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AVERAGE DISTANCE CONSTANTS FOR POLYGONS IN SPACES WITH NON-POSITIVE CURVATURE

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In this paper we show that the average distance constant of a general polygon which is a subset of an M-space with non-positive curvature can be expressed as the extreme value of either of two nonlinear programs and discuss the practical application of one of these nonlinear programs for the determination of the average distance constant for a polygon in general, and in particular for a planar triangle.

1. INTRODUCTION

Recently the work of Gross [4] concerning the average distance property of compact connected metric spaces has interested many authors [**2, 6, 7, 8, 11, 13, 14**]. Gross and Stadje [**10**] independently proved the following remarkable result:

THEOREM 1. *Let (X, d) be a compact connected metric space with metric d. Then there exists a unique positive real number $a(X, d)$ (called the average distance constant of (X, d)) such that for any finite collection of points $x_1, x_2, \ldots, x_n \in X$ (not necessarily distinct), there is $y \in X$ with $\left(\sum_{i=1}^{n} d(x_i, y)\right)/n = a(X, d)$.*

The evaluation of the average distance constant of a given compact connected metric space is an important problem in numerical geometry. Explicit formulae for the constant have only been found in simple cases many of which are detailed in [**2**]. In particular a result is given for the perimeter of a regular polygon in the two dimensional Euclidean plane. Recently a result for general planar polygons has been obtained in [**5**] which expresses the average distance constant of the polygon as the extreme value of either of two nonlinear programs. In this paper we establish the same result for general polygons which are subsets of an M-space with not necessarily constant non-positive curvature. To date all non-trivial results concerning average distance constants of special subsets have been confined to spaces with constant curvature.

We now introduce some definitions which are essential to our work. Following [**3**] we have:

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DEFINITION 1: Let (X, d) be a metric space. Then (X, d) is an M-space if for each pair of distinct points $x, y \in X$ and $\forall \alpha \in (0, 1)$ there is a unique $z(\alpha) \in X$ such that

$$d(x, z(\alpha)) = \alpha d(x, y) \text{ and } d(y, z(\alpha)) = (1 - \alpha)d(x, y). \tag{1.1}$$

We now define the concepts of line segment and midpoint for M-spaces as follows

DEFINITION 2: Let x, y be any pair of distinct points in an M-space (X, d) and for any $\alpha \in [0, 1]$ let $z(\alpha)$ denote the unique point in X such that (1.1) holds. Then

$$\{z(\alpha) : \alpha \in [0, 1]\} \tag{1.2}$$

defines the line segment xy joining x and y.

DEFINITION 3: Let x, y be any pair of distinct points in an M-space (X, d). Then the midpoint of xy is the unique point $m \in X$ such that $d(x, m) = d(y, m) = d(x, y)/2$.

Following [1] we can now define an M-space with non-positive curvature.

DEFINITION 4: Let (X, d) be an M-space. Then (X, d) is said to have non-positive curvature if for all triplets of distinct points $x, y, z \in X$ the midpoints m, n of xy and xz respectively, satisfy $d(m, n) \leqslant d(y, z)/2$.

2. Preliminary results

In this section we introduce some lemmas similar to those of [5] which are required for the proof of the main theorem.

LEMMA 1. *For any positive integer n let $\mathbf{r}_n$ be n-tuple $(r_1, r_2, \ldots, r_n)$ and let*

$$C(n) = \left\{ \mathbf{r}_n \in R^n : \sum_{i=1}^{n} r_i = 1,\ r_i \geqslant 0 \right\}. \tag{2.1}$$

Then if $x_1, x_2, \ldots, x_n$ is any set of distinct points in a compact connected metric space (X, d) and $\mathbf{r}_n \in C(n)$,

$$\min_{x \in X} \sum_{i=1}^{n} r_i d(x_i, x) \leqslant a(X, d) \leqslant \max_{x \in X} \sum_{i=1}^{n} r_i d(x_i, x). \tag{2.2}$$

PROOF: If $r_1, r_2, \ldots, r_n$ are all rational numbers then we can find a positive integer m and a set of points $y_1, y_2, \ldots, y_m \in X$ which is formed by suitable repetitions of the points x_i, $i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} r_i d(x_i, x) = \frac{1}{m} \sum_{j=1}^{m} d(y_j, x).$$

