q_n}{\beta_n}\biggr)\biggr|<+\infty$$

and $\{q_n\}\in\ell^2$.

https://doi.org/10.1017/S0013091502000925 Published online by Cambridge University Press

Proof. (a) Let $\lambda = \rho z + 1/z \in E$, where $|z| = 1$, then $z_- = z$, $z_+ = (\rho z)^{-1}$. According to the formula (4.12) for w_n^- we have

$$w_n^- = \tfrac{1}{2}[|z_+ - A_n^+|^2 + |A_n^-|^2 + |z_- + A_n^-|^2 + |A_n^+|^2 - \sqrt{(|z_+ - A_n^+|^2 + |A_n^-|^2 - (|z_- + A_n^-|^2 + |A_n^+|^2))^2 + 4|E_n|^2}].$$

Because $|z_+| = \rho^{-1} > |z_-| = 1$ we have

$$w_n^- = \tfrac{1}{2}[|z_+ - A_n^+|^2 + |A_n^-|^2 + |z_- + A_n^-|^2 + |A_n^+|^2 - (|z_+ - A_n^+|^2 + |A_n^-|^2 - (|z_- + A_n^-|^2 + |A_n^+|^2))\sqrt{1 + O(|E_n|^2)}].$$

Since $\sqrt{1+x} \leqslant 1 + x/2$ for $x \geqslant 0$, so

$$w_n^- \geqslant |z_- + A_n^-|^2 + |A_n^+|^2 - O(|E_n|^2)$$

and

$$w_n^- \geqslant 1 + 2\,\mathrm{Re}(\bar{z}_- A_n^-) + |A_n^-|^2 + |A_n^+|^2 - O(|E_n|^2).$$

Because $\{\epsilon_n\}, \{\delta_n\}, \{q_n\} \in \ell^2$, by the estimates (4.14)–(4.16)

$$w_n^- \geqslant 1 + 2\,\mathrm{Re}(\bar{z}_- A_n^-) + R_n,$$

where $\{R_n\}_{n=1}^{\infty} \in \ell^1$.

Computing the expression $\mathrm{Re}(\bar{z}_- A_n^-)$ in an equivalent form, we have

$$\begin{aligned}\mathrm{Re}(\bar{z}_- A_n^-) &= \mathrm{Re}\left[\frac{\rho}{1-\rho z^2}\epsilon_n + \frac{\rho(1+\rho z^2)}{1-\rho z^2}\delta_n\right] + \mathrm{Re}\left[\frac{q_n}{\beta_n}\frac{z\rho}{(1-\rho z^2)}\right] \\ &= \mathrm{Re}\left[\frac{\rho(1-\rho\bar{z}^2)}{|1-\rho z^2|^2}\epsilon_n + \frac{\rho(1-\rho^2)}{|1-\rho z^2|^2}\delta_n + \frac{2\rho^2\mathrm{i}(\mathrm{Im}\, z^2)}{|1-\rho z^2|^2}\delta_n\right] + \mathrm{Re}\left[\frac{q_n}{\beta_n}\frac{z\rho}{(1-\rho z^2)}\right].\end{aligned}$$

Because $z = \cos\theta + \mathrm{i}\sin\theta$, for some $\theta \in [0, 2\pi]$ we obtain

$$\mathrm{Re}(\bar{z}_- A_n^-) = \frac{\rho(1-\rho\cos 2\theta)}{|1-\rho z^2|^2}\,\mathrm{Re}\,\epsilon_n + \frac{\rho(1-\rho^2)}{|1-\rho z^2|^2}\,\mathrm{Re}\,\delta_n - \frac{\rho^2\sin 2\theta}{|1-\rho z^2|^2}(\mathrm{Im}\,\epsilon_n + 2\,\mathrm{Im}\,\delta_n) + \mathrm{Re}\left(\frac{q_n}{\beta_n}\frac{z\rho}{(1-\rho z^2)}\right).$$

Therefore, by assumption (a),

$$\mathrm{Re}(\bar{z}_- A_n^-) = \tau_n + R'_n,$$

where $\tau_n \geqslant 0$ and $\{R'_n\} \in \ell^1$. Thus, $w_n^- \geqslant 1 + 2R'_n + R_n$, and so

$$\sum_{n=p}^{\infty}\prod_{k=p}^{n} w_k^- = +\infty$$

(see the estimate of w_n^- from two lines beneath (4.16)), which completes the proof of case (a).

