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SUPER-REFLEXIVE BANACH SPACES

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Introduction. A super-reflexive Banach space is defined to be a Banach space B which has the property that no non-reflexive Banach space is finitely representable in B. Super-reflexivity is invariant under isomorphisms; a Banach space B is super-reflexive if and only if B^* is super-reflexive. This concept has many equivalent formulations, some of which have been studied previously. For example, two necessary and sufficient conditions for super-reflexivity are: (i) There exist positive numbers $\delta < \frac{1}{3}$, A, and r such that $1 < r < \infty$ and $A[\sum |a_i|^r]^{1/r} \leqq ||\sum a_i e_i||$ for every normalized basic sequence $\{e_i\}$ with $\text{char}\{e_i\} \geqq \delta$ and all numbers $\{a_i\}$; (ii) There exist positive numbers $\delta < \frac{1}{2}$, B, and s such that $1 < s < \infty$ and $||\sum a_i e_i|| \leqq B[\sum |a_i|^r]^{1/r}$ for every normalized basic sequence $\{e_i\}$ with $\text{char}\{e_i\} \geqq \delta$ and all numbers $\{a_i\}$.

Definition 1. A normed linear space X being *finitely representable* in a normed linear space Y means that, for each finite-dimensional subspace X_n of X and each number $\lambda > 1$, there is an isomorphism T_n of X_n into Y for which

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \quad \text{if} \quad x \in X_n.$$

Definition 2. A normed linear space X being *crudely finitely representable* in a normed linear space Y means that there is a number $\lambda > 1$ such that, for each finite-dimensional subspace X_n of X, there is an isomorphism T_n of X_n into Y for which

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \quad \text{if} \quad x \in X_n.$$

Definition 3. A *super-reflexive* Banach space is a Banach space B which has the property that no non-reflexive Banach space is finitely representable in B.

It follows directly from known facts that a Banach space is super-reflexive if it is isomorphic to a Banach space that is uniformly non-square [**3**, Lemma C]. Clearly, all super-reflexive spaces are reflexive. The next theorem will enable us to prove easily that super-reflexivity is isomorphically invariant.

THEOREM 1. *A Banach space B is super-reflexive if and only if no non-reflexive Banach space is crudely finitely representable in B.*

Proof. Clearly, a Banach space B is super-reflexive if no non-reflexive Banach space is crudely finitely representable in B. We must show that if a non-reflexive space X is crudely finitely representable in B, then there is a

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non-reflexive space Y that is finitely representable in B. Since X is non-reflexive, there is an $\epsilon > 0$ and a sequence $\{x_n\}$ in the unit ball of X such that

$$\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_k\}, \operatorname{conv}\{x_{k+1}, \ldots\}) > \epsilon$$

for every $k \geqq 1$ [**2**, Theorem **7**, p. 114]. Let $\lambda > 1$ be a number such that, for each n, there is an isomorphism T_n of $\operatorname{lin}\{x_1, \ldots, x_n\}$ into B with

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \text{ if } x \in \operatorname{lin}\{x_1, \ldots, x_n\}.$$

Let $y_i^n = \lambda^{-1}T_n(x_i)$ for $i \leqq n$. Then $||y_i^n|| \leqq 1$ and, if $1 \leqq k < n$,

$$\operatorname{dist}(\operatorname{conv}\{y_1^n, \ldots, y_k^n\}, \operatorname{conv}\{y^n_{k+1}, \ldots, y_n^n\}) \geqq \lambda^{-2}\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_k\}, \operatorname{conv}\{x_{k+1}, \ldots, x_n\}),$$

so that

$$\operatorname{dist}(\operatorname{conv}\{y_1^n, \ldots, y_k^n\}, \operatorname{conv}\{y^n_{k+1}, \ldots, y_n^n\}) \geqq \lambda^{-2}\epsilon.$$

Now the procedure used in the proof of Lemma B in [**3**] gives a space Y that is finitely representable in B and is non-reflexive by virtue of having a sequence $\{\eta_n\}$ for which $||\eta_n|| \leqq 1$ and, for every $k \geqq 1$,

$$\operatorname{dist}(\operatorname{conv}\{\eta_1, \ldots, \eta_k\}, \operatorname{conv}\{\eta_{k+1}, \ldots\}) \geqq \lambda^{-2}\epsilon.$$

