BIORTHOGONAL SYSTEMS IN l^p-SPACES

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1. Introduction. Our aim in this paper is to generalize certain ideas and results of Bary (**1**) on biorthogonal systems in separable Hilbert spaces to their counterparts in separable l^p-spaces, $1 < p$. The main result of Bary is to characterize a natural generalization of the concept of orthonormal basis for a Hilbert space. That of this paper is to characterize the concept of a Bary basis which is a generalization of the idea of standard basis of an l^p-space. The result is interesting for l^p-spaces because of the paucity of standard bases in these spaces.

Before summarizing our results, we shall introduce some notation and recall a few pertinent definitions and facts. The symbols $\mathfrak{X}$ and $\mathfrak{Y}$ denote mutually conjugate l^p-spaces, where $\mathfrak{X}$ is the space l^r and $\mathfrak{Y}$ the space l^s with $1 < r < 2$ and $2 < s = r/(r-1)$. The *standard bases*, $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, of the spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, consist of infinite sequences of complex numbers where for each particular $\mathbf{x}_i$ and $\mathbf{y}_i$ the corresponding sequence is determined by the rule: $\mathbf{x}_i = \mathbf{y}_i = \{\delta_{ij}\}$, $j = 1, 2, \ldots$. The value of the linear functional $\mathbf{y}$ (belonging to $\mathfrak{Y}$) at the point $\mathbf{x}$ (belonging to $\mathfrak{X}$) is denoted by the symbol $(\mathbf{x}, \mathbf{y})$. Two sequences, $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, are said to form a *biorthogonal system* if and only if $(\mathbf{x}_i, \mathbf{y}_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. Each member of a biorthogonal system is said to be *dual* or *adjoint* to the other member. A member of a biorthogonal system is called an *O-system* if and only if it is fundamental. All systems in this paper are O-systems unless the contrary is specified. The sequence $\{\alpha_i\}$, $i = 1, 2, \ldots$, is commonly denoted merely by $\{\alpha_i\}$. Limits are usually omitted from summation signs so that $\sum \alpha_i$ denotes $\sum_{i=1}^{\infty} \alpha_i$. When the range of summation is limited, $\sum_{i=n}^{m} \alpha_i$ is denoted by

$$\sum \alpha_i, \qquad i = n, \ldots, m.$$

This last convention is ignored when the context indicates the limits.

Bary (**1**) introduced three basic definitions for biorthogonal sequences, $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, each member of which is assumed to be fundamental in some separable Hilbert space $\mathfrak{H}$. The O-system $\{\mathbf{x}_i\}$ is a *Bessel system*, denoted by B-system, if and only if for every $\mathbf{z}$ belonging to $\mathfrak{H}$ the sequence $\{(\mathbf{z}, \mathbf{y}_i)\}$, $i = 1, 2, \ldots$, belongs to l^2. The O-system $\{\mathbf{x}_i\}$ is a *Hilbert system*, denoted by H-system, if and only if given any sequence $\{\gamma_i\}$ of l^2 there exists a *unique* element $\mathbf{z}$ belonging to $\mathfrak{H}$ such that $(\mathbf{z}, \mathbf{y}_i) = \gamma_i$, $i = 1, 2, \ldots$. The O-system $\{\mathbf{x}_i\}$ is a *Fischer-Riesz system*, denoted by F-R-system, if and only if it is both a B-system and an H-system.

Received December 21, 1967.

The present investigation extends these notions in a direct fashion to the cases of l^p-spaces. Let $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$ be a pair of biorthogonal sequences, each of which is fundamental, in the mutually conjugate spaces $\mathfrak{X}$ $(=l^r)$ and $\mathfrak{Y}$ $(=l^s)$, respectively. The O-system $\{\mathbf{x}_i\}$ of $\mathfrak{X}$ is a *Bessel system* if and only if for each $\mathbf{z}$ belonging to $\mathfrak{X}$ the sequence $\{(\mathbf{z}, \mathbf{y}_i)\}$, $i = 1, 2, \ldots$, belongs to l^r. The O-system $\{\mathbf{x}_i\}$ of $\mathfrak{X}$ is a *Hilbert system* if and only if given any sequence $\{\gamma_i\}$ of l^r there exists a unique element $\mathbf{z}$ belonging to $\mathfrak{X}$ such that $(\mathbf{z}, \mathbf{y}_i) = \gamma_i$, $i = 1, 2, \ldots$. The O-system $\{\mathbf{x}_i\}$ is a *Fischer-Riesz system* if and only if it is both a B-system and an H-system. The analogous definitions hold, of course, for the O-system $\{\mathbf{y}_i\}$.

It is well known that every basis $\{\mathbf{x}_i\}$ of the space $\mathfrak{X}$ is an O-system with a dual basis $\{\mathbf{y}_i\}$ in the conjugate space $\mathfrak{Y}$. A basis $\{\mathbf{x}_i\}$ of $\mathfrak{X}$ is called a *Bary basis* if and only if there exist positive constants m and M such that if $\mathbf{z}$ of $\mathfrak{X}$ has the expansion

$$\mathbf{z} = \zeta_1\mathbf{x}_1 + \ldots + \zeta_j\mathbf{x}_j + \ldots,$$

then

$$m||\mathbf{z}|| \leqq (\sum|\zeta_i|^r)^{1/r} \leqq M||\mathbf{z}||.$$

A Bary basis of $\mathfrak{X}$ is commonly denoted by the use of a symbol such as $\{\mathbf{x}_i\}(m, M)$, although the specification of the bounds by (m, M) is sometimes omitted. The principal result of this paper is that an O-system $\{\mathbf{x}_i\}$ is an F-R-system in $\mathfrak{X}$ if and only if it is a Bary basis of $\mathfrak{X}$.

2. Preliminary theorems. This section is devoted to the proofs of several theorems which are useful in establishing the main result. Some of these have independent interest in developing the concept of a Bary basis which has not previously appeared in the literature.

THEOREM 2.1. *Let $\{\mathbf{e}_i\}$ be a fundamental sequence in $\mathfrak{X}$. Then there exists a Bary basis $\{\mathbf{u}_i\}$, each of whose elements is a finite linear combination of elements belonging to the fundamental sequence $\{\mathbf{e}_i\}$.*

Proof. Let $\{\mathbf{x}_i\}$ be the standard basis of $\mathfrak{X}$. Given a real number δ, $0 < \delta < 1$, there exists a finite linear combination $\mathbf{u}_j$ of elements of $\{\mathbf{e}_i\}$ such that

$$||\mathbf{x}_j - \mathbf{u}_j|| < \delta(\tfrac{1}{2})^{j/s}, \qquad j = 1, 2, \ldots.$$