Thus by Theorem 1 (2.2) holds for any $\mathbf{r}_n \in C(n)$ with rational components but since the set of rational numbers is dense in R it follows that (2.2) holds for all $\mathbf{r}_n \in C(n)$. ☐

LEMMA 2. *Let $\mathbf{x}_n$ denote an n-tuple $x_1, x_2, \ldots, x_n$ and let X^n denote the set of all n-tuples with elements in a compact connected metric space (X, d) and let*

$$H(n) = \max_{\mathbf{x}_n \in X^n} \min_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x), \quad \text{and}$$

$$L(n) = \min_{\mathbf{x}_n \in X^n} \max_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x).$$

Then

$$a(X, d) = \sup_n H(n) = \inf_n L(n). \tag{2.3}$$

PROOF: From Theorem 1 we clearly have

$$\min_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x) \leqslant a(X, d) \leqslant \max_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x).$$

Hence $H(n) \leqslant a(X, d) \leqslant L(n)$ and thus

$$\sup_n H(n) \leqslant a(X, d) \leqslant \inf_n L(n).$$

Now for each positive integer n and each $\mathbf{x}_n \in X^n$ since we have

$$\min_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x) \leqslant \sup_n H(n) \text{ and } \max_{x \in X} \frac{1}{n} \sum_{i=1}^{n} d(x_i, x) \geqslant \inf_n L(n)$$

from the connectivity of X it follows that there is $y \in X$ such that $\left(\sum_{i=1}^{n} d(x_i, y)\right)/n = a$ for any $a \in [\sup_n H(n), \inf_n L(n)]$ and the result of the lemma follows since the existence of more than one such a would contradict Theorem 1. ☐

For the remaining lemmas in this section let (X, d) denote an M-space with non-positive curvature.

LEMMA 3. *Let ab be a line segment in (X, d) with midpoint m. Then for any $y \in X$*

$$d(a, y) + d(b, y) \geqslant 2d(m, y). \tag{2.4}$$

PROOF: Let p be the midpoint of ay. Then $d(m, y) \leqslant d(p, m) + d(p, y)$ since d is a metric, and $d(p, m) \leqslant d(b, y)/2$ from Definition 4 since (X, d) has non-positive curvature, and $d(p, y) = d(a, y)/2$ from Definition 3. Hence $d(m, y) \leqslant d(b, y)/2 + d(a, y)/2$ and (2.4) follows. □

LEMMA 4. *Let ab be a line segment in (X, d) and let x be any point on ab. Then for any $y \in X$,*

$$d(x, y)d(a, b) \leqslant d(x, a)d(y, b) + d(x, b)d(y, a). \tag{2.5}$$

PROOF: Let $\alpha = d(a, b)$ and let $f\colon [0, \alpha] \to R$ be such that $f(d(x, a)) = d(x, y) \forall x \in ab$. Then, from Lemma 3, $f(\alpha/2) \leqslant (f(0) + f(\alpha))/2$, and since f is clearly continuous it follows from [**12**, p.7] that f is a convex function. Thus

$$f(\lambda t_1 + (1 - \lambda)t_2) \leqslant \lambda f(t_1) + (1 - \lambda)f(t_2)$$

for any $t_1, t_2 \in [0, \alpha]$ and $0 \leqslant \lambda \leqslant 1$. On setting $\lambda = d(x, a)/\alpha$, $t_1 = \alpha$ and $t_2 = 0$ (2.5) follows. □

LEMMA 5. *For any positive integer n let $\mathbf{r}_n$ and $C(n)$ be as defined in Lemma 1. Then for any $x_1, x_2, \ldots, x_n$ in a line segment ab of $(X, d) \forall y \in X$ and $\forall \mathbf{r}_n \in C(n)$ there is $x \in X$ such that*

$$d(x, y) \leqslant \sum_{i=1}^{n} r_i d(x_i, y). \tag{2.6}$$

PROOF: Take $x \in ab$ such that $d(a, x) = \sum_{i=1}^{n} r_i d(a, x_i)$; then the result follows from the convexity of the function f defined in Lemma 4. □