THEOREM 2. *Super-reflexivity is invariant under isomorphisms. A Banach space B is super-reflexive if and only if B^* is super-reflexive.*

Proof. It follows from Theorem 1 that super-reflexivity is invariant under isomorphisms. Now suppose that X is non-reflexive and finitely representable in B. Since X^* is non-reflexive, there is an $\epsilon > 0$ and a sequence of linear functionals $\{f_n\}$ in the unit ball of X^* for which

$$\operatorname{dist}(\operatorname{conv}\{f_1, \ldots, f_k\}, \operatorname{conv}\{f_{k+1}, \ldots\}) > \epsilon \quad \text{if} \quad k \geqq 1.$$

For a positive integer n and a finite-dimensional subspace X_p of X, let T map X_p into B as described in Definition 1. Define ϕ_k^n for $k \leqq n$ by letting $\phi_k^n[T(x)] = f_k(x)$ if $x \in X_p$, and then extending ϕ_k to all of B. If X_p is chosen suitably and λ is close enough to 1, then $||\phi_k|| < 2$ and

$$\operatorname{dist}(\operatorname{conv}\{\phi_1^n, \ldots, \phi_k^n\}, \operatorname{conv}\{\phi^n_{k+1}, \ldots, \phi_n^n\}) > \epsilon \tag{1}$$

if $1 \leqq k < n$. Again, the procedure of [**3**, Lemma B] gives a space Y that is finitely representable in B^* and is non-reflexive by virtue of containing a bounded sequence $\{\eta_n\}$ for which

$$\operatorname{dist}(\operatorname{conv}\{\eta_1, \ldots, \eta_k\}, \operatorname{conv}\{\eta_{k+1}, \ldots\}) \geqq \epsilon \quad \text{if} \quad k \geqq 1.$$

Conversely, suppose Y is non-reflexive and finitely representable in B^*. As in the proof of Theorem 1, it then follows that there is an $\epsilon > 0$ such that, for every positive integer n, there is a subset $\{\phi_1^n, \ldots, \phi_n^n\}$ of the unit ball of B^* for which (1) is satisfied. The procedure of [**3**, Lemma B] then gives a

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space X that is finitely representable in B and is non-reflexive by virtue of there being a bounded sequence of linear functionals $\{f_n\}$ in X^* for which

$$\text{dist}(\text{conv}\{f_1, \ldots, f_k\}, \text{conv}\{f_{k+1}, \ldots\}) \geqq \epsilon \quad \text{if} \quad k \geqq 1.$$

The next two lemmas are needed to develop some characterizations of reflexivity that will be useful in establishing characterizations of super-reflexivity. It is known that every non-reflexive Banach space has an infinite-dimensional subspace with a non-shrinking basis and an infinite-dimensional subspace with a basis that is not boundedly complete [**5**, p. 374; **6**, p. 362]. We shall need quantitative measures of how "good" these bases can be, as described by means of the characteristic of the basis. This is given by Lemmas 1 and 2. The proofs of Lemmas 1 and 2 are similar to the argument on pages 116–117 of [**2**], but these lemmas give more information. In fact, Lemma 2 is a combination of (31) and (35) in [**2**].

It is known that a sequence $\{x_i\}$ in a Banach space is a basis for its closed linear span if and only if there is a positive number ϵ such that

$$\left\|\sum_1^{n+p} a_i x_i\right\| \geqq \epsilon \left\|\sum_1^n a_i x_i\right\|$$

for all positive integers n and p and all numbers $\{a_i\}$. The largest such number ϵ is the *characteristic* of the basis.

The proofs of Lemmas 1 and 2 make repeated use of the following form of *Helly's condition*. "Given linear functionals $f_1, \ldots, f_n$ on a Banach space B and numbers $c_1, \ldots, c_n$ and M, the following two statements are equivalent.

(i) $\left|\sum_1^n a_i c_i\right| \leqq M\left\|\sum_1^n a_i f_i\right\|$ for all numbers $\{a_i\}$.