For any sequence $\{\xi_i\}$ belonging to l^r, the series

$$\mathbf{s} = \xi_1(\mathbf{x}_1 - \mathbf{u}_1) + \ldots + \xi_j(\mathbf{x}_j - \mathbf{u}_j) + \ldots$$

converges. To see this, note that

$$||\sum \xi_i(\mathbf{x}_i - \mathbf{u}_i)|| \leqq \sum |\xi_i| \, ||\mathbf{x}_i - \mathbf{u}_i||$$
$$\leqq (\sum |\xi_i|^r)^{1/r}(\sum ||\mathbf{x}_i - \mathbf{u}_i||^s)^{1/s} \leqq \delta(\sum |\xi_i|^r)^{1/r}, \quad i = n, \ldots, m,$$

where this final expression tends to zero as n increases without limit. Furthermore, one obtains:

$$(2.1) \qquad ||\sum \xi_i(\mathbf{x}_i - \mathbf{u}_i)|| \leqq \delta||\mathbf{x}||, \qquad x = \sum \xi_i\mathbf{x}_i.$$

https://doi.org/10.4153/CJM-1969-071-6 Published online by Cambridge University Press

An element $\mathbf{w}$ belonging to $\mathfrak{X}$ has the unique expansion,

$$\mathbf{w} = \omega_1\mathbf{x}_1 + \ldots + \omega_j\mathbf{x}_j + \ldots,$$

with $\sum |\omega_i|^r$ finite. Let T be the mapping from $\mathfrak{X}$ to $\mathfrak{X}$ defined by

$$T\mathbf{w} = \omega_1(\mathbf{x}_1 - \mathbf{u}_1) + \ldots + \omega_j(\mathbf{x}_j - \mathbf{u}_j) + \ldots.$$

It follows from the definition that T is linear and from (2.1) that T is bounded by δ, with $0 < \delta < 1$. Let R denote the bounded linear transformation $I - T$, where I is the identity transformation on $\mathfrak{X}$. Then R has a bounded inverse S given by the convergent series,

$$I + T + \ldots + T^j + \ldots.$$

Given any $\mathbf{w}$ belonging to $\mathfrak{X}$, the element $S\mathbf{w}$ has the unique expansion,

$$S\mathbf{w} = \tau_1\mathbf{x}_1 + \ldots + \tau_j\mathbf{x}_j + \ldots.$$

Consequently, $\mathbf{w}$ itself has the unique expansion,

$$\mathbf{w} = R(S\mathbf{w}) = \tau_1\mathbf{u}_1 + \ldots + \tau_j\mathbf{u}_j + \ldots,$$

so that $\{\mathbf{u}_i\}$ is a Schauder basis of $\mathfrak{X}$. In addition,

$$||\mathbf{w}|| \leqq ||R||\ ||S\mathbf{w}|| \leqq ||R||\ ||S||\ ||\mathbf{w}||,$$

which implies that

$$(1/||R||)||\mathbf{w}|| \leqq (\sum |\tau_i|^r)^{1/r} \leqq ||S||\ ||\mathbf{w}||.$$

This demonstrates that $\{\mathbf{u}_i\}$ is a Bary basis of $\mathfrak{X}$.

THEOREM 2.2. *The basis $\{\mathbf{u}_i\}$ of $\mathfrak{X}$ is a Bary basis if and only if the dual basis $\{\mathbf{v}_i\}$ of $\mathfrak{Y}$ is a Bary basis.*

Proof. Let $\{\mathbf{u}_i\}(m, M)$ be a Bary basis of $\mathfrak{X}$ and let $\{\mathbf{v}_i\}$ be the dual basis of $\mathfrak{Y}$. Let the elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, have the expansions

(2.2) $\mathbf{x} = \xi_1\mathbf{u}_1 + \ldots + \xi_j\mathbf{u}_j + \ldots$ and $\mathbf{y} = \eta_1\mathbf{v}_1 + \ldots + \eta_j\mathbf{v}_j + \ldots.$

It follows that

(2.3) $$|\sum \xi_i\eta_i| = |(\mathbf{x}, \mathbf{y})| \leqq ||\mathbf{x}||\ ||\mathbf{y}|| \leqq (1/m)(\sum |\xi_i|^r)^{1/r}||\mathbf{y}||.$$

Relation (2.3) holds for each $\{\xi_i\} \in l^r$ which implies that the sequence $\{\eta_i\}$ is an element of l^s. Consequently,

$$||\{\eta_i\}||_s = (\sum |\eta_i|^s)^{1/s} \leqq (1/m)||\mathbf{y}||.$$

On the other hand,

$$|(\mathbf{x}, \mathbf{y})| = |\sum \xi_i\eta_i| \leqq (\sum |\xi_i|^r)^{1/r}(\sum |\eta_i|^s)^{1/s} \leqq M||\mathbf{x}||(\sum |\eta_i|^s)^{1/s}.$$

Thus, we have:

$$(1/M)||\mathbf{y}|| \leqq (\sum |\eta_i|^s)^{1/s} \leqq (1/m)||\mathbf{y}||,$$

so that $\{\mathbf{v}_i\}$ is a Bary basis $\{\mathbf{v}_i\}(1/M, 1/m)$. The converse is the same.

Recall that a matrix A with elements $\{\alpha_{ij}\}(i, j = 1, 2, \ldots)$ is said to be *bounded* or have the bound $|A|$ in $[r, s]$ if and only if

$$\begin{aligned} |A_n(\mathbf{x}, \mathbf{y})| &= |\sum \alpha_{ij}\xi_i\eta_j| \\ &\leqq |A|(\sum |\xi_i|^r)^{1/r}(\sum |\eta_j|^s)^{1/s}, \qquad i, j = 1, \ldots, n, \\ &\leqq |A| \, ||\mathbf{x}||_r ||\mathbf{y}||_s \end{aligned}$$

for $\mathbf{x} = \{\xi_i\}$ and $\mathbf{y} = \{\eta_i\}$ in l^r and l^s, respectively. When A is bounded, the double and iterated forms of the above sum converge to the same limit. Furthermore, the sequence $\{\zeta_i\}$, whose elements are defined by

$$\zeta_j = \sum \alpha_{ij}\xi_i, \qquad j = 1, 2, \ldots, \tag{2.4}$$

is an element of l^r whenever $\{\xi_i\}$ is an element of l^r. Similar results hold for elements of l^s.

THEOREM 2.3. *Let $\{\mathbf{u}_i\}(m, M)$ and $\{\mathbf{v}_i\}(1/M, 1/m)$ be biorthogonal Bary bases for the spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let A be the matrix $\{\alpha_{ij}\}$, defined by*

$$\alpha_{ij} = (T\mathbf{u}_i, \mathbf{v}_j), \qquad i, j = 1, 2, \ldots, \tag{2.5}$$

where T is a bounded linear transformation on $\mathfrak{X}$. Then A is a bounded matrix in $[r, s]$. Conversely, if A is a bounded matrix in $[r, s]$, there exists a bounded linear transformation T on $\mathfrak{X}$ whose matrix with respect to the basis $\{\mathbf{u}_i\}$ is given by (2.5).