3. THEOREM FOR POLYGONS

THEOREM 2. *Let (X, d) be an M-space with non-positive curvature and let*

$$P = \bigcup_{i=1}^{n} q_i q_{i+1}$$

be the polygon with n vertices $q_1, q_2, \ldots, q_n \in X$ where $q_{n+1} \equiv q_1$. Then

$$a(P, d) = \max_{\mathbf{r}_n \in C(n)} \min_{x \in P} \sum_{i=1}^{n} r_i d(q_i, x), \tag{3.1}$$

and there exist $x_1, x_2, \ldots, x_n \in P$ such that

$$a(P, d) = \min_{\mathbf{r}_n \in C(n)} \max_{x \in P} \sum_{j=1}^{n} r_j d(x_j, x), \tag{3.2}$$

where $\mathbf{r}_n$ *and* $C(n)$ *are as defined in Lemma 1.*

PROOF: Since P is compact, for any positive integer m we can choose $x_1, x_2, \ldots, x_m \in P$ such that

$$\min_{x \in P} \frac{1}{m} \sum_{i=1}^{m} d(x_i, x) = H(m),$$

where $H(m)$ is as defined in Lemma 2. Now let $q_0 \equiv q_n$ and for $j = 1, 2, \ldots, n$ let

$$r_j(m) = \frac{1}{m} \left(\sum_{x_i \in q_{j-1} q_j \setminus q_j} \frac{d(q_{j-1}, x_i)}{d(q_{j-1}, q_j)} + \sum_{x_i \in q_j q_{j+1}} \frac{d(q_{j+1}, x_i)}{d(q_j, q_{j+1})} \right),$$

for the chosen x_i; then $\mathbf{r}_n(m) \in C(n)$, and by Lemma 4 for $x_i \in q_j q_{j+1}$ and $x \in P$ we have

$$d(x_i, x) \leqslant \frac{d(q_j, x_i)}{d(q_j, q_{j+1})} d(q_{j+1}, x) + \frac{d(q_{j+1}, x_i)}{d(q_j, q_{j+1})} d(q_j, x),$$

and hence

$$\min_{x \in P} \sum_{j=1}^{n} r_j(m) d(q_j, x) \geqslant \min_{x \in P} \frac{1}{m} \sum_{i=1}^{m} d(x_i, x) = H(m). \tag{3.3}$$

From (2.2) it follows that if there exists $\mathbf{r}_n^* = (r_1^*, \ldots, r_n^*) \in C(n)$ such that

$$\min_{x \in P} \sum_{j=1}^{n} r_j^* d(q_j, x) = a(P, d), \tag{3.4}$$

then the left hand side of (3.4) is equal to the right hand side of (3.1). Now if there is a finite positive integer m such that $H(m) = a(P, d)$, then from (2.2) and (3.3) it follows that (3.4) holds with $\mathbf{r}_n^* = \mathbf{r}_n(m)$ and hence (3.1) is valid. Otherwise it follows from Lemma 2 that there is a subsequence $\{H(m_k)\}$ of $\{H(m)\}$ such that $\lim_{m_k \to \infty} H(m_k) = a(P, d)$, and thus from (2.2) and (3.3)

$$\lim_{m_k \to \infty} \min_{x \in P} \sum_{j=1}^{n} r_j(m_k) d(q_j, x) = a(P, d).$$

Since $C(n)$ is compact we can again find $\mathbf{r}_n^* \in C(n)$ such that (3.4) holds and thus (3.1) is valid in this case also.

Also since P is compact, for any positive integer m there are $x_1^*, x_2^*, \ldots, x_m^* \in P$ such that

$$\max_{x \in P} \frac{1}{m} \sum_{i=1}^{m} d(x_i^*, x) = L(m),$$

where $L(m)$ is as defined in Lemma 2.

Suppose the points x_i^* are numbered so that for $j = 1, 2, \ldots, n$ the segment $q_{j-1}q_j$ (excluding q_j) contains the t_j points $x_{b_j+1}^*, x_{b_j+2}^*, \ldots, x_{b_j+t_j}^*$ where $b_1 = 0$ and $b_{j+1} = b_j + t_j$, $j = 1, 2, \ldots, n-1$. Then for each $j = 1, 2, \ldots, n$ choose point $x_j(m) = q_j$ and weight $s_j(m) = 0$ if $t_j = 0$, and otherwise choose $x_j(m) \in q_{j-1}q_j$ such that

$$d(q_{j-1}, x_j(m)) = \frac{1}{t_j}\sum_{i=1}^{t_j} d\left(q_{j-1}, x_{b_j+i}^*\right),$$

and choose $s_j(m) = t_j/m$. Then by Lemma 5

$$d(x_j(m), x) \leqslant \frac{1}{t_j}\sum_{i=1}^{t_j} d\left(x_{b_j+i}^*, x\right),$$

and hence

$$\max_{x\in P}\sum_{j=1}^{n} s_j(m)d(x_j(m), x) \leqslant \max_{x\in P}\frac{1}{m}\sum_{i=1}^{m} d(x_i^*, x) = L(m). \tag{3.5}$$

Also clearly $\mathrm{s}_n(m) \equiv (s_1(m), s_2(m), \ldots, s_n(m)) \in C(n)$.