(ii) For every $\epsilon > 0$, there is an x in B such that $\|x\| < M + \epsilon$ and $f_i(x) = c_i$ if $1 \leqq i \leqq n$."

LEMMA 1. *Let B be a non-reflexive Banach space. If $0 < \theta < 1$ and $0 < \epsilon < 1$, then there are sequences $\{z_i\}$ and $\{g_i\}$ in the interiors of the unit balls of B and B^* such that*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j, \tag{2}$$

and, for all positive integers n and p and all numbers $\{a_i\}$,

$$\left\|\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})\right\| \geqq \tfrac{1}{3}\epsilon\left\|\sum_1^n a_i z_i\right\|. \tag{3}$$

Proof. Let θ and ϵ satisfy $0 < \theta < 1$ and $0 < \epsilon < 1$. Let F be a member of B^{**} for which $\|F\| < 1$ and

$$\text{dist}(F, B^c) > \max\{\theta, \epsilon^{\frac{1}{2}}\},$$

where B^c is the canonical image of B in B^{**}. We shall show that a sequence $\{(z_n, g_n, H_n)\}$ can be chosen inductively so that $z_n \in B$, $g_n \in B^*$, $\{H_n\}$ is an increasing sequence of finite sets of linear functionals with B as their domains, and:

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(a) $||z_n|| < 1$, $||g_n|| < 1$;
(b) $F(g_n) = \theta$ for all n;
(c) $g_i(z_j) = \theta$ if $i \leqq j$ and $g_i(z_j) = 0$ if $i > j$;
(d) $||h|| < 3\epsilon^{-\frac{1}{2}}$ and $F(h) = h(z_i)$ if $h \in H_n$ and $i \geqq n$;
(e) if $z \in \text{lin}\{z_1, \ldots, z_n\}$, then there is an h in H_n with $|h(z)| \geqq \epsilon^{\frac{1}{2}}||z||$.

Since $||F|| > \theta$, we can choose g_1 so that $||g_1|| < 1$ and $F(g_1) = \theta$. Then $||g_1|| > \theta$ and we can choose z_1 so that $g_1(z_1) = \theta$ and $||z_1|| < 1$. Let H_1 contain a single member chosen by the procedure described below for determining H_{p+1}. Suppose that (z_i, g_i, H_i) have been chosen to satisfy (a)-(e) when $i \leqq p$, where $p \geqq 1$. Then g_{p+1} must satisfy

$$||g_{p+1}|| < 1, \quad F(g_{p+1}) = \theta, \quad g_{p+1}(z_j) = z_j^c(g_{p+1}) = 0 \quad \text{if} \quad j \leqq p.$$

For the last two of these three conditions, Helly's condition (i) becomes

$$\theta \leqq M||\textstyle\sum_1^p a_i z_i^c + F|| \quad \text{for all } \{a_i\}.$$

Since this is satisfied if $M = \theta/\text{dist}(F, B^c) < 1$, g_{p+1} can be chosen to satisfy $||g_{p+1}|| < 1$. Now z_{p+1} must satisfy

$$||z_{p+1}|| < 1, \quad g_i(z_{p+1}) = \theta \quad \text{if} \quad i \leqq p+1, \quad h(z_{p+1}) = h(z_p) \quad \text{if} \quad h \in H_p.$$

For the last two of these three conditions, Helly's condition (i) becomes

$$|\theta \textstyle\sum_1^{p+1} a_i + h(z_p)| \leqq M||\textstyle\sum_1^{p+1} a_i g_i + h||$$

for all $\{a_i\}$ and all $h \in \text{lin}(H_p)$. Since

$$|\theta \textstyle\sum_1^{p+1} a_i + h(z_p)| = |F(\textstyle\sum_1^{p+1} a_i g_i + h)| \leqq ||F||\,||\textstyle\sum_1^{p+1} a_i g_i + h||$$

and $||F|| < 1$, we can let $M = ||F||$ and choose z_{p+1} so that $||z_{p+1}|| < 1$. Now let G_p be a finite set of linear functionals with unit norms and domains B which contains suitable linear functionals so that, for each z in $\text{lin}\{z_1, \ldots, z_{p+1}\}$, there is a g in G_p with $|g(z)| \geqq \epsilon^{\frac{1}{2}}||z||$. Let us now show that, for each g in G_p, there is an h in B^* such that

$$(4) \qquad ||h|| < 3\epsilon^{-\frac{1}{2}}, \quad F(h) = g(z_{p+1}), \quad z_i^c(h) = z_i^c(g) \quad \text{if} \quad i \leqq p+1.$$

