

# COALESCENCE IN SUBCRITICAL BELLMAN–HARRIS AGE-DEPENDENT BRANCHING PROCESSES

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## Abstract

We consider a continuous-time, single-type, age-dependent Bellman–Harris branching process. We investigate the limit distribution of the point process  $A(t) = \{a_{t,i} : 1 \leq i \leq Z(t)\}$ , where  $a_{t,i}$  is the age of the  $i$ th individual alive at time  $t$ ,  $1 \leq i \leq Z(t)$ , and  $Z(t)$  is the population size of individuals alive at time  $t$ . Also, if  $Z(t) \geq k$ ,  $k \geq 2$ , is a positive integer, we pick  $k$  individuals from those who are alive at time  $t$  by simple random sampling without replacement and trace their lines of descent backward in time until they meet for the first time. Let  $D_k(t)$  be the coalescence time (the death time of the last common ancestor) of these  $k$  random chosen individuals. We study the distribution of  $D_k(t)$  and its limit distribution as  $t \rightarrow \infty$ .

*Keywords:* Branching process; coalescence; subcritical; Bellman; Harris; age dependent; line of descent

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## 1. Introduction

### 1.1. Definition of Bellman–Harris branching processes

We consider a single-type, continuous-time, age-dependent branching process such that each individual lives for a random amount of time, say  $L$ , with distribution function  $G$  and, upon death, produces a random number  $\xi$  of children according to the offspring distribution  $\{p_j\}_{j \geq 0}$  with  $L$  and  $\xi$  independent. All individuals live and produce children independently of each other and with the same distributions. (See [3, Chapter 4].)

Let  $Z(t)$  be the population at time  $t$ , i.e. the number of individuals alive at time  $t$ . Then  $\{Z(t) : t \geq 0\}$  is called a *continuous-time, single-type, age-dependent Bellman–Harris branching process* with lifetime distribution  $G(\cdot)$  and offspring distribution  $\{p_j\}_{j \geq 0}$ .

Let  $m \equiv \sum_{j=1}^{\infty} j p_j$ . The Bellman–Harris branching process is called a supercritical, critical, or subcritical process according to whether  $1 < m < \infty$ ,  $m = 1$ , or  $m < 1$ .

For the lifetime distribution  $G$ , we assume throughout that  $G(0+) = 0$ . This together with finite mean (i.e.  $m < \infty$ ) guarantees the almost-sure finiteness of the process for all time  $t > 0$ , i.e.  $\mathbb{P}(Z(t) < \infty) = 1$  for all  $0 < t < \infty$ .

**Definition 1.1.** The Malthusian parameter for  $m$  and  $G$  is the root  $\alpha$  in  $\mathbb{R}$  (provided it exists) such that

$$m \int_0^{\infty} e^{-\alpha x} dG(x) = 1.$$

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Due to the monotonicity of the left-hand side of the above equation as a function of  $\alpha$ , such a root, when it exists, is always unique. Also, such a Malthusian parameter  $\alpha$  always exists and is necessarily nonnegative when  $m \geq 1$ .

Let  $f$  be the generating function of the offspring distribution, i.e.

$$f(s) = \sum_{j=0}^{\infty} p_j s^j, \quad 0 \leq s \leq 1.$$

Let

$$F(s, t) \equiv \sum_{j=0}^{\infty} \mathbb{P}(Z(t) = j \mid Z(0) = 1) s^j, \quad 0 \leq s \leq 1.$$

Thus,

$$F(s, t) \equiv \mathbb{E}(s^{Z(t)} \mid \text{a newborn ancestor at time } 0) = \mathbb{E}(s^{Z(t)} : L_0 > t) + \mathbb{E}(s^{Z(t)} : L_0 \leq t),$$

where  $L_0$  is the lifetime of the ancestor. On the event  $\{L_0 \leq t\}$ ,

$$Z(t) = \sum_{j=1}^{\xi_0} Z_j(t - L_0),$$

where  $\xi_0$  is the number of offspring of the ancestor and  $\{Z_j(t - L_0) : t \geq L_0\}$  is the branching process initiated by the  $j$ th offspring of the ancestor,  $j = 1, 2, \dots, \xi_0$ . Thus,

$$F(s, t) = s(1 - G(t)) + \int_{[0,t]} f(F(s, t - x)) \, dG(x);$$

$F(s, t)$  can be shown to be the unique bounded solution of the above integral equation (see [3, pp. 139–140]). Thus,  $F$  is fully determined by the pair  $(f, G)$ .