Proof. Let T be a bounded linear transformation on $\mathfrak{X}$. Let $\{\xi_i\}$ and $\{\eta_i\}$ be two sequences of complex numbers which belong to l^r and l^s, respectively. Then there exist elements,

$$\mathbf{x} = \xi_1\mathbf{u}_1 + \ldots + \xi_j\mathbf{u}_j + \ldots \quad \text{and} \quad \mathbf{y} = \eta_1\mathbf{v}_1 + \ldots + \eta_j\mathbf{v}_j + \ldots,$$

which belong to $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let $\mathbf{x}^n$ and $\mathbf{y}^n$ be the elements defined by

$$\mathbf{x}^n = \xi_1\mathbf{u}_1 + \ldots + \xi_n\mathbf{u}_n \quad \text{and} \quad \mathbf{y}^n = \eta_1\mathbf{v}_1 + \ldots + \eta_n\mathbf{v}_n,$$

$n = 1, 2, \ldots$. Then one has

$$\begin{aligned} |A_n(\{\xi_i\}, \{\eta_i\})| &= |\sum \alpha_{ij}\xi_i\eta_j|, \qquad i, j = 1, \ldots, n, \\ &= |(T\mathbf{x}^n, \mathbf{y}^n)| \\ &\leqq ||T|| \, ||\mathbf{x}^n|| \, ||\mathbf{y}^n|| \\ &\leqq ||T||(1/m)(\sum \xi_i^r)^{1/r}M(\sum \eta_i^s)^{1/s} \\ &\leqq ||T||(M/m)||\{\xi_i\}||_r||\{\eta_i\}||_s. \end{aligned}$$

Conversely, let $\{\alpha_{ij}\}$ $(i, j = 1, 2, \ldots)$ be a bounded matrix in $[r, s]$. If $\{\xi_i\}$ belongs to l^r, then the sequence $\{\zeta_j\}$ whose elements are defined by (2.4) is also an element of l^r. Let T be the mapping with domain and range $\mathfrak{X}$ such that if $\mathbf{x}$ of $\mathfrak{X}$ has the expansion (2.2), then the image $T\mathbf{x}$ is given by the series,

$$\zeta_1\mathbf{u}_1 + \ldots + \zeta_j\mathbf{u}_j + \ldots = (\sum \alpha_{k1}\xi_k)\mathbf{u}_1 + \ldots + (\sum \alpha_{kj}\xi_k)\mathbf{u}_j + \ldots.$$

T is a linear transformation on $\mathfrak{X}$. Furthermore,

$$||T\mathbf{x}|| \leqq (1/m)(\sum |\zeta_i|^r)^{1/r} \leqq (1/m)|A|(\sum |\xi_i|^r)^{1/r} \leqq (M/m)|A|\ ||\mathbf{x}||.$$

Thus, T is a bounded linear transformation on $\mathfrak{X}$. In addition,

$$T\mathbf{u}_i = \alpha_{i1}\mathbf{u}_1 + \ldots + \alpha_{ij}\mathbf{u}_j + \ldots,$$

so that

$$(T\mathbf{u}_i, \mathbf{v}_j) = \alpha_{ij},$$

as was to be shown.

We note the following useful fact. Let $\{\mathbf{e}_i\}$ and $\{\mathbf{g}_i\}$ be biorthogonal systems lying in the mutually conjugate l^p-spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let $\{\mathbf{u}_i\}$ and $\{\mathbf{v}_i\}$ be another such biorthogonal pair. Let T be a bounded linear transformation on $\mathfrak{X}$ such that

$$T\mathbf{u}_i = \mathbf{e}_i, \qquad i = 1, 2, \ldots.$$

Then the adjoint T^* of T is a bounded linear transformation on $\mathfrak{Y}$ such that

$$T^*\mathbf{g}_i = \mathbf{v}_i, \qquad i = 1, 2, \ldots.$$

3. Principal theorems. This section contains certain results on B-systems and H-systems and the theorem that every Fischer-Riesz system in $\mathfrak{X}$ is a Bary basis of $\mathfrak{X}$.

THEOREM 3.1. *Let $\{\mathbf{e}_i\}$ and $\{\mathbf{g}_i\}$ be O-systems in the mutually conjugate l^p-spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. In order that $\{\mathbf{e}_i\}$ be a* B*-system, it is necessary and sufficient that there exist a bounded linear transformation T on $\mathfrak{X}$ such that*

$$T\mathbf{e}_i = \mathbf{u}_i, \qquad i = 1, 2, \ldots,$$

where $\{\mathbf{u}_i\}(m, M)$ is a Bary basis of $\mathfrak{X}$.

Proof. Let $\{\mathbf{e}_i\}$ be a B-system in $\mathfrak{X}$ and $\{\mathbf{u}_i\}(m, M)$ a Bary basis of $\mathfrak{X}$. If $\mathbf{z}$ is any element of $\mathfrak{X}$, then the sequence $\{(\mathbf{z}, \mathbf{g}_i)\}$ is an element of l^r. It follows that there exists a uniquely defined element $\mathbf{Z}$ of $\mathfrak{X}$ such that

$$\mathbf{Z} = (\mathbf{z}, \mathbf{g}_1)\mathbf{u}_1 + \ldots + (\mathbf{z}, \mathbf{g}_j)\mathbf{u}_j + \ldots.$$

Let T be the mapping from $\mathfrak{X}$ to $\mathfrak{X}$ defined by $T\mathbf{z} = \mathbf{Z}$. It is easy to see that T is a linear transformation with domain $\mathfrak{X}$. We show that T is closed. Let the sequence $\{\mathbf{z}_n\}$ of elements of $\mathfrak{X}$ converge to the element $\mathbf{z}$ and let the set $\{\mathbf{Z}_n\}$ of images $\{T\mathbf{z}_n\}$ converge to $\mathbf{Z}$. Let $\{\mathbf{v}_i\}(1/M, 1/m)$ denote the sequence in $\mathfrak{Y}$ biorthogonal to $\{u_i\}$. Then for each integer i, one has

$$(\mathbf{Z}_n, \mathbf{v}_i) = (\mathbf{z}_n, \mathbf{g}_i), \qquad n = 1, 2, \ldots.$$

It follows by continuity of the inner product that

$$(\mathbf{Z}, \mathbf{v}_i) = (\mathbf{z}, \mathbf{g}_i), \qquad i = 1, 2, \ldots,$$

and, consequently, that

$$\mathbf{Z} = (\mathbf{z}, \mathbf{g}_1)\mathbf{u}_1 + \ldots + (\mathbf{z}, \mathbf{g}_j)\mathbf{u}_j + \ldots.$$

Thus, one finds $T\mathbf{z} = \mathbf{Z}$, so that T is a closed linear transformation with domain $\mathfrak{X}$; hence, T is bounded. From the definition of T, we have:

$$(\mathbf{z}, \mathbf{g}_i) = (T\mathbf{z}, \mathbf{v}_i), \qquad i = 1, 2, \ldots,$$

for all $\mathbf{z}$. In particular,

$$\delta_{ij} = (\mathbf{e}_j, \mathbf{g}_i) = (T\mathbf{e}_j, \mathbf{v}_i),$$

which implies by uniqueness of the adjoint system of $\{\mathbf{v}_i\}$ that $T\mathbf{e}_j = \mathbf{u}_j$, as was to be shown.