For the last two of these conditions, Helly's condition (i) becomes

$$(5) \quad |a \cdot g(z_{p+1}) + \textstyle\sum_1^{p+1} a_i z_i^c(g)| \leqq M||aF + \textstyle\sum_1^{p+1} a_i z_i^c|| \quad \text{for all } \{a_i\} \text{ and } a.$$

Since

$$\begin{aligned} |a \cdot g(z_{p+1}) + \textstyle\sum_1^{p+1} a_i z_i^c(g)| &= |g(az_{p+1} + \textstyle\sum_1^{p+1} a_i z_i)| \leqq ||az_{p+1} + \textstyle\sum_1^{p+1} a_i z_i|| \\ &\leqq ||aF + \textstyle\sum_1^{p+1} a_i z_i^c|| + ||aF - az^c_{p+1}|| \\ &\leqq (1 + [||F - z^c_{p+1}||/||F + \textstyle\sum_1^{p+1} a_i z_i^c/a||]) \\ &\qquad \times ||aF + \textstyle\sum_1^{p+1} a_i z_i^c|| \\ &\leqq (1 + 2\epsilon^{-\frac{1}{2}})||aF + \textstyle\sum_1^{p+1} a_i z_i^c||, \end{aligned}$$

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we can satisfy (5) with $M = 1 + 2\epsilon^{-\frac{1}{2}}$ and choose h so that $||h|| < 3\epsilon^{-\frac{1}{2}}$. It follows from (4) that $h \equiv g$ on $\text{lin}\{z_1, \ldots, z_{p+1}\}$. Let each member of G_p be replaced in this way and then let H_{p+1} be the union of H_p and all such replacements of members of G_p. Clearly the sequence $\{(z_i, g_i)\}$ satisfies (2). It follows from (e) that, for any sum $\sum_1^n a_i z_i$, there is an h in H_n such that

$$|h(\textstyle\sum_1^n a_i z_i)| \geqq \epsilon^{\frac{1}{2}} ||\textstyle\sum_1^n a_i z_i||.$$

Since $||h|| < 3\epsilon^{-\frac{1}{2}}$ and $h(z_i - z_{i-1}) = 0$ if $i > n$, we have

$$\begin{aligned} ||\textstyle\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})|| &\geqq \tfrac{1}{3}\epsilon^{\frac{1}{2}} |h[\textstyle\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})]| \\ &= \tfrac{1}{3}\epsilon^{\frac{1}{2}} |h(\textstyle\sum_1^n a_i z_i)| \geqq \tfrac{1}{3}\epsilon ||\textstyle\sum_1^n a_i z_i||. \end{aligned}$$

LEMMA 2. *Let B be a non-reflexive Banach space. If $0 < \theta < 1$ and $0 < \epsilon < 1$, then there are sequences $\{z_i\}$ and $\{g_i\}$ in the interiors of the unit balls of B and B^* such that*

$$g_1(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j,$$

and, for all positive integers n and p and all numbers $\{a_i\}$,

$$(6) \qquad ||\textstyle\sum_1^{n+p} a_i z_i|| \geqq \tfrac{1}{2}\epsilon ||\textstyle\sum_1^n a_i z_i||.$$

Proof. Let θ and ϵ satisfy $0 < \theta < 1$ and $0 < \epsilon < 1$. Let F be a member of B^{**} for which $||F|| < 1$ and

$$\text{dist}(F, B^c) > \max\{\theta, \epsilon^{\frac{1}{2}}\},$$

where B^c is the canonical image of B in B^{**}. We shall show that a sequence $\{(z_n, g_n, H_n)\}$ can be chosen inductively so that $z_n \in B$, $g_n \in B^*$, $\{H_n\}$ is an increasing sequence of finite sets of linear functionals with B as their domains, and:

(a) $||z_n|| < 1$, $||g_n|| < 1$;
(b) $F(g_n) = \theta$ for all n;
(c) $g_i(z_j) = \theta$ if $i \leqq j$ and $g_i(z_j) = 0$ if $i > j$;
(d) $||h|| < 2\epsilon^{-\frac{1}{2}}$ and $F(h) = h(z_i) = 0$ if $h \in H_n$ and $i > n$;
(e) If $z \in \text{lin}\{z_1, \ldots, z_n\}$, then there is an h in H_n with $|h(z)| \geqq \epsilon^{\frac{1}{2}}||z||$.

Assuming that (z_i, g_i, H_i) have been chosen to satisfy (a)-(e) for $i \leqq p$, the choice of g_{p+1} is made exactly as in the proof of Lemma 1. Then z_{p+1} must satisfy

$$||z_{p+1}|| < 1, \quad g_i(z_{p+1}) = \theta \quad \text{if} \quad i \leqq p+1, \quad h(z_{p+1}) = 0 \quad \text{if} \quad h \in H_p.$$

For the last two of these conditions, Helly's condition (i) becomes

$$|\theta \textstyle\sum_1^{p+1} a_i| \leqq M||\textstyle\sum_1^{p+1} a_i g_i + h||$$

for all $\{a_i\}$ and all $h \in \text{lin}(H_p)$. Since

$$|\theta \textstyle\sum_1^{p+1} a_i| = |F(\textstyle\sum_1^{p+1} a_i g_i + h)| \leqq ||F||\,||\textstyle\sum_1^{p+1} a_i g_i + h||$$

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and $||F|| < 1$, we can let $M = ||F||$ and choose z_{p+1} so that $||z_{p+1}|| < 1$. The remaining argument is similar to that for Lemma 1, with (4) replaced by

$$||h|| < 2\epsilon^{\frac{1}{2}}, \quad F(h) = 0, \quad z_i^c(h) = z_i^c(g) \quad \text{if} \quad i \leqq p + 1,$$

and (5) replaced by

$$|\sum_1^{p+1} a_i z_i^c(g)| \leqq M||F + \sum_1^{p+1} a_i z_i^c||.$$

The coefficient $\frac{1}{2}$ in (6) is the best possible. To see this, suppose θ is a positive number and $\{x^n\}$ is a normalized basic sequence in c_0 for which there is a continuous linear functional g such that $g(x^n) \geqq \theta$ for every n. We shall show that $\operatorname{char}\{x^n\} \leqq \frac{1}{2}$. Let $\{y^n\}$ be a subsequence of $\{x^n\}$ for which

$$\lim_{n\to\infty} y^n(i) = \alpha_i$$

exists for each i. Then $|\alpha_i| \leqq 1$ for every i. Also $g(x^n) \geqq \theta$ for every n implies $\sup\{|\alpha_i|\} > 0$. For an arbitrary $\epsilon > 0$, let $\{z^n\}$ be a subsequence of $\{y^n\}$ such that, for every n,

$$|z^n(i) - \alpha_i| < \epsilon \quad \text{if} \quad i \leqq p(n) < p(n + 1),$$

where $p(n)$ is an integer for which $|z^k(i)| < \epsilon$ if $k < n$ and $i \geqq p(n)$. Then, for every k and r,

$$||\sum_{i=1}^k z^{r+i} - \omega|| < k\epsilon + 1,$$

where $\omega(i) = k\alpha_i$ if $1 \leqq i \leqq p(r+1)$, $\omega(i) = (k-j)\alpha(i)$ if $p(r+j) < i \leqq p(r+j+1)$, and $\omega(i) = 0$ if $i > p(r+k)$. Choose r such that $\sup\{|\alpha_i| : i \leqq p(r)\} > M - \epsilon$, where $M = \sup\{|\alpha_i|\}$. Then choose $s > k + r$. It follows that

$$||\sum_{i=1}^k z^{r+i} - \tfrac{1}{2}\sum_{i=1}^k z^{s+i}|| < \tfrac{1}{2}kM + 2(k\epsilon + 1),$$
$$||\sum_{i=1}^k z^{r+i}|| > k(M - \epsilon) - k\epsilon.$$

Thus, $\operatorname{char}\{x^n\} \leqq \operatorname{char}\{z^n\} < [\frac{1}{2}M + 2(\epsilon + 1/k)]/[M - 2\epsilon]$. Since k and ϵ were arbitrary, $\operatorname{char}\{x^n\} \leqq \frac{1}{2}$.