Case (b) is easier. Indeed, write

$$\operatorname{Re}(\bar{z}_- A_n^-) = \operatorname{Re}\left[\frac{\rho}{1-\rho z^2}\epsilon_n\right] + \operatorname{Re}\left[\frac{\rho(1+\rho z^2)}{1-\rho z^2}\delta_n\right] + \operatorname{Re}\left[\frac{z\rho}{1-\rho z^2}\frac{q_n}{\beta_n}\right]$$

and notice that

$$\begin{aligned} s_N := \prod_{n=1}^{N} w_n^- &\geqslant \exp\biggl(2\sum_{n=1}^{N}\operatorname{Re}(\bar{z}_- A_n^-) + O((\operatorname{Re}(\bar{z}_- A_n^-))^2) + R_n\biggr) \\ &\geqslant C\exp\biggl(2\sum_{n=1}^{N}\operatorname{Re}(\bar{z}_- A_n^-)\biggr) \\ &= C\exp\biggl(2\biggl(\operatorname{Re}\left[\frac{\rho}{1-\rho z^2}\sum_{n=1}^{N}\epsilon_n\right] + \operatorname{Re}\left[\frac{\rho(1+\rho z^2)}{1-\rho z^2}\sum_{n=1}^{N}\delta_n\right] \\ &\qquad + \operatorname{Re}\left[\frac{z\rho}{1-\rho z^2}\sum_{n=1}^{N}\frac{q_n}{\beta_n}\right]\biggr)\biggr), \end{aligned}$$

for some constant $C > 0$ because $\ln(1+x) = x + O(x^2)$. But $|z| = 1$ and the above inequality implies that

$$s_N \geqslant C\exp\biggl(\frac{-2\rho}{1-\rho}\biggl((1+\rho)\biggl|\sum_{n=1}^{N}\delta_n\biggr| + \biggl|\sum_{n=1}^{N}\epsilon_n\biggr| + \biggl|\sum_{n=1}^{N}\frac{q_n}{\beta_n}\biggr|\biggr)\biggr).$$

Theorem 4.1 concludes the proof of (b) and of the theorem as well. □

When $\rho = 1$, J is a compact perturbation of $S + S^*$ so $\sigma_{\text{ess}}(J) = [-2, 2]$. If $\lambda \in [-2, 2]$, then $\lambda = z + (1/z)$, where $z \in \mathbb{C}$ and $|z| = 1$ (for example, $z = z_+ = (\lambda/2) + \mathrm{i}\sqrt{1-(\lambda^2/4)}$ and so $z_- = \bar{z}$) and using Corollary 4.2 we obtain the following theorem.

Theorem 4.6. *Let* $\lambda \in (-2, 2)$ *and*

$$\sum_{n=p}^{\infty}\exp\biggl(-q\sum_{k=p}^{n}\biggl(\frac{|\epsilon_k| + |q_k/\beta_k|}{\sqrt{4-\lambda^2}} + \frac{|\lambda|}{\sqrt{4-\lambda^2}}|\delta_k|\biggr)\biggr) = +\infty$$

for some $p \geqslant 1$ *and* $q > 1$, *then* $\lambda \notin \sigma_{\text{p}}(J)$.

The case $\alpha_n = \beta_n \in \mathbb{R}$, $q_n \in \mathbb{R}$ was investigated in [**12**] and the result obtained there by Janas and Naboko is stronger than Theorem 4.6.

5. Jacobi operators of the Atzmon–Sodin type

In the present section we consider the Jacobi operator (given below by the formula (5.3)) of the Atzmon–Sodin type. It was extensively studied by Atzmon and Sodin because of the particular structure of its invariant subspaces (see [**2**]). They have found an analytic model for it and the structure of the spectrum and the point spectrum in a more general situation. Below we shall analyse the point spectrum of such operators in a special case because it relates strongly to the context of this paper.

Let us fix $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \neq 0$, and assume that $\lim_{n\to\infty} \alpha_n = \alpha$, where $\alpha \in C$, $|\alpha| \geqslant 1$. Define the operator L by

$$Le_n = \alpha_n e_{n+1}, \quad n = 1, 2, \dots, \tag{5.1}$$

and the operator $L^{(-1)}$ by

$$L^{(-1)} e_1 = 0, \quad L^{(-1)} e_n = \frac{1}{\alpha_{n-1}} e_{n-1}, \quad n = 2, 3, \dots. \tag{5.2}$$

Consider the operator

$$A = L + L^{(-1)} \tag{5.3}$$

associated with the sequence $\{\alpha_n\}$. The operator A is a compact perturbation of the operator $\alpha S + (1/\alpha) S^*$ and

$$\sigma\left(\alpha S + \frac{1}{\alpha} S^*\right) = \left\{\lambda = z + \frac{1}{z} : 1 \leqslant |z| \leqslant |\alpha|\right\}.$$