Conversely, let $\{\mathbf{e}_i\}$ be an O-system in $\mathfrak{X}$ and suppose that there exists a bounded linear transformation T on $\mathfrak{X}$ such that

$$T\mathbf{e}_i = \mathbf{u}_i, \qquad i = 1, 2, \ldots,$$

where $\{\mathbf{u}_i\}(m, M)$ is a Bary basis of $\mathfrak{X}$ with dual basis $\{\mathbf{v}_i\}(1/M, 1/m)$ in $\mathfrak{Y}$. For any $\mathbf{x}$ in $\mathfrak{X}$, one has

$$(\mathbf{x}, \mathbf{g}_i) = (\mathbf{x}, T^*\mathbf{v}_i) = (T\mathbf{x}, \mathbf{v}_i), \qquad i = 1, 2, \ldots.$$

Thus, the expansion coefficients of $\mathbf{x}$ with respect to the O-system $\{\mathbf{e}_i\}$ coincide with the expansion coefficients of $T\mathbf{x}$ with respect to the Bary basis $\{\mathbf{u}_j\}$. It follows that

$$(\sum |(\mathbf{x}, \mathbf{g}_i)|^r)^{1/r} \leqq M||T\mathbf{x}|| \leqq M||T||\,||\mathbf{x}||.$$

Consequently, the O-system $\{\mathbf{e}_i\}$ is a B-system. Furthermore, one discovers the existence of a constant K such that

$$(\sum |(\mathbf{x}, \mathbf{g}_i)|^r)^{1/r} \leqq K||\mathbf{x}||$$

whenever the O-system $\{\mathbf{e}_i\}$ is a B-system with the adjoint system $\{\mathbf{g}_i\}$.

Corollary 3.2. *Let $\{\mathbf{e}_i\}$ be an O-system in $\mathfrak{X}$ with $\{\mathbf{g}_i\}$ its dual in $\mathfrak{Y}$. Let A be the matrix whose elements are the coefficients $(\mathbf{u}_i, \mathbf{g}_j)$ of the expansions,*

$$\mathbf{g}_j = (\mathbf{u}_1, \mathbf{g}_j)\mathbf{v}_1 + \ldots + (\mathbf{u}_k, \mathbf{g}_j)\mathbf{v}_k + \ldots,$$

of the members of the adjoint system $\{\mathbf{g}_i\}$ with respect to a Bary basis $\{\mathbf{v}_i\}$ of $\mathfrak{Y}$. In order that the O-system $\{\mathbf{e}_i\}$ be a B*-system, it is necessary and sufficient that A be bounded in $[r, s]$ for a suitable choice of the Bary basis $\{\mathbf{v}_i\}$.*

Proof. Let $\{\mathbf{e}_i\}$ be a Bessel system in $\mathfrak{X}$; then there exists a bounded linear transformation T on $\mathfrak{X}$ such that

$$T\mathbf{e}_i = \mathbf{u}_i, \qquad i = 1, 2, \ldots,$$

where $\{\mathbf{u}_i\}(m, M)$ is a Bary basis of $\mathfrak{X}$. Let $\{\mathbf{v}_i\}(1/M, 1/m)$ be the Bary basis of $\mathfrak{Y}$ which is biorthogonal to $\{\mathbf{u}_i\}$. We note that T^* is a bounded linear transformation on $\mathfrak{Y}$ such that

$$T^*\mathbf{v}_j = \mathbf{g}_j, \qquad j = 1, 2, \ldots .$$

The expansion of $\mathbf{g}_i$ with respect to the Bary basis $\{\mathbf{v}_i\}$ is

$$\begin{aligned} \mathbf{g}_i &= (\mathbf{u}_1, \mathbf{g}_i)\mathbf{v}_1 + \ldots + (\mathbf{u}_j, \mathbf{g}_i)\mathbf{v}_j + \ldots \\ &= (\mathbf{u}_1, T^*\mathbf{v}_i)\mathbf{v}_1 + \ldots + (\mathbf{u}_j, T^*\mathbf{v}_i)\mathbf{v}_j + \ldots \\ &= (T\mathbf{u}_1, \mathbf{v}_i)\mathbf{v}_1 + \ldots + (T\mathbf{u}_j, \mathbf{v}_i)\mathbf{v}_j + \ldots . \end{aligned}$$

Thus, the expansion coefficients of the sequence of elements $\{\mathbf{g}_i\}$ with respect to the basis $\{\mathbf{v}_i\}$ are the elements of the matrix A of T with respect to the basis $\{\mathbf{u}_i\}$. It follows by our previous results that A is bounded in $[r, s]$.

Conversely, suppose that the matrix $\{\alpha_{ji}\}$ $(j, i = 1, 2, \ldots)$ is bounded in $[r, s]$. When $\{\xi_i\}$ is an element of l^r, the series $\sum \alpha_{ji}\xi_j$ converges to a limit η_i, where the sequence $\{\eta_i\}$ is an element of l^r. Each element $\mathbf{x}$ of $\mathfrak{X}$ has the unique expansion,

$$\mathbf{x} = \xi_1\mathbf{u}_1 + \ldots + \xi_k\mathbf{u}_k + \ldots ,$$

while each element $\mathbf{g}_i$ of the adjoint system has the unique expansion,

$$\mathbf{g}_i = \alpha_{1i}\mathbf{v}_1 + \ldots + \alpha_{ji}\mathbf{v}_j + \ldots ,$$

so that

$$(\mathbf{x}, \mathbf{g}_i) = \sum \alpha_{ji}\xi_j = \eta_i, \qquad i = 1, 2, \ldots ,$$

where the sequence $\{\eta_i\}$ belongs to l^r. It follows that the O-system $\{\mathbf{e}_i\}$ is a B-system.

THEOREM 3.3. *Let $\{\mathbf{e}_i\}$ be an O-system in $\mathfrak{X}$ with adjoint system $\{\mathbf{g}_i\}$ in Y. In order that $\{\mathbf{e}_i\}$ be an* H-*system, it is necessary and sufficient that there exist a bounded linear transformation T on $\mathfrak{X}$ such that*

$$T\mathbf{u}_j = \mathbf{e}_j, \qquad j = 1, 2, \ldots ,$$

where $\{\mathbf{u}_i\}(m, M)$ is a Bary basis of $\mathfrak{X}$.

Proof. Let $\{\mathbf{e}_i\}$ be an H-system and let $\{\mathbf{u}_i\}(m, M)$ be a Bary basis of $\mathfrak{X}$. Let $\mathbf{z}$ belong to $\mathfrak{X}$, then

$$\mathbf{z} = \zeta_1\mathbf{u}_1 + \ldots + \zeta_k\mathbf{u}_k + \ldots ,$$

where $(\sum |\zeta_i|^r)^{1/r}$ is finite. There exists a unique $\mathbf{Z}$ belonging to $\mathfrak{X}$ such that

$$(\mathbf{Z}, \mathbf{g}_i) = \zeta_i = (\mathbf{z}, \mathbf{v}_i), \qquad i = 1, 2, \ldots ,$$

where $\{\mathbf{v}_i\}(1/M, 1/m)$ is the Bary basis of $\mathfrak{Y}$ biorthogonal to $\{\mathbf{u}_i\}$. We define a mapping T on $\mathfrak{X}$ by $T\mathbf{z} = \mathbf{Z}$. The mapping T is a linear transformation with domain $\mathfrak{X}$. We show that T is closed. Let $\mathbf{z}_n$ tend to $\mathbf{z}$ and $\mathbf{Z}_n$ tend to $\mathbf{Z}$, where

$$(\mathbf{Z}_n, \mathbf{g}_i) = (\mathbf{z}_n, \mathbf{v}_i), \qquad i = 1, 2, \ldots .$$

It follows that

$$(\mathbf{Z}, \mathfrak{g}_i) = (\mathbf{z}, \mathbf{v}_i), \qquad i = 1, 2, \ldots,$$

so that T is a closed linear transformation, and hence is bounded.