THEOREM 3. *Each of the following is a necessary and sufficient condition for a Banach space B to be non-reflexive. (Equivalent conditions are obtained if the introductory phrases for* (I), (II) *and* (III) *are replaced by "For some positive numbers θ and ϵ," or the introductory phrases for* (IV) *and* (V) *are replaced by "For some positive number θ".)*

(I) *For all θ and ϵ such that $0 < \theta < 1$ and $0 < \epsilon < 1$, there is a basic sequence $\{x_i\}$ in B such that $||x_i|| \geqq \theta$ for every i, $||\sum_1^k x_i|| < 1$ for every k, and* $\operatorname{char}\{e_i\} \geqq \frac{1}{3}\epsilon$.

(II) *For all θ and ϵ such that $0 < \theta < 1$ and $0 < \epsilon < 1$, there are sequences $\{z_n\}$ and $\{g_n\}$ in the unit balls of B and B^*, respectively, such that $\{z_i\}$ is a basic sequence with* $\operatorname{char}\{e_i\} \geqq \frac{1}{2}\epsilon$ *and*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j.$$

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(III) *For all* θ *and* ϵ *such that* $0 < \theta < 1$ *and* $0 < \epsilon < 1$, *there is a basic sequence* $\{z_n\}$ *in the unit ball of* B *such that* $\text{char}\{z_n\} \geqq \frac{1}{2}\epsilon$ *and*

$$||z|| \geqq \theta \quad \text{if} \quad z \in \text{conv}\{z_n\}.$$

(IV) *For all* θ *such that* $0 < \theta < 1$, *there is a sequence* $\{z_n\}$ *in the unit ball of* B *such that, for every sequence of numbers* $\{a_i\}$ *such that* $\sum_1^\infty a_i z_i$ *is convergent,*

$$\theta \cdot \sup\{|\textstyle\sum_k^\infty a_i| : k \leqq n\} \leqq ||\textstyle\sum_1^\infty a_i z_i||. \tag{7}$$

(V) *For all* θ *such that* $0 < \theta < 1$, *there is a sequence* $\{x_n\}$ *in* B *such that, for every sequence of numbers* $\{a_i\}$ *for which* $\sum_1^\infty a_i x_i$ *is convergent and* $a_i \to 0$,

$$\theta \cdot \sup\{|a_i|\} \leqq ||\textstyle\sum_1^\infty a_i x_i|| \leqq \textstyle\sum_1^\infty |a_i - a_{i+1}|. \tag{8}$$

Proof. Suppose first that B is not reflexive. Let $\{(z_i, g_i)\}$ be as described in Lemma 1. Let $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ if $i > 1$. Then, for every i, $g_i(x_i) = \theta$ and therefore $||x_i|| \geqq \theta$. Also, $\sum_1^k x_i = z_k$, so that $||\sum_1^k x_i|| < 1$ for every k. Inequality (3) is equivalent to $\text{char}\{x_i\} \geqq \frac{1}{3}\epsilon$. Thus (I) is satisfied. Clearly, (II) follows from Lemma 2 and (II) implies (III). Also, (II) implies (IV), since if $\{(z_i, g_i)\}$ are as described in (II), then

$$\theta \cdot \sup\{|\textstyle\sum_n^\infty a_i|\} = \sup\{|g_n(\sum_1^\infty a_i z_i)|\} \leqq ||\textstyle\sum_1^\infty a_i z_i||.$$

Let us now show that (IV) implies (V). To do this, let $\{z_n\}$ and θ be as described in (IV). Let $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ if $i > 1$. Then $\sum_1^\infty a_i x_i = \sum_1^\infty (a_i - a_{i+1}) z_i$, so that (7) and $||z_i|| \leqq 1$ imply (8).