We now present some well-known results for Bellman–Harris processes. (See [3] for proofs.) Let  $q$  be the probability of extinction, i.e.

$$q \equiv \mathbb{P}(Z(t) = 0 \text{ for some } 0 < t < \infty \mid Z(0) = 1).$$

It is known that  $q = 1$  in the critical and subcritical cases ( $0 < m \leq 1$ ).

**Theorem 1.1.** *If  $m \neq 1$ ,  $0 < \gamma = f'(q)$ ,  $G$  is nonlattice, the Malthusian parameter  $\alpha$  for  $\gamma$  and  $G$  exists, and  $\mu_\alpha = \gamma \int t e^{-\alpha t} \, dG(t) < \infty$ , then*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} (q - F(s, t)) \equiv Q(s) \quad \text{exists for } 0 \leq s \leq 1.$$

Furthermore,

$$Q(s) \equiv 0 \iff m < 1 \quad \text{and} \quad \sum_{j=1}^{\infty} (j \log j) p_j = \infty.$$

**Theorem 1.2.** *Let  $0 < m < 1$  and  $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$ . Assume that the Malthusian parameter  $\alpha$  for  $m$  and the lifetime distribution  $G$  exists and that  $\int_0^\infty t e^{-\alpha t} \, dG(t) < \infty$ . Then*

(a) for all  $j \geq 1$ ,

$$\lim_{t \rightarrow \infty} \mathbb{P}(Z(t) = j \mid Z(t) > 0) = b_j$$

exists,  $\sum_{j=1}^{\infty} b_j = 1$ , and  $\sum_{j=1}^{\infty} j b_j < \infty$ ,

(b)  $\mathbb{P}(Z(t) > 0) \sim c e^{\alpha t}$  for some  $0 < c < \infty$ .

**1.2. The age chart**

Since, by Theorem 1.2(a), conditioned on the event of nonextinction, the population  $Z(t)$  of a single-type, continuous-time, age-dependent subcritical branching process will converge to a proper real-valued random variable in distribution as  $t \rightarrow \infty$ , the question of the convergence of the age chart of the individuals alive at time  $t$  is of interest. Let  $a_{t,i}$  be the age of the  $i$ th individual alive at time  $t$ ,  $1 \leq i \leq Z(t)$ . Then,  $\{A(t) \equiv \{a_{t,i} : 1 \leq i \leq Z(t)\}, t \geq 0\}$  is a point process. In this paper, the limit distribution of  $A(t)$  as  $t \rightarrow \infty$  conditioned on the event  $\{Z(t) > 0\}$  will be discussed.

**1.3. The coalescence problem**

Suppose that, for  $t > 0$ ,  $Z(t) \geq k$ . Now, pick  $k$  individuals at random from the population alive at time  $t$  by simple random sampling without replacement. Trace their lines of descent backward in time till they meet. Let  $D_k(t)$  be the coalescence time of these  $k$  individuals alive at time  $t$ . We call this common ancestor who died at time  $D_k(t)$  their *last common ancestor*. In this paper, the limit behaviors of the distributions of  $D_k(t)$  as  $t \rightarrow \infty$  for  $k \geq 2$  is studied for the subcritical age-dependent Bellman–Harris branching process.

The analog of Theorem 2.2 (below, the result on the coalescence time) for the discrete-time, single-type Galton–Watson branching processes has been addressed in [1] (for supercritical and explosive cases) and [2] (for critical and subcritical cases). Also, Lambert [7] considered the discrete and continuous (time and state space) settings for subcritical cases and Hong dealt with continuous-time, age-dependent supercritical Bellman–Harris branching processes in [5]. For the results on multitype discrete-time processes, see Hong [5] for supercritical, critical, and subcritical cases.

**2. Main results**

The first result we establish for the subcritical case is the convergence of the age chart of the population.