Furthermore,

$$(\mathbf{e}_j, \mathfrak{g}_i) = \delta_{ji} = (\mathbf{u}_j, \mathbf{v}_i), \qquad i, j = 1, 2, \ldots,$$

which implies that

$$T\mathbf{u}_j = \mathbf{e}_j, \qquad j = 1, 2, \ldots. \tag{3.1}$$

Conversely, let T be a bounded linear transformation on $\mathfrak{X}$ such that (3.1) holds for some Bary basis $\{\mathbf{u}_i\}$ (m, M) of $\mathfrak{X}$. Given any sequence $\{\zeta_i\}$ of complex numbers belonging to l^r, there exists a unique element $\mathbf{z}$ of $\mathfrak{X}$ such that

$$\mathbf{z} = \zeta_1\mathbf{u}_1 + \ldots + \zeta_k\mathbf{u}_k + \ldots.$$

Let $\mathbf{x}$ be the image of $\mathbf{z}$ under T, that is,

$$\mathbf{x} = \zeta_1\mathbf{e}_1 + \ldots + \zeta_k\mathbf{e}_k + \ldots,$$

which implies that

$$(\mathbf{x}, \mathfrak{g}_i) = \zeta_i, \qquad i = 1, 2, \ldots,$$

which shows that $\{\mathbf{e}_i\}$ is an H-system. In addition,

$$||\mathbf{x}|| = ||T\mathbf{z}|| \leqq ||T||\,||\mathbf{z}|| \leqq ||T||(1/m)(\sum |\zeta_i|^r)^{1/r}.$$

COROLLARY 3.4. *Let $\{\mathbf{e}_i\}$ be an O-system in $\mathfrak{X}$. In order that $\{\mathbf{e}_i\}$ be an* H-*system, it is necessary and sufficient that the matrix A, whose elements are the coefficients of the expansion,*

$$\mathbf{e}_i = \alpha_{1i}\mathbf{u}_1 + \ldots + \alpha_{ji}\mathbf{u}_j + \ldots, \qquad i = 1, 2, \ldots,$$

with respect to some Bary basis $\{\mathbf{u}_i\}$ (m, M) of $\mathfrak{X}$, be bounded in $[r, s]$.

The above Corollary is equivalent to the fact that in order that $\{\mathbf{e}_i\}$ be an H-system in $\mathfrak{X}$, it is necessary and sufficient that the matrix A, whose elements are $\{(\mathbf{e}_i, \mathbf{v}_j)\}$ $(i, j = 1, 2, \ldots)$, be bounded in $[r, s]$ for every Bary basis $\{\mathbf{v}_i\}$ of the conjugate space $\mathfrak{Y}$. This last result has an interesting interpretation in the case of l^2, where $\mathfrak{X}$ and $\mathfrak{Y}$ can be identified. One notes that the matrix A, whose elements are given by

$$\alpha_{ij} = (\mathbf{e}_i, \mathbf{v}_j), \qquad i, j = 1, 2, \ldots,$$

$\{\mathbf{v}_k\}$ an orthonormal basis of the space, is bounded if and only if the matrix A^* is bounded, where the elements of A^* are given by

$$\gamma_{ij} = (\mathbf{v}_i, \mathbf{e}_j) = \bar{\alpha}_{ji}; \qquad i, j = 1, 2, \ldots.$$

However, A and A^* are bounded if and only if AA^* is a bounded matrix B, where the elements of B are given by

$$\beta_{ij} = \sum (\mathbf{e}_i, \mathbf{v}_k)(\mathbf{v}_k, \mathbf{e}_j) = (\mathbf{e}_i, \mathbf{e}_j).$$

The matrix B is the Gram matrix of the O-system $\{\mathbf{e}_i\}$. Thus, the system is an H-system if and only if its Gram matrix is bounded in [2, 2].

THEOREM 3.5. *Let* $\{\mathbf{e}_i\}$ *and* $\{\mathbf{g}_i\}$ *be O-systems belonging to the mutually conjugate spaces* $\mathfrak{X}$ *and* $\mathfrak{Y}$, *respectively. Then* $\{\mathbf{e}_i\}$ *is a* B-*system in* $\mathfrak{X}$ *if and only if* $\{\mathbf{g}_i\}$ *is an* H-*system in* $\mathfrak{Y}$.

Proof. Let $\{\mathbf{e}_i\}$ be a B-system in $\mathfrak{X}$. Then there exists a Bary basis $\{\mathbf{u}_i\}$ (m, M) and a bounded linear transformation T on $\mathfrak{X}$ such that

$$T\mathbf{e}_j = \mathbf{u}_j, \qquad j = 1, 2, \ldots .$$

Let $\{\mathbf{v}_i\}$ $(1/M, 1/m)$ be the Bary basis of $\mathfrak{Y}$ biorthogonal to $\{\mathbf{u}_i\}$ and let T^* be the bounded linear transformation on $\mathfrak{Y}$ adjoint to T. Then one has

$$\delta_{ij} = (\mathbf{u}_i, \mathbf{v}_j) = (T\mathbf{e}_i, \mathbf{v}_j) = (\mathbf{e}_i, T^*\mathbf{v}_j), \qquad i, j = 1, 2, \ldots,$$

which implies that

$$T^*\mathbf{v}_j = \mathbf{g}_j, \qquad j = 1, 2, \ldots .$$

Thus, $\{\mathbf{g}_j\}$ is an H-system. The converse is similar.