If there is a finite m such that $L(m) = a(P, d)$ then from (3.5) and (2.2) it follows that (3.2) is established with $x_i = x_i(m)$, $i = 1, \ldots, n$. Otherwise from Lemma 2 it follows that there is a subsequence $\{L(m_k)\}$ of $\{L(m)\}$ such that $\lim_{m_k\to\infty} L(m_k) = a(P, d)$ and thus from (2.2) and (3.5) it follows that

$$\lim_{m_k\to\infty}\max_{x\in P}\sum_{j=1}^{n} s_j(m_k)d(x_j(m_k), x) = a(P, d).$$

Since both P and $C(n)$ are compact it follows that there is $\mathrm{r}_n \in C(n)$ and $x_1, x_2, \ldots, x_n \in P$ such that

$$\max_{x\in P}\sum_{j=1}^{n} r_j d(x_j, x) = a(P, d),$$

and thus from (2.2) it again follows that (3.2) is valid. □

4. Practical considerations

In this section we discuss the practical implementation of the nonlinear programming problem (3.1) for the computation of the average distance constant $a(P, d)$ of a polygon P with vertices $q_1, q_2, \ldots, q_n$ as defined in Theorem 2.

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For $\mathbf{r}_n \in C(n)$ and $x \in P$ let

$$g(\mathbf{r}_n, x) = \sum_{i=1}^{n} r_i d(q_i, x) \tag{4.1}$$

and consider $g(\mathbf{r}_n, x)$ for $x \in q_j q_{j+1}$. Firstly let $l_j = d(q_j, q_{j+1})$ and let $x(t) \in q_j q_{j+1}$ be such that $t = d(q_j, x(t))$. Now let $f_{jk}\colon [0, l_j] \to R$ be such that $f_{jk}(t) = d(q_k, x(t))$ for $k = 1, 2, \ldots, n$. Note by analogy with Lemma 4 that each f_{jk} is convex and also that for $x(t) \in q_j q_{j+1}$ we have $g(\mathbf{r}_n, x(t)) \equiv F_j(\mathbf{r}_n, t)$ where

$$F_j(\mathbf{r}_n, t) = (r_j - r_{j+1})t + r_{j+1} l_j + \sum_{k \neq j, j+1}^{n} r_k f_{jk}(t). \tag{4.2}$$

Clearly $F_j(\mathbf{r}_n, t)$ is a convex function of t for fixed $\mathbf{r}_n \in C(n)$ and thus has a unique minimum in $[0, l_j]$.

Now let

$$g_j(\mathbf{r}_n) = \min_{x \in q_j q_{j+1}} g(\mathbf{r}_n, x); \tag{4.3}$$

then clearly $g_j(\mathbf{r}_n) = \min_{t \in [0, l_j]} F_j(\mathbf{r}_n, t)$. If $f_{jk}(t)$ can be evaluated then $g_j(\mathbf{r}_n)$ can be found using a one dimensional optimisation search technique which uses function evaluations, for example [9, Section 8] on $F_j(\mathbf{r}_n, t)$. In certain cases (for example see Section 5) we can differentiate $F_j(\mathbf{r}_n, t)$ with respect to t and find the minimum analytically.

Now let

$$G(\mathbf{r}_n) = \min_{x \in P} g(\mathbf{r}_n, x); \tag{4.4}$$

then clearly $G(\mathbf{r}_n) = \min_{1 \leqslant j \leqslant n} g_j(\mathbf{r}_n)$. From (3.1), (4.3) and (4.4) it now follows that

$$a(P, d) = \max_{\mathbf{r}_n \in C(n)} G(\mathbf{r}_n). \tag{4.5}$$

Now let $\mathbf{r}_n^* \in C(n)$ and $\mathbf{s}_n \in R^n$ be such that

$$\mathbf{r}_n = \mathbf{r}_n^* + t\mathbf{s}_n \in C(n) \text{ for } t \in [t_1, t_2]; \tag{4.6}$$

then for $\mathbf{r}_n$ given by (4.6) we can write

$$g(\mathbf{r}_n, x) = \alpha(x) + t\beta(x) \equiv h(t, x) \tag{4.7}$$

where

$$\alpha(x) = \sum_{i=1}^{n} \mathbf{r}_i^* d(q_i, x) \text{ and } \beta(x) = \sum_{i=1}^{n} s_i d(q_i, x). \tag{4.8}$$

Now let $F(t) = \min_{x \in P} h(t, x)$; then $F(t) = \alpha(x(t)) + t\beta(x(t))$ where $x(t) \in P$ is a point which minimises $h(t, x)$ for a given t. Note that $x(t)$ may not be unique.