**Theorem 2.1.** *Let  $0 < m < 1$  and  $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$ . Assume that the lifetime distribution  $G$  is nonlattice,  $G(0+) = 0$  and such that the Malthusian parameter  $\alpha$  exists, and  $\int_0^{\infty} te^{-\alpha t} dG(t) < \infty$ . Then the following statements hold.*

- (a) *Conditioned on the event  $\{Z(t) > 0\}$ , the point process*

$$A(t) \equiv \{a_{t,i} : 1 \leq i \leq Z(t)\}$$

*converges in distribution as  $t \rightarrow \infty$  to a point process*

$$\tilde{A} \equiv \{\tilde{a}_i : 1 \leq i \leq Y\}, \tag{2.1}$$

*where  $Y$  is the random variable with distribution  $\{b_j\}_{j \geq 0}$  as defined in Theorem 1.2. The distribution of  $\tilde{A}$  is determined by its Laplace functional  $\varphi(s)$  in (3.10) below.*

- (b) *Moreover, let  $r_{t,i}$  be the remaining lifetime of the  $i$ th individual alive at time  $t$ . Let*

$$R(t) \equiv \{r_{t,i} : 1 \leq i \leq Z(t)\}.$$

*Then, conditioned on the event  $\{Z(t) > 0\}$ , the point process  $R(t)$  converges in distribution as  $t \rightarrow \infty$  to a point process*

$$\tilde{R} \equiv \{\tilde{r}_i : 1 \leq i \leq Y\},$$

*where  $Y$  is as in (a) above. The distribution of  $\tilde{R}$  is determined by its Laplace functional in (3.2) below.*

The above theorem can be used to prove Theorem 2.2 below on the coalescence problem for a subcritical Bellman–Harris branching process.

**Theorem 2.2.** *Let  $0 < m < 1$  and  $\sum_{j=1}^\infty (j \log j) p_j < \infty$ . Assume that the lifetime distribution  $G$  is nonlattice,  $G(0+) = 0$ , the Malthusian parameter  $\alpha$  exists, and  $\int_0^\infty te^{-\alpha t} dG(t) < \infty$ . Let  $D_k(t)$  be as defined in Section 1.3. Then, conditioned on  $\{Z(t) \geq 2\}$ ,*

$$t - D_2(t) \xrightarrow{D} \tilde{D}_2 \quad \text{as } t \rightarrow \infty,$$

where  $\tilde{D}_2$  is a positive random variable such that  $\mathbb{P}(0 < \tilde{D}_2 < \infty) = 1$ . For any  $u \geq 0$ ,

$$\mathbb{P}(\tilde{D}_2 \leq u) = 1 - \frac{1}{e^{\alpha u} \mathbb{P}(Y \geq 2)} \mathbb{E}(\phi(\tilde{A}, u)) \equiv H_2(u),$$

where  $\tilde{A}$  and  $Y$  are as defined in Theorem 2.1,

$$\phi((a_1, a_2, \dots, a_k), u) = \mathbb{E} \left( \frac{\sum_{i \neq j=1}^k \tilde{Z}_i(a_i + u) \tilde{Z}_j(a_j + u)}{(\sum_{i=1}^k \tilde{Z}_i(a_i + u))(\sum_{i=1}^k \tilde{Z}_i(a_i + u) - 1)} \mathbf{1}_{\{\sum_{i=1}^k \tilde{Z}_i(a_i + u) \geq 2\}} \right)$$

for any positive integer  $k$  and any positive real numbers  $a_1, a_2, \dots, a_k$ , and  $\{\tilde{Z}_i(t) : t \geq 0\}_{i \geq 1}$  are i.i.d. copies of  $\{Z(t) : t \geq 0\}$  with newborn initial ancestors.

**Remark 2.1.** Coalescence thus takes place close to the present.

By the same lines as the proof of Theorem 2.2, we can extend the result to any integer  $k \geq 2$ .