THEOREM 3.6. *Let* $\{\mathbf{e}_i\}$ *and* $\{\mathbf{g}_i\}$ *be O-systems in the mutually conjugate* l^p-*spaces* $\mathfrak{X}$ $(=l^r)$ *and* $\mathfrak{Y}$ $(=l^s)$, $1 < r < s$, *respectively. If* $\{\mathbf{e}_i\}$ *is a* B-*system, then there exists a bounded linear transformation* T *from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *such that*

$$T\mathbf{e}_j = \mathbf{g}_j, \qquad j = 1, 2, \ldots .$$

Proof. Let $\{\mathbf{u}_i\}$ (m, M) be any Bary basis of $\mathfrak{X}$. There exists a bounded linear transformation Q on $\mathfrak{X}$ such that

$$Q\mathbf{e}_j = \mathbf{u}_j, \qquad j = 1, 2, \ldots .$$

Denote by $\{\mathbf{v}_i\}$ $(1/M, 1/m)$ the Bary basis of $\mathfrak{Y}$ dual to $\{\mathbf{u}_i\}$. Every $\mathbf{x}$ of $\mathfrak{X}$ has the unique expansion

$$\mathbf{x} = \xi_1\mathbf{u}_1 + \ldots + \xi_j\mathbf{u}_j + \ldots \tag{3.2}$$

with

$$\left(\sum |\xi_i|^r\right)^{1/r} \leqq M\|\mathbf{x}\|.$$

It follows that

$$\left(\sum |\xi_i|^s\right)^{1/s} \leqq \left(\sum |\xi_i|^r\right)^{1/r} \leqq M\|\mathbf{x}\|.$$

Consequently, the series

$$\xi_1\mathbf{v}_1 + \ldots + \xi_j\mathbf{v}_j + \ldots$$

converges to an element $\mathbf{z}$ of $\mathfrak{Y}$ with

$$\|\mathbf{z}\| \leqq M\left(\sum |\xi_i|^s\right)^{1/s} \leqq M\left(\sum |\xi_i|^r\right)^{1/r} \leqq M^2\|\mathbf{x}\|.$$

The mapping R from $\mathfrak{X}$ to $\mathfrak{Y}$ defined for the element $\mathbf{x}$ of (3.2) by

$$R\mathbf{x} = \xi_1\mathbf{v}_1 + \ldots + \xi_j\mathbf{v}_j + \ldots$$

is a bounded linear transformation from $\mathfrak{X}$ to $\mathfrak{Y}$. Furthermore, there exists a bounded linear transformation S on $\mathfrak{Y}$ such that

$$S\mathbf{v}_j = \mathfrak{g}_j, \qquad j = 1, 2, \ldots .$$

The product SRQ is the desired bounded linear transformation from $\mathfrak{X}$ to $\mathfrak{Y}$.

THEOREM 3.7. *Let $\{\mathbf{e}_i\}$ and $\{\mathfrak{g}_i\}$ be O-systems belonging to the mutually conjugate spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Then the following four conditions are equivalent:*

(i) *The O-system $\{\mathbf{e}_i\}$ is a Bary basis of $\mathfrak{X}$;*
(ii) *Each of the O-systems $\{\mathbf{e}_i\}$ and $\{\mathfrak{g}_i\}$ is an* H-*system;*
(iii) *Each of the O-systems $\{\mathbf{e}_i\}$ and $\{\mathfrak{g}_i\}$ is a* B-*system; and*
(iv) *The O-system $\{\mathbf{e}_i\}$ is an* F-R-*system.*

Proof. Let $\{\mathbf{e}_i\}$ be a Bary basis corresponding to the bounds (m, M). Then, given any sequence $\{\gamma_i\}$ belonging to l^r, there is an $\mathbf{x}$ in $\mathfrak{X}$ with the expansion,

$$\mathbf{x} = \gamma_1\mathbf{e}_1 + \ldots + \gamma_j\mathbf{e}_j + \ldots ,$$

and, furthermore,

$$\gamma_i = (\mathbf{x}, \mathfrak{g}_i), \qquad i = 1, 2, \ldots .$$

It follows that $\{\mathbf{e}_i\}$ is an H-system. A similar argument shows that $\{\mathfrak{g}_i\}$ is an H-system. Suppose that both $\{\mathbf{e}_i\}$ and $\{\mathfrak{g}_i\}$ are H-systems. Then, by Theorem 3.5, each of them is also a B-system. If $\{\mathbf{e}_i\}$ and $\{\mathfrak{g}_i\}$ are each B-systems, then Theorem 3.5 implies that $\{\mathbf{e}_i\}$ is an F-R-system. Suppose that $\{\mathbf{e}_i\}$ is an F-R-system in the space $\mathfrak{X}$ with $\{\mathfrak{g}_i\}$ its dual in $\mathfrak{Y}$. Given any $\mathbf{x}$ of $\mathfrak{X}$, the sequence $\{(\mathbf{x}, \mathfrak{g}_i)\}$ is an element of l^r, since $\{\mathbf{e}_i\}$ is a B-system. Thus, there exists an $\mathbf{x}'$ of $\mathfrak{X}$ such that

$$\mathbf{x}' = (\mathbf{x}, \mathfrak{g}_1)\mathbf{e}_1 + \ldots + (\mathbf{x}, \mathfrak{g}_j)\mathbf{e}_j + \ldots , \tag{3.3}$$

since $\{\mathbf{e}_i\}$ is an H-system. The equalities,

$$(\mathbf{x}', \mathfrak{g}_i) = (\mathbf{x}, \mathfrak{g}_i), \qquad i = 1, 2, \ldots ,$$

imply that $\mathbf{x}'$ coincides with $\mathbf{x}$. Consequently, (3.3) is the biorthogonal expansion of $\mathbf{x}$. Thus, $\{\mathbf{e}_i\}$ is a Schauder basis of $\mathfrak{X}$. By Theorem 3.3, there exists a constant m such that

$$m||\mathbf{x}|| \leqq (\sum |(\mathbf{x}, \mathfrak{g}_i)|^r)^{1/r}.$$

Furthermore, by Theorem 3.1, there exists a constant M such that

$$(\sum |(\mathbf{x}, \mathfrak{g}_i)|^r)^{1/r} \leqq M||\mathbf{x}||.$$

It follows that $\{\mathbf{e}_i\}$ is a Bary basis of $\mathfrak{X}$.

4. Applications and examples. This section includes two applications of the preceding theorems and an example to show that a reasonably good O-system need not be a basis.

https://doi.org/10.4153/CJM-1969-071-6 Published online by Cambridge University Press

We extend a standard definition of Hilbert space theory. A sequence $\{\mathbf{e}_i\}$ of vectors in $\mathfrak{X}$ is said to be *r-near* a Bary basis $\{\mathbf{u}_i\}$ (m, M) of $\mathfrak{X}$ if and only if

$$\sum ||\mathbf{u}_i - \mathbf{e}_i||^s = D < \infty.$$

The following result is a generalization of a theorem of Brauer (**2**) from Hilbert spaces to l^p-spaces with a slight weakening of the hypothesis.