Because $L^{(-1)} L = I$, then for $\lambda = z + (1/z)$ we have

$$A - \lambda = L^{(-1)} (L - z)\left(L - \frac{1}{z}\right).$$

Using ideas of [**2**] it is not difficult to prove that

$$\sigma(A) = \left\{\lambda = z + \frac{1}{z} : 1 \leqslant |z| \leqslant |\alpha|\right\}.$$

Proposition 5.1. *If*

$$\left\{d_n = \prod_{k=1}^{n} \alpha \alpha_k^{-1}\right\}_{n=1}^{\infty} \quad \textit{and} \quad \{d_n^{-1}\}_{n=1}^{\infty}$$

are bounded, then A is similar to $\alpha S + (1/\alpha) S^*$.

Proof. Let

$$\left\{d_n = \prod_{k=1}^{n} \frac{\alpha}{\alpha_k}\right\}_{n=1}^{\infty}.$$

Then the diagonal operator D with $d(D) = \{d_n\}$ is bounded and boundedly invertible; moreover, $DAD^{-1} = \alpha S + (1/\alpha) S^*$ and the result holds. □

Notice that both $\{d_n\}_n$ and $\{1/d_n\}_n$ are bounded, if for instance $\{\alpha_n - \alpha\}_{n=1}^{\infty} \in \ell^1$, and as a simple consequence of Proposition 5.1 we have that $\sigma_{\rm p}(A) = \emptyset$. If $|\alpha| \geqslant 1$, then $\sigma_{\rm p}(\alpha S + (1/\alpha)S^*) = \emptyset$. As we will see in the next proposition, we do not need any conditions on the rate of convergence of $\{\alpha_n\}$ to assure that $\sigma_{\rm p}(A) = \emptyset$, if $|\alpha| > 1$. Since

$$A - \lambda = -z\left(L^{(-1)} - \frac{1}{z}\right)\left(L - \frac{1}{z}\right); \tag{5.4}$$

therefore, to decide when $\ker(A-\lambda) = \{0\}$ it is enough to show that $ker(L-(1/z)) = \{0\}$ and $\ker(L^{(-1)} - (1/z)) = \{0\}$.

Since L is the unilateral weighted shift, it is well known that $\ker(L - \omega) = \{0\}$ for any $\omega \in \mathbb{C}$.

Let us examine $\ker(L^{(-1)} - \omega)$. By direct computations one can prove the following lemma.

Lemma 5.2.

(1) *If $f = \{f_n\}_{n=1}^{\infty}$ satisfies $(L^{(-1)} - \omega)f = 0$, then*

$$f_n = \omega^{n-1}\alpha_{n-1}\dots\alpha_1 f_1, \quad n = 2, 3, \dots, \tag{5.5}$$

where $f_1 \in C$ is any number.

(2) *If $g = \{g_n\}_{n=1}^{\infty}$ satisfies $(L - \omega)g = f$, where f is as in the previous point, then*

$$g_1 = -\omega^{-1} f_1 \tag{5.6}$$

and

$$g_n = -\alpha_{n-1}\dots\alpha_1 f_1 \omega^{n-2}(1 + \omega^{-2} + \dots + (\omega^{-2})^{n-1}) \tag{5.7}$$

$$= -\alpha_{n-1}\cdots\alpha_1 f_1 \omega^{n-2}\frac{1 - (\omega^{-2})^n}{1 - \omega^{-2}}, \quad n = 2, 3, \dots. \tag{5.8}$$

(3) *If f and g are given by (5.5), (5.6), (5.7) or (5.8), then they satisfy $(L^{(-1)} - \omega)f = 0$ and $(L - \omega)g = f$.*

Proposition 5.3. *If $|\alpha| > 1$, then $\sigma_{\rm p}(A) = \emptyset$.*

Proof. Let $\lambda = z + (1/z) \in \sigma_{\rm p}(A)$. There exists $g \in \ell^2$ such that

$$0 = (A - \lambda)g = -z\left(L^{(-1)} - \frac{1}{z}\right)\left(L - \frac{1}{z}\right)g.$$

Define $f = (L - (1/z))g \in \ell^2$. Because f and g are as in Lemma 5.2, so

$$f_n = z^{1-n}\alpha_1\cdots\alpha_{n-1}f_1 \quad \text{and} \quad g_n = -\alpha_{n-1}\cdots\alpha_1 f_1 z^{2-n}(1 - (z^2)^n)(1 - z^2)^{-1}.$$

If $z \neq \pm 1$ and $f_1 \neq 0$, then

$$|g_n| = |f_1|\,|\alpha_{n-1}\cdots\alpha_1|\,|z^{-n} - z^n|/|z^{-2} - 1|$$

does not converge to 0, this implies that $g \notin \ell^2$. Hence $f = g = 0$.