Now take any $t^* \in (t_1, t_2)$.

(a) If there is $x(t^*) \in P$ which mininises $h(t^*, x)$ and is such that $\beta(x(t^*)) < 0$ then let β_1 be the smallest such $\beta(x(t^*))$ and let α_1 be the corresponding $\alpha(x(t^*))$. Then it follows that for $t \in (t^*, t_2]$

$$F(t) \leqslant \alpha_1 + t\beta_1 < \alpha_1 + t^*\beta_1 = F(t^*). \tag{4.9}$$

(b) If there is $x(t^*) \in P$ which minimises $h(t^*, x)$ and is such that $\beta(x(t^*)) > 0$ then let β_2 be the largest such $\beta(x(t^*))$ and let α_2 be the corresponding $\alpha(x(t^*))$. Then it follows that for $t \in [t_1, t^*)$

$$F(t) \leqslant \alpha_2 + t\beta_2 < \alpha_2 + t^*\beta_2 = F(t^*). \tag{4.10}$$

(c) If $\beta(x(t^*)) = 0 \,\forall\, x(t^*) \in P$ which minimise $h(t^*, x)$ let α_3 be the common value of all corresponding $\alpha(x(t^*))$. Then it follows that for $t \in [t_1, t_2]; t \neq t^*$

$$F(t) \leqslant \alpha_3 = F(t^*). \tag{4.11}$$

Since at least one of (a), (b), (c) must hold, it follows from (4.9), (4.10), and (4.11) that $F(t)$ cannot have a local minimum for any $t \in (t_1, t_2)$. It now follows that a one dimensional optimisation search technique such as that of [9] can be used to find $\max_{t \in [t_1, t_2]} F(t)$. Furthermore it is clear that any optimisation method which carries out a sequence of suitably chosen one dimensional optimisations can be used to solve (4.5). Since $G(\mathbf{r}_n)$ is not differentiable for all $\mathbf{r}_n \in C(n)$ and may not be differentiable at its maximum, a method which requires only function evaluations is preferable for solving this problem. One such method which has been tried successfully for planar triangles (see Section 5) is a modification of the method of [9]. In order to take into account the constraints on $\mathbf{r}_n$, the one dimensional optimisations in this method were limited to line segments lying in the constraint region $C(n)$.

5. Average distance constants of planar triangles

In this section we discuss the evaluation of the function $G(\mathbf{r}_3)$ to be maximised over $\mathbf{r}_3 \in C(3)$ in order to find the average distance constant $a(T, d)$ of a triangle T with vertices $q_1, q_2, q_3 \in R^2$ where d is the Euclidean metric, and present computed results for a variety of triangles.

From (3.1) we have

$$a(T, d) = \max_{\mathbf{r}_3 \in C(3)} \min_{x \in T} \sum_{i=1}^{3} r_i d(q_i, x). \tag{5.1}$$

Now for $j = 1, 2, 3$ define

$$l_j = d(q_j, q_{j+1}) \text{ and } \theta_j = \angle q_{j-1} q_j q_{j+1} \tag{5.2}$$

where $q_0 \equiv q_3$ and $q_4 \equiv q_1$. Also for convenience we assume $l_0 \equiv l_3$, $l_4 \equiv l_1$, $r_0 \equiv r_3$ and $r_4 \equiv r_1$.