To complete the proof, it is sufficient to show that B is non-reflexive if (I), (III) or (V) is satisfied (note that the following arguments use only the existence of positive numbers θ and ϵ as described in (I)-(V), rather than the possibility of using arbitrary θ and ϵ in the interval (0,1)). If (I) or (III) is satisfied, then a subspace of B has a basis that is not boundedly complete or is not shrinking, so that B is not reflexive [**1**, Theorem 3, p. 71]. Now suppose θ and $\{x_n\}$ are as described in (V). For each n, let

$$K_n = \text{cl}\{\textstyle\sum_1^p \alpha_i x_i : p \geqq n \quad \text{and} \quad 1 = \alpha_1 = \ldots = \alpha_n \geqq \alpha_{n+1} \geqq \ldots \geqq \alpha_p \geqq 0\}.$$

Then K_n is bounded, closed and convex, with $K_n \supset K_{n+1}$. Thus we can show B is non-reflexive by showing that $\bigcap K_n$ is empty [**1**, Theorem 1, p. 48]. Suppose $x \in \bigcap K_n$. Then there exist sequences $\{\alpha_i\}$ and $\{\beta_i\}$ that decrease monotonically to 0 for which

$$||x - \textstyle\sum_1^p \alpha_i x_i|| < \frac{1}{2}\theta, \qquad ||x - \textstyle\sum_1^q \beta_i x_i|| < \frac{1}{2}\theta,$$

and $\beta_i = 1$ if $i \leqq p + 1$. Then $||\sum_1^p \alpha_i x_i - \sum_1^q \beta_i x_i|| < \theta$, but from (8) we have

$$||\textstyle\sum_1^p \alpha_i x_i - \textstyle\sum_1^q \beta_i x_i|| \geqq \theta\beta_{p+1} = \theta.$$

There are many properties of Banach spaces whose equivalence to non-super-reflexivity follows easily from the definition of super-reflexivity, but

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which will not be discussed in this paper (see Lemmas B and C and Theorem 6 of [**3**]). The first five characterizations in the next theorem are closely related to (I)-(V) of Theorem 3. Characterizations (vi) and (viii) are known [**4**, Theorem 6], but are included here to show their relation to (vii).

THEOREM 4. *Each of the following is a necessary and sufficient condition for a Banach space B not to be super-reflexive. (Equivalent conditions are obtained if the introductory phrases for (i), (ii) and (iii) are replaced by "For some positive numbers θ and ϵ," or the introductory phrases for (iv) and (v) are replaced by "For some positive number θ".)*

(i) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there is a subset $\{x_1, \ldots, x_n\}$ of B such that $\|x_i\| \geqq \theta$ for every i, $\|\sum_1^k x_i\| < 1$ if $k \leqq n$, and, for every sequence of numbers $\{a_i\}$,*

$$\left\|\sum_1^n a_i x_i\right\| \geqq \tfrac{1}{3}\epsilon \left\|\sum_1^k a_i x_i\right\| \quad \text{if} \quad k \leqq n.$$

(ii) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there are subsets $\{z_1, \ldots, z_n\}$ and $\{g_1, \ldots, g_n\}$ of the unit balls of B and B^*, respectively, such that*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j,$$

and, for every sequence of numbers $\{a_i\}$ and every $k \leqq n$,

$$\left\|\sum_1^n a_i z_i\right\| \geqq \tfrac{1}{2}\epsilon \left\|\sum_1^k a_i z_i\right\|.$$

(iii) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there is a subset $\{z_1, \ldots, z_n\}$ of the unit ball of B such that $\|z\| > \theta$ if $z \in \operatorname{conv} \{z_1, \ldots, z_n\}$, and, for every sequence of numbers $\{a_i\}$ and every $k \leqq n$,*

$$\left\|\sum_1^n a_i z_i\right\| \geqq \tfrac{1}{2}\epsilon \left\|\sum_1^k a_i z_i\right\|.$$