If $|z| = 1$ and $g \in \ell^2$, then $f \in \ell^2$, and so

$$\sum_{n=1}^{\infty} |z^{-n}\alpha_1 \cdots \alpha_n|^2 < +\infty$$

or $f_1 = 0$.

Notice that

$$+\infty > \sum_{n=1}^{\infty} |z^{-n}\alpha_1 \cdots \alpha_n|^2 = \sum_{n=1}^{\infty} |\alpha_1 \cdots \alpha_n|^2,$$

but this is impossible since $\alpha_n \to \alpha$, $|\alpha| > 1$. Thus $f_1 = 0$ implies $f = g = 0$ and the proof is complete. □

In the case $|\alpha| = 1$ the situation looks different and is described by the following proposition.

Proposition 5.4. *Assume that* $|\alpha| = 1$.

(1) *If*

$$\sum_{n=1}^{\infty} |\alpha_1 \cdots \alpha_n|^2 = +\infty, \tag{5.9}$$

then $\sigma_{\mathrm{p}}(A) = \emptyset$.

(2) *If*

$$\sum_{n=1}^{\infty} |\alpha_1 \cdots \alpha_n|^2 < +\infty, \tag{5.10}$$

then $(-2, 2) \subset \sigma_{\mathrm{p}}(A)$.

(3)

$$\sum_{n=1}^{\infty} n^2 |\alpha_1 \cdots \alpha_n|^2 < +\infty \iff \pm 2 \in \sigma_{\mathrm{p}}(A).$$

Proof. (1) Let $\lambda = z + (1/z) \in \sigma_{\mathrm{p}}(A) \subset [-2, 2]$, where $|z| = 1$ and $\omega = z^{-1}$. By Lemma 5.2 we have $(L^{-1} - \omega)f = 0$ and so $f_n = \alpha_1 \cdots \alpha_{n-1}\omega^{n-1} f_1$. Because of (5.9) and $|\omega| = 1$, $f \in \ell^2$ only if $f_1 = 0$, which implies $f = 0$. Thus $\ker(L^{-1} - \omega) = 0$, combining with $\ker(L - \omega) = 0$ we get that $\ker(A - \lambda) = 0$, a contradiction.

(2) If $\lambda = z + (1/z) \in (-2, 2)$, then z can be chosen with $|z| = 1$ and $z^2 \neq 1$. If $(A - \lambda)g = 0$ and $f = (L - (1/z))g$, then by Lemma 5.2 we have relations for f and g. Using them we see that f and g can be chosen from ℓ^2 and not equal to 0.

(3) Consider now $\lambda = \pm 2$, then $z = \omega^{-1} = \pm 1$. Using Lemma 5.2 again the result follows easily. □

The results included in Propositions 5.3 and 5.4 have also been proved by Atzmon and Sodin in [**2**].

6. An example

We are going to construct an operator J such that it is still a 'small' perturbation of J_0 but for which $1+\rho \in \sigma_{\mathrm{p}}(J)$; therefore J is not similar to the operator T defined by (2.2).

Assume that $q_n = 0$ and let us start to define β_n and α_n. Let $\beta_1 = 1+\rho$, $\zeta = 1$ and for the eigenvector f assume that $f_1 = f_2 = 1$ and so (4.2) for $n = 1$ is satisfied.

Using (4.4) and (4.6) we have

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = W B_n \cdots B_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n = 2, 3, \dots,$$

because

$$W^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume that $A_n^+ = 0$ (defined by (4.9)) for $n \geqslant 2$. Then

$$\epsilon_n = -(1+\rho)/\rho\delta_n \tag{6.1}$$

and $A_n^- = -(1+\rho)\delta_n$, where $n \geqslant 2$.