Now let $x(t)$ be a point on $q_j q_{j+1}$ such that $t = d(q_j, x(t))$ and let $f_j\colon [0, l_j] \to R$ be such that $f_j(t) = d(q_{j-1}, x(t))$; then by analogy with (4.1) and (4.2), after expressing $f_j(t)$ using the cosine rule, we find that $g(\mathbf{r}_3, x(t)) \equiv F_j(\mathbf{r}_3, t)$ where

$$F_j(\mathbf{r}_3, t) = (r_j - r_{j+1})t + r_{j+1} l_j + r_{j-1}\sqrt{l_{j-1}^2 + t^2 - 2 l_{j-1} t \cos\theta_j}. \tag{5.3}$$

By differentiation of $F_j(\mathbf{r}_3, t)$ with respect to t and noting its convexity, the following expressions were obtained in [5] for the function $g_j(\mathbf{r}_3)$ defined by (4.3) with $n = 3$

(5.4)

$$g_j(\mathbf{r}_3) = \begin{cases} r_{j+1} l_j + r_{j-1} l_{j-1} & \text{for } r_j - r_{j+1} \geqslant r_{j-1}\cos\theta_j, \\ r_j l_j + r_{j-1} l_{j+1} & \text{for } r_j - r_{j+1} \leqslant -r_{j-1}\cos\theta_{j+1}, \\ r_j l_{j-1}\cos\theta_j + r_{j+1} l_{j+1} & \cos\theta_{j+1} + l_{j-1}\sin\theta_j \sqrt{r_{j-1}^2 - (r_j - r_{j+1})^2}, \\ & \text{otherwise.} \end{cases}$$

Using (5.4), the function $G(\mathbf{r}_3) = \min_{1 \leqslant j \leqslant 3} g_j(\mathbf{r}_3)$ to be maximised can be readily evaluated for any $\mathbf{r}_3 \in C(3)$ and the values can then be used in the optimisation method referred to in Section 4.

Using this method the average distance constants $a(T, d)$ were successfully computed for a number of planar triangles T. The results are presented in Table 1 which along with $a(T, d)$, gives the side lengths $l_1, \ldots, l_3$ used for each T, the value of $\mathbf{r}_3$ at which the maximum of $G(\mathbf{r}_3)$ was found, and references to notes on the results given below the table.

Table 1. Average Distance Constants of Planar Triangles

l_1	l_2	l_3	$a(T, d)$	r_1	r_2	r_3	Notes
1.00	1.00	1.00	0.6220085	0.3333333	0.3333333	0.3333333	1,2
2.00	1.50	1.50	1.0351435	0.3511297	0.3511297	0.2977407	2
2.00	3.00	3.00	1.6666667	0.3000000	0.3000000	0.4000000	2
2.00	5.00	5.00	2.6000000	0.2692308	0.2692308	0.4615385	2
2.00	10.00	10.00	5.0500000	0.2549505	0.2549505	0.4900990	2
2.00	100.00	100.00	50.0050000	0.2500501	0.2500501	0.4998998	2,3
3.00	4.00	5.00	2.5000000	0.2604160	0.4062503	0.3333338	4
5.00	12.00	13.00	6.5000000	0.1965602	0.4710648	0.3333330	4
1.10	0.90	1.00	0.6213553	0.3352839	0.3187589	0.3459473	5
0.61	1.40	2.00	1.0000000	0.0000000	0.5000000	0.5000000	4

Notes on Table 1:

1. Value of $a(T, d)$ agrees with the value $(2+\sqrt{3})/6$ for an equilateral triangle with unit length sides given in [**2**].
2. Values of $a(T, d)$ given agree with the following formulae given in [**5**] for an isosceles triangle with two sides of equal length l and base of length 2 .

$$a(T, d) = \begin{cases} \dfrac{l^2+2l-\sqrt{l^2-1}-2\sqrt{\left(1-\sqrt{l^2-1}\right)l(l+1)}}{l^2+3l-1-l\sqrt{l^2-1}-2\sqrt{\left(l-\sqrt{l^2-1}\right)l(l+1)}} & \text{for } \sqrt{2} \leqslant l \leqslant l^* \\ (l^2+1)/2l & \text{for } l \geqslant l^* \end{cases}$$

where $l^* \approx 2.321285$ is the unique real root of $2l^5 - 4l^4 - 5l^2 + 4l - 1 = 0$.

3. The average distance constant of a line segment is half its length, and the average distance of any triangle is greater than or equal to half the length of its longest side and the result for this triangle confirms that the average distance constant for an acute angled triangle which is almost a line segment is slightly greater than half the length of its longest side.
4. The computations for these triangles confirm the well known result that the average distance constant of an obtuse or right angled triangle is equal to half the length of its longest side.
5. As expected the average distance constant of this triangle which is close to an equilateral triangle with unit length sides is close to that of the latter triangle.

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