THEOREM 4.1. *Let $\{\mathbf{e}_i\}$ and $\{\mathbf{g}_i\}$ be biorthogonal sequences in the mutually conjugate l^p-spaces, $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, with neither assumed to be fundamental. Then $\{\mathbf{e}_i\}$ is a Bary basis of $\mathfrak{X}$ if and only if it is r-near a Bary basis of $\mathfrak{X}$.*

Proof. If $\{\mathbf{e}_i\}$ is a Bary basis of $\mathfrak{X}$, it is clearly r-near itself. Conversely, suppose that $\{\mathbf{e}_j\}$ is r-near the Bary basis $\{\mathbf{u}_i\}$ (m, M) of $\mathfrak{X}$. Then each $\mathbf{x}$ of $\mathfrak{X}$ has the unique expansion

$$\mathbf{x} = \xi_1\mathbf{u}_1 + \ldots + \xi_j\mathbf{u}_j + \ldots, \text{ where } (\sum |\xi_i|^r)^{1/r} < \infty.$$

Let the sequence $\{R_i\}$ of linear transformations be defined by

$$R_n\mathbf{x} = \xi_1(\mathbf{u}_1 - \mathbf{e}_1) + \ldots + \xi_n(\mathbf{u}_n - \mathbf{e}_n).$$

One notes that

$$\begin{aligned} ||(R_m - R_n)\mathbf{x}|| &\leqq ||\sum \xi_i(\mathbf{u}_i - \mathbf{e}_i)|| \\ &\leqq (\sum |\xi_i|^r)^{1/r}(\sum ||\mathbf{u}_i - \mathbf{e}_i||^s)^{1/s} \\ &\leqq ||\mathbf{x}||M(\sum ||\mathbf{u}_i - \mathbf{e}_i||^s)^{1/s}, \qquad i = m + 1, \ldots, n. \end{aligned}$$

Consequently, the sequence $\{R_i\}$ of compact linear transformations on $\mathfrak{X}$ converges uniformly to a compact linear transformation R on $\mathfrak{X}$ such that

$$R\mathbf{x} = \xi_1(\mathbf{u}_1 - \mathbf{e}_1) + \ldots + \xi_j(\mathbf{u}_j - \mathbf{e}_j) + \ldots.$$

In addition, one has

$$(I - R)\mathbf{x} = \xi_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots.$$

Suppose that 1 belongs to the spectrum of R; then there exists a non-zero $\mathbf{x}$, given by (3.2), such that

$$0 = (I - R)\mathbf{x} = \xi_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots.$$

Using the biorthogonality of $\{\mathbf{g}_i\}$ and $\{\mathbf{e}_i\}$, one finds that

$$0 = (\xi_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots, \mathbf{g}_k) = \xi_k, \qquad k = 1, 2, \ldots,$$

contradicting the assumption that $\mathbf{x}$ is not the zero vector. Consequently, 1 belongs to the resolvent set of R and the linear transformation $I - R$ has a bounded inverse S. Since $I - R$ is a bounded automorphism of $\mathfrak{X}$ and $\mathbf{e}_j = (I - R)\mathbf{u}_j$ $(j = 1, 2, \ldots)$, it follows that the sequence $\{\mathbf{e}_i\}$ is a Schauder basis of $\mathfrak{X}$. Let $\mathbf{x}$ have the expansion:

$$\mathbf{x} = \xi_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots = (I - R)(\xi_1\mathbf{u}_1 + \ldots + \xi_j\mathbf{u}_j + \ldots).$$

One sees that

$$||\mathbf{x}|| = ||(I - R)S\mathbf{x}|| \leqq ||I - R|| \, ||S\mathbf{x}|| \leqq ||I - R|| \, ||S|| \, ||\mathbf{x}||$$

and

$$(1/||I - R||)||\mathbf{x}|| \leqq ||S\mathbf{x}|| \leqq ||S|| \, ||\mathbf{x}||.$$

It follows that

$$(m/||I - R||)||\mathbf{x}|| \leqq (\sum |\xi_i|^r)^{1/r} \leqq M||S|| \, ||\mathbf{x}||. \tag{4.1}$$

Relation (4.1) implies that $\{\mathbf{e}_i\}$ is a Bary basis of $\mathfrak{X}$ from which it follows that $\{\mathfrak{g}_i\}$ is a Bary basis of $\mathfrak{Y}$.

The concept of a Bary basis leads to the consideration of a semisimple Banach algebra with underlying space $\mathfrak{X}$ having a dual Banach algebra whose underlying space is the conjugate space $\mathfrak{Y}$ of $\mathfrak{X}$.

Example 4.2. Let $\{\mathbf{e}_i\}(m, M)$ be a Bary basis of $\mathfrak{X}$ with $\{\mathfrak{g}_i\}(1/M, 1/m)$ its dual basis in $\mathfrak{Y}$. Let $\mathfrak{R}$ be the algebra obtained by defining the product of two elements,

$$\mathbf{x} = \xi_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots$$

and

$$\mathbf{z} = \zeta_1\mathbf{e}_1 + \ldots + \xi_j\mathbf{e}_j + \ldots$$

of $\mathfrak{X}$ to be

$$\mathbf{xz} = \xi_1\zeta_1\mathbf{e}_1 + \ldots + \xi_j\zeta_j\mathbf{e}_j + \ldots.$$

One finds that

$$\begin{aligned} ||\mathbf{xz}|| &\leqq (1/m)(\sum |\xi_i\zeta_i|^r)^{1/r} \\ &\leqq (1/m)(\sum |\xi_i|^r)^{1/r}(\sum |\zeta_i|^r)^{1/r} \\ &\leqq (M^2/m)||\mathbf{x}|| \, ||\mathbf{z}||, \end{aligned}$$

so that $\mathfrak{R}$ is a Banach algebra. Each set $\mathfrak{M}_i$ of $\mathfrak{R}$ consisting of all elements of $\mathfrak{R}$ with zero as ith component is clearly a maximal modular ideal of $\mathfrak{R}$. It can be seen that every maximal modular ideal of $\mathfrak{R}$ is of this nature. Since $\mathbf{0}$ is the only element common to all of these, $\mathfrak{R}$ is semisimple. The multiplicative linear functional defined by $\mathfrak{M}_i$ is the element $\mathfrak{g}_i$ of the dual basis. These algebras are closely related to earlier ones introduced in (**4**; **5**). An argument of Gel'fand (**3**) extends to the case of CSSR Banach algebras, see (**5**), in the sense that the set $\{\mathbf{e}_i\}$ of minimal idempotents of a CSSR algebra is a Fischer-Riesz basis if and only if it is homogeneous. This condition is realized if and only if the set of minimal idempotents $\{\mathbf{e}_i\}$ of $\mathfrak{R}$ is uniformly bounded in norm. If the space $\mathfrak{Y}$ is made into a Banach algebra $\mathfrak{R}^*$ by taking the basis $(\mathfrak{g}_i\}$ to be a set of minimal idempotents, there exists both a vector space and an algebraic duality between $\mathfrak{R}$ and $\mathfrak{R}^*$. Each closed ideal $\mathfrak{J}$ of $\mathfrak{R}$ corresponds to a closed dual ideal $a(\mathfrak{J})$ of $\mathfrak{R}^*$ consisting of all functionals in $\mathfrak{R}^*$ which vanish on $\mathfrak{J}$. Similarly, each closed ideal $\mathfrak{J}'$ of $\mathfrak{R}^*$ corresponds to a closed ideal $n(\mathfrak{J}')$ of $\mathfrak{R}$ consisting of the subspace on which all elements of $\mathfrak{J}'$ vanish. These correspondences are dual in the sense that

$$n(a(\mathfrak{J})) = \mathfrak{J} \quad \text{and} \quad a(n(\mathfrak{J}')) = \mathfrak{J}'.$$

https://doi.org/10.4153/CJM-1969-071-6 Published online by Cambridge University Press

One can also define an adjoint mapping $*$ of $\mathfrak{R}$ into $\mathfrak{R}^*$ such that

$$(\xi_1 \mathbf{e}_1 + \ldots + \xi_j \mathbf{e}_j + \ldots)^* = \xi_1 \mathfrak{g}_1 + \ldots + \xi_j \mathfrak{g}_j + \ldots.$$

This mapping is a continuous mapping of $\mathfrak{R}$ into $\mathfrak{R}^*$ which sends a closed ideal $\mathfrak{J}$ of $\mathfrak{R}$ into an ideal $\mathfrak{J}^*$ which is not necessarily closed. Let $c(\mathfrak{J}^*)$ be the closure of the image $\mathfrak{J}^*$ of the closed ideal $\mathfrak{J}$ under the adjoint mapping $*$. One notes that if $\mathfrak{J}$ is any closed ideal of $\mathfrak{R}$, then there exists a decomposition of $\mathfrak{R}^*$ as the direct sum,

$$\mathfrak{R}^* = c(\mathfrak{J}^*) + a(\mathfrak{J}).$$

We hope to return to the analysis of Banach algebras with such a double duality in a later paper.