Define $a_n = A_n^-$. Then the transfer matrix B_n has the form (see (4.11))

$$B_n = \begin{pmatrix} 1/\rho & -a_n \\ 0 & 1 + a_n \end{pmatrix}.$$

Hence

$$B_{2l+1} B_{2l} = \begin{pmatrix} 1/\rho^2 & -a_{2l}/\rho - a_{2l+1}(1 + a_{2l}) \\ 0 & (1 + a_{2l+1})(1 + a_{2l}) \end{pmatrix}, \quad l = 1, 2, \dots.$$

If we assume that $-a_{2l}/\rho - a_{2l+1}(1 + a_{2l}) = 0$, then

$$a_{2l+1} = -\frac{a_{2l}}{\rho(1 + a_{2l})} \tag{6.2}$$

for all $l \geqslant 1$. Computing by pairs of B_n we have

$$\begin{pmatrix} f_{2k+1} \\ f_{2(k+1)} \end{pmatrix} = W \begin{pmatrix} 1/\rho^{2k} & 0 \\ 0 & \displaystyle\prod_{l=1}^{k} (1 + a_{2l+1})(1 + a_{2l}) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6.3}$$

where $k = 1, 2, \dots$.

Due to (6.2) we have $(1 + a_{2l+1})(1 + a_{2l}) = 1 - [(1-\rho)/\rho]a_{2l}$ so

$$f_{2k+1} = f_{2(k+1)} = \prod_{l=1}^{k} (1 - [(1-\rho)/\rho]a_{2l}),$$

where $k \geqslant 1$ and additionally $f_1 = f_2 = 1$. Therefore, it is not difficult to choose the sequence $\{a_{2l}\}_{l=1}^{\infty}$ to obtain $\{f_n\} \in \ell^2$ and $\{\epsilon_n\}, \{\delta_n\} \notin \ell^1$ but $\{\epsilon_n\}, \{\delta_n\} \in \ell^p$ for all $p > 1$.

For example, if

$$a_{2l} = \rho/(1-\rho)\frac{1}{l+1}, \quad l = 1, 2, \dots, \tag{6.4}$$

then

$$a_{2l+1} = -\frac{1}{(1-\rho)l+1} \tag{6.5}$$

for $l = 1, 2, \dots,$ and

$$f_{2k+1} = f_{2(k+1)} = \prod_{l=1}^{k}\left(1 - \frac{1}{l+1}\right) = \frac{1}{1+k},$$

where $k = 1, 2, \dots$. It is obvious that $\{f_n\} \in \ell^2$ and we can calculate $\delta_n = -a_n/(1+\rho)$ and then β_n for $n \geqslant 2$. Moreover, it is easy to check that $\{\beta_n - \rho\} \in \ell^p \setminus \ell^1$, $p > 1$. Finally, (by (6.1) and (4.7)) we obtain $\{\epsilon_n\}$ with similar properties.

Acknowledgements. The research of J.J. and M.M. was supported by the grant PB 2 PO3 002 13 of Komitet Badań Naukowych.

References

1. A. I. Aptekarev, V. Kaliaguine and W. Van Assche, Criterion for the resolvent set of nonsymmetric tridiagonal operators, *Proc. Am. Math. Soc.* **123** (1995), 2423–2430.
2. A. Atzmon and M. Sodin, Completely indecomposable operators and a uniqueness theorem of Cartwright–Levinson type, *J. Funct. Analysis* **169** (1999), 164–188.
3. B. Beckermann, On the classification of the spectrum of second-order difference operators, *Math. Nachr.* **216** (2000), 45–59.
4. B. Beckermann and V. Kaliaguine, The diagonal of the Padé table and the approximation of the Weyl function of second-order difference operators, *Constr. Approx.* **13** (1997), 481–510.
5. P. Cojuhari, The absence of eigenvalues of the perturbated discrete Wiener–Hopf operator, *Bull. Acad. Ştiinţe Repub. Mold. Mat.* **3** (1990), 26–35 (in Russian).
6. R. G. Douglas, *Banach algebra techniques in operator theory* (Academic, 1972).
7. S. N. Elaydi, *An introduction to difference equations* (Springer, 1999).
8. J. M. Freeman, Volterra operators similar to $J : f \to \int_0^x f(t)\,\mathrm{d}t$, *Trans. Am. Math. Soc.* **116** (1965), 181–192.
9. K. O. Friedrichs, *Perturbation of spectra in Hilbert space*, Lectures in Applied Mathematics, vol. 3 (American Mathematical Society, Providence, RI, 1965).
10. W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Rev.* **9** (1967), 24–82.
11. I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators* (State Publishing of Mathematics–Physics Literature, Moscow, 1963) (in Russian).
12. J. Janas and S. N. Naboko, On the point spectrum some Jacobi matrices, *J. Operat. Theory* **40** (1998), 113–132.
13. B. S. Pavlov, *On non-self-adjoint Schrödinger operators*, Problems of Mathematical Physics, vol. 1, (Leningrad University Press, 1966) (in Russian).