(iv) *If $0 < \theta < 1$, then for every positive integer n there is a subset $\{y_1, \ldots, y_n\}$ of the unit ball of B such that, for every sequence of numbers $\{a_i\}$,*

$$\theta \cdot \sup\left\{\left|\sum_k^n a_i\right| : k \leqq n\right\} \leqq \left\|\sum_1^n a_i y_i\right\|.$$

(v) *If $0 < \theta < 1$, then for every positive integer n there is a subset $\{x_1, \ldots, x_n\}$ of B such that, for every sequence of numbers $\{a_i\}$ for which $a_{n+1} = 0$,*

$$\theta \cdot \sup\{|a_i| : 1 \leqq i \leqq n\} \leqq \left\|\sum_1^n a_i x_i\right\| \leqq \sum_1^n |a_i - a_{i+1}|.$$

(vi) *For every A, δ and B such that $0 < 2A < \delta \leqq 1 < B$, there exist numbers r and s for which $1 < r < \infty$, $1 < s < \infty$, and, if $\{e_i\}$ is any normalized basic sequence in B with $\operatorname{char}\{e_i\} \geqq \delta$, then*

$$A\left[\sum |a_i|^r\right]^{1/r} \leqq \left\|\sum a_i e_i\right\| \leqq B\left[\sum |a_i|^s\right]^{1/s},$$

for every sequence of numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

(vii) *There exist positive numbers δ, A and r such that $\delta < 1$, $1 < r < \infty$, and*

(9) $$A\left[\sum |a_i|^r\right]^{1/r} \leqq \left\|\sum a_i e_i\right\|,$$

for every normalized basic sequence $\{e_i\}$ *with* $\operatorname{char}\{e_i\} \geqq \frac{1}{3}\delta$ *and every sequence of numbers* $\{a_i\}$ *such that* $\sum a_i e_i$ *is convergent.*

(viii) *There exist positive numbers* δ, B *and* s *such that* $\delta < 1$, $1 < s < \infty$, *and*

$$|\sum a_i e_i| \leqq B[\sum |a_i|^s]^{1/s}, \tag{10}$$

for every normalized basic sequence $\{e_i\}$ *with* $\operatorname{char}\{e_i\} \geqq \frac{1}{2}\delta$ *and every sequence of numbers* $\{a_i\}$ *such that* $\sum a_i e_i$ *is convergent.*

Proof. Observe first that if a Banach space B is not super-reflexive, then there is a non-reflexive space X that is finitely representable in B. The fact that X has each of properties (I)-(V) of Theorem 3 implies that B has each of properties (i)-(v). The proof that each of (i)-(v) implies there is a non-reflexive space X that is finitely representable in B is essentially the same as a known process that will not be repeated here (see the proof of Lemma B in [**3**]). This completes the proof of (i)-(v). It is known that (vi) is implied by super-reflexivity [**4**, Theorem 4]. Clearly (vi) implies both (vii) and (viii). Let us suppose that B is not super-reflexive, but that (vii) is satisfied. For δ, A and r as described in (vii), choose ϵ and n so that $\delta < \epsilon < 1$ and

$$n^{1/r}\delta A > 1.$$

For this ϵ and for $\theta = \delta$, choose $\{x_1, \ldots, x_n\}$ as described in (i). Since $\{x_1, \ldots, x_n\}$ can be extended to a basic sequence with characteristic greater than $\frac{1}{3}\delta$, (9) gives the contradiction:

$$n^{1/r}\delta A \leqq A[\sum_1^n ||x_i||^r]^{1/r} \leqq ||\sum_1^n x_i|| < 1.$$

Similarly, if B is not super-reflexive, but (viii) is satisfied, choose ϵ and n so that $\delta < \epsilon < 1$ and

$$\theta n > Bn^{1/s}.$$

For this ϵ and for $\theta = \delta$, choose $\{z_1, \ldots, z_n\}$ as described in (iii). Since $\{z_1, \ldots, z_n\}$ can be extended to a basic sequence with characteristic greater than $\frac{1}{2}\delta$, (10) gives the contradiction

$$\theta n < ||\sum_1^n z_1|| \leqq B[\sum ||z_i||^s]^{1/s} \leqq Bn^{1/s}.$$

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