**Corollary 2.1.** *Let  $0 < m < 1$  and  $\sum_{j=1}^\infty (j \log j) p_j < \infty$ . Then, under the same hypotheses as in Theorem 2.2, for any  $k \geq 2$ ,  $t - D_k(t) \xrightarrow{D} \tilde{D}_k$  as  $t \rightarrow \infty$ , where  $\tilde{D}_k$  is a positive random variable such that  $\mathbb{P}(0 < \tilde{D}_k < \infty) = 1$ .*

### 3. Proofs of the main results

#### 3.1. Proof of Theorem 2.1(a)

Let  $Z(t)$  be a continuous-time, single-type, age-dependent Bellman–Harris branching process with  $Z(0) = 1$ . Let  $a_{t,i}$  be the age of the  $i$ th individual alive at time  $t, i = 1, 2, \dots, Z(t)$ .

To establish Theorem 2.1, it suffices (see Theorem 4.2 of [6]) to show that, for any bounded and continuous function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $s \geq 0$ ,

$$\varphi(s) \equiv \lim_{t \rightarrow \infty} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \middle| Z(t) > 0 \right)$$

exists and  $\varphi(0+) = 1$ .

Now,  $0 \leq s < \infty, t > 0$ ,

$$\begin{aligned} \tilde{H}(s, t) &\equiv \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \middle| Z(t) > 0 \right) \\ &= \frac{1}{\mathbb{P}(Z(t) > 0)} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \mathbf{1}_{\{Z(t) > 0\}} \right) \\ &= \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) - \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \mathbf{1}_{\{Z(t)=0\}} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) - \mathbb{E}(\mathbf{1}_{\{Z(t)=0\}}) \right] \\
 &= \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) - 1 + 1 - \mathbb{P}(Z(t) = 0) \right] \\
 &= \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) - 1 + \mathbb{P}(Z(t) > 0) \right] \\
 &= \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) - 1 \right] + 1.
 \end{aligned} \tag{3.1}$$

Let the ancestor be newborn. Then

$$\begin{aligned}
 H(s, t) &\equiv \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) \\
 &= \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] : L_0 > t \right) + \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] : L_0 \leq t \right) \\
 &= e^{-sh(t)} \mathbb{P}(L_0 > t) + \int_{[0,t]} f(H(s, t - u)) \, dG(u) \\
 &= e^{-sh(t)} (1 - G(t)) + \int_{[0,t]} f(H(s, t - u)) \, dG(u),
 \end{aligned} \tag{3.2}$$

which implies that, for any  $s \geq 0$ ,  $H(s, t)$  satisfies the integral equation

$$H(s, t) = e^{-sh(t)} (1 - G(t)) + \int_{[0,t]} f(H(s, t - u)) \, dG(u), \quad H(s, 0) = e^{-sh(0)}.$$

Moreover,

$$H(\infty, t) \equiv \lim_{s \rightarrow \infty} H(s, t) = \mathbb{P}(Z(t) = 0).$$

Then, by (3.1) and (3.2),

$$\begin{aligned}
 &\mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \mid Z(t) > 0 \right) \\
 &= 1 - \frac{1}{\mathbb{P}(Z(t) > 0)} [1 - H(s, t)] \\
 &= 1 - \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ 1 - e^{-sh(t)} (1 - G(t)) - \int_{[0,t]} f(H(s, t - u)) \, dG(u) \right] \\
 &= 1 - \frac{1}{\mathbb{P}(Z(t) > 0)} \left[ (1 - e^{-sh(t)}) (1 - G(t)) + \int_{[0,t]} [1 - f(H(s, t - u))] \, dG(u) \right].
 \end{aligned}$$

For any fixed  $s \geq 0$ , let

$$H(t) \equiv 1 - H(s, t), \tag{3.3}$$

$$\xi_1(t) \equiv (1 - e^{-sh(t)})(1 - G(t)), \tag{3.4}$$

$$\xi_2(t) \equiv \int_0^t [1 - f(H(s, t - u)) - mH(t - u)] dG(u), \tag{3.5}$$

$$\xi_3(t) \equiv \xi_1(t) + \xi_2(t). \tag{3.6}$$

Then

$$H(t) = \xi_3(t) + m \int_{[0,t]} H(t - u) dG(u).$$

Before we proceed to the proof of the theorem, we require the introduction of direct Riemann integrability and some lemmas about the properties of the functions defined above.