We conclude with an example of a reasonable O-system which is not a basis. It is a minor modification of one given by Bary in the case of Hilbert spaces. According to Levin (**6**), a fundamental sequence $\{\mathbf{e}_i\}$ in $\mathfrak{X}$ is *minimal* if and only if for each choice of i, the element $\mathbf{e}_i$ does not belong to the closed linear hull $[\mathbf{e}_j]$, $j \neq i$, of the remaining elements of the sequence. It follows from the Hahn-Banach theorem that a minimal sequence $\{\mathbf{e}_i\}$ is an O-system with an adjoint system $\{\mathfrak{g}_i\}$ in $\mathfrak{Y}$. It is clear that an O-system is minimal. An O-system $\{\mathbf{e}_i\}$ is said to be *uniformly minimal* if and only if there exists a positive number δ such that each element $\mathbf{e}_j$, $i = 1, 2, \ldots$, is not less than the distance δ from the closed linear hull $[\mathbf{e}_j]$, $i \neq j$. An O-system $\{\mathbf{e}_i\}$ is uniformly minimal if and only if the elements of the adjoint system $\{\mathfrak{g}_i\}$ are uniformly bounded in norm. One notes that if the O-system $\{\mathbf{e}_i\}$ is a normalized basis of $\mathfrak{X}$, then it is uniformly minimal. For given $\mathbf{x}$ of $\mathfrak{X}$ with the expansion

$$\mathbf{x} = (\mathbf{x}, \mathfrak{g}_1)\mathbf{e}_1 + \ldots + (\mathbf{x}, \mathfrak{g}_j)\mathbf{e}_j + \ldots,$$

the convergence of the series implies that the terms $(\mathbf{x}, \mathfrak{g}_j)$ tend to zero. It follows from the principal of uniform boundedness that the sequence $\{\mathfrak{g}_i\}$ is uniformly bounded. Let $\{\mathbf{e}_i\}$ be an O-system in $\mathfrak{X}$ and let $\{\mathbf{u}_i\}(m, M)$ be a Bary basis of $\mathfrak{X}$ such that

$$\mathbf{u}_i = (\mathbf{u}_i, \mathfrak{g}_1)\mathbf{e}_1 + \ldots + (\mathbf{u}_i, \mathfrak{g}_{i'})\mathbf{e}_{i'},$$

where each sum is finite. Such a basis exists by Theorem 2.1. Let $\{\mathfrak{g}_i\}$ and $\{\mathbf{v}_i\}$ be the corresponding adjoint systems in $\mathfrak{Y}$ of $\{\mathbf{e}_i\}$ and $\{\mathbf{u}_i\}$, respectively. Then one obtains the following extension of a theorem of Pell (**7**), namely, when $\{\mathbf{e}_i\}$ is uniformly minimal, one has

$$\mathfrak{g}_j = (\mathbf{u}_1, \mathfrak{g}_j)\mathbf{v}_1 + \ldots + (\mathbf{u}_k, \mathfrak{g}_j)\mathbf{v}_k + \ldots$$

with

$$(4.2) \qquad \left(\sum |(\mathbf{u}_i, \mathfrak{g}_j)|^s\right)^{1/s} \leqq (1/m)||\mathfrak{g}_j|| \leqq K$$

for some constant K. This condition (4.2) holds, in particular, when $\{\mathbf{e}_i\}$ is a basis of $\mathfrak{X}$. In order that the dual system $\{\mathfrak{g}_i\}$ be fundamental, it is necessary that

$$(\mathbf{z}, \mathfrak{g}_i) = 0, \qquad i = 1, 2, \ldots,$$

https://doi.org/10.4153/CJM-1969-071-6 Published online by Cambridge University Press

imply that $\mathbf{z}$ is zero. When $\mathbf{z}$ has the expansion

$$\mathbf{z} = \zeta_1\mathbf{u}_1 + \ldots + \zeta_j\mathbf{u}_j + \ldots,$$

this last condition leads to the requirement that

$$\sum(\mathbf{u}_i, \mathbf{g}_j)\zeta_i = 0, \qquad j = 1, 2, \ldots, \tag{4.3}$$

must have only the trivial solution,

$$\zeta_1 = \ldots = \zeta_j = \ldots = 0.$$

Thus, one arrives at the set of necessary conditions, given by Bary, in order that $\{\mathbf{e}_i\}$ be a basis, namely, that (4.2) and (4.3) hold. We are now prepared to give a modification of Bary's example.

Example 4.3. Let $\{\mathbf{u}_i\}$ be the standard basis of $\mathfrak{X}$. Let the sequence $\{\mathbf{e}_i\}$ be defined by

$$\mathbf{e}_{2n-1} = \mathbf{u}_{2n-1}, \qquad \mathbf{e}_{2n} = (1/d_n)(q_n\mathbf{u}_{2n-1} + \mathbf{u}_{2n}),$$

where

$$d_n = (1 + q_n{}^r)^{1/r}.$$

The system $\{\mathbf{e}_i\}$ is normalized, fundamental, and minimal. The Bary basis $\{\mathbf{u}_i\}$ can be expressed in the form

$$\mathbf{u}_{2n-1} = \mathbf{e}_{2n-1}, \qquad \mathbf{u}_{2n} = -q_n\mathbf{e}_{2n-1} + d_n\mathbf{e}_{2n},$$

while the adjoint system $\{\mathbf{g}_i\}$ of $\{\mathbf{e}_i\}$ can be expressed as

$$\mathbf{g}_{2n-1} = \mathbf{v}_{2n-1} - q_n\mathbf{v}_{2n}, \qquad \mathbf{g}_{2n} = d_n\mathbf{v}_{2n},$$

where $\{\mathbf{v}_i\}$ if the Bary basis of $\mathfrak{Y}$ dual to $\{\mathbf{u}_i\}$. It is easy to verify that condition (4.3) is satisfied. However, one finds that the norms of the set of elements $\{\mathbf{g}_i\}$ are unbounded whenever the sequence $\{q_n\}$ diverges to plus infinity. For such a choice of this sequence, the O-system $\{\mathbf{e}_i\}$ is not a basis of $\mathfrak{X}$.

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