**Definition 3.1.** A function  $\xi$  is directly Riemann integrable if

- (a)  $\sum_{n=0}^\infty \delta(\sup_{n\delta \leq t < (n+1)\delta} \xi(t))$  and  $\sum_{n=0}^\infty \delta(\inf_{n\delta \leq t < (n+1)\delta} \xi(t))$  converge absolutely for sufficient small  $\delta > 0$ ; and
- (b)  $(\sum_{n=0}^\infty \delta(\sup_{n\delta \leq t < (n+1)\delta} \xi(t)) - \sum_{n=0}^\infty \delta(\inf_{n\delta \leq t < (n+1)\delta} \xi(t))) \rightarrow 0$  as  $\delta \rightarrow 0$ .

**Remark 3.1.** Some sufficient conditions for the direct Riemann integrability of  $\xi$  are

- (a)  $\xi \geq 0$ , bounded, continuous, and  $\sum_{n=0}^\infty (\sup_{n \leq t < n+1} \xi(t)) < \infty$ ;
- (b)  $\xi \geq 0$ , nonincreasing, and Riemann integrable in the ordinary sense;
- (c)  $\xi$  is bounded by a directly Riemann integrable function;
- (d)  $\xi$  is constant on the intervals  $(n, n + 1)$  and absolutely integrable.

Lemma 3.1 is a well-known result in renewal theory. See [4, pp. 362–363].

**Lemma 3.1.** Let  $G$  be a probability distribution function, and let  $G^{*n}$  denote its  $n$ -fold convolution. Let  $U = \sum_{n=0}^\infty G^{*n}$ . If  $\xi$  is directly Riemann integrable and  $G$  is nonlattice, then

$$\lim_{t \rightarrow \infty} (\xi * U)(t) = \frac{\int_0^\infty \xi(u) du}{\int_0^\infty u dG(u)}.$$

**Lemma 3.2.** If the Mathusian parameter  $\alpha$  of  $m$  and  $G$  exists, if  $e^{-\alpha t} \xi(t)$  is directly Riemann integrable, and if  $G$  is nonlattice, then the solution  $H$  of the integral equation

$$H(t) = \xi(t) + m \int_0^t H(t - u) dG(u), \quad t \geq 0,$$

satisfies

$$H(t)e^{-\alpha t} \rightarrow \frac{\int_0^\infty e^{-\alpha u} \xi(t) du}{m \int_0^\infty u e^{-\alpha u} dG(u)}$$

as  $t \rightarrow \infty$ .

The proof of Lemma 3.2 can be found in [3].

**Lemma 3.3.** *Let  $H$  be the function defined in (3.3). Then, under the hypotheses of Theorem 2.1,*

$$\sup_{s,t \geq 0} e^{-\alpha t} H(t) < \infty.$$

*Proof.* For any fixed  $s \geq 0$  and any  $t \geq 0$ , we have

$$\begin{aligned} |H(t)| &= |1 - H(s, t)| \\ &= \left| (1 - e^{-sh(t)})(1 - G(t)) + \int_{[0,t]} [1 - f(H(s, t - u))] dG(u) \right| \\ &\leq |1 - e^{-sh(t)}| |1 - G(t)| + \left| \int_{[0,t]} [1 - f(H(s, t - u))] dG(u) \right|. \end{aligned}$$

Note that  $f(1) = 1$ ,  $0 < H(s, t - u) < 1$ , and  $f$  is a continuous function. Then, by the mean value theorem, there exists  $c$  such that  $H(s, t - u) < c < 1$  and

$$f'(c) = \frac{f(1) - f(H(s, t - u))}{1 - H(s, t - u)}.$$

Therefore,

$$\begin{aligned} |H(t)| &\leq |1 - e^{-sh(t)}| |1 - G(t)| + \left| \int_{[0,t]} f'(c)(1 - H(s, t - u)) dG(u) \right| \\ &\leq |1 - G(t)| + \int_{[0,t]} |f'(c)| |1 - H(s, t - u)| dG(u) \\ &\leq (1 - G(t)) + m \int_{[0,t]} |H(t - u)| dG(u) \end{aligned} \tag{3.7}$$

since  $f'$  is nondecreasing.

Let  $me^{-\alpha t} dG(t) = dG_\alpha(t)$  and  $g_\alpha(t) \equiv e^{-\alpha t}(1 - G(t))$ . Note that  $\int_0^\infty te^{-\alpha t} dG(t) < \infty$  implies the Riemann integrability of  $g_\alpha$ . So,  $g_\alpha \geq 0$  is nonincreasing and Riemann integrable, and, hence,  $g_\alpha$  is directly Riemann integrable by Remark 3.1(b). Moreover, that  $G$  is nonlattice implies that  $G_\alpha$  is also nonlattice.

Let  $G_\alpha^{*n}$  be the  $n$ -fold convolution of  $G_\alpha$ , and let  $U_\alpha = \sum_{n=0}^\infty G_\alpha^{*n}$ . Then, by Lemma 3.1, we have

$$\lim_{t \rightarrow \infty} g_\alpha * U_\alpha(t) = \frac{\int_0^\infty g_\alpha(u) du}{\int_0^\infty u dG_\alpha(u)} < \infty.$$

Multiply both sides of (3.7) by  $e^{-\alpha t}$ . Then

$$\begin{aligned} e^{-\alpha t} |H(t)| &\leq e^{-\alpha t}(1 - G(t)) + m \int_{[0,t]} e^{-\alpha t} |H(t - u)| dG(u) \\ &= g_\alpha(t) + \int_{[0,t]} e^{-\alpha(t-u)} |H(t - u)| dG_\alpha(u) \\ &= g_\alpha(t) + H_\alpha * G_\alpha(t) \\ &\leq g_\alpha(t) + (g_\alpha + H_\alpha * G_\alpha) * G_\alpha(t) \\ &= \dots \\ &= g_\alpha(t) + g_\alpha * G_\alpha(t) + g_\alpha * G_\alpha^{*2}(t) + g_\alpha * G_\alpha^{*3}(t) + \dots \\ &= g_\alpha * U_\alpha(t), \end{aligned}$$

and, hence,  $\lim_{t \rightarrow \infty} e^{-\alpha t} |H(t)|$  is bounded by a constant for any  $s \geq 0$ . So,

$$\sup_{s, t \geq 0} e^{-\alpha t} H(t) < \infty.$$

**Lemma 3.4.** *Let  $\xi_1$  be the function defined in (3.4). Then, under the hypotheses of Theorem 2.1,  $e^{-\alpha t} \xi_1(t)$  is directly Riemann integrable.*

*Proof.* Note that

$$|e^{-\alpha t} \xi_1| = |e^{-\alpha t} (1 - e^{-sh(t)})(1 - G(t))| \leq e^{-\alpha t} (1 - G(t)) \equiv g_\alpha(t),$$

where  $g_\alpha$  is known as a directly Riemann integrable function from the proof of Lemma 3.3.

So,  $e^{-\alpha t} \xi_1$  is directly Riemann integrable by Remark 3.1(c).

**Lemma 3.5.** *Let  $\xi_2$  be the function defined in (3.5). Then, under the hypotheses of Theorem 2.1,*

$$\int_0^\infty e^{-\alpha t} |\xi_2(t)| dt < \infty.$$

*Proof.* Recall that

$$H(t) = \xi_1(t) + \xi_2(t) + m \int_{[0,t]} H(t - u) dG(u)$$

implies that

$$e^{-\alpha t} H(t) = e^{-\alpha t} \xi_1(t) + e^{-\alpha t} \xi_2(t) + m \int_{[0,t]} e^{-\alpha t} H(t - u) dG(u).$$

Let  $H_\alpha(t) = e^{-\alpha t} H(t)$ ,  $\xi_{1\alpha}(t) = e^{-\alpha t} \xi_1(t)$ , and  $\xi_{2\alpha}(t) = e^{-\alpha t} \xi_2(t)$ . Then

$$\begin{aligned} H_\alpha(t) &= \xi_{1\alpha}(t) + \xi_{2\alpha}(t) + \int_{[0,t]} H_\alpha(t - u) dG_\alpha(u) \\ &= \xi_{1\alpha}(t) + \xi_{2\alpha}(t) + H_\alpha * G_\alpha(t). \end{aligned} \tag{3.8}$$

We know that  $\xi_{1\alpha}$  is bounded by 1 and, by Lemma 3.3,  $H_\alpha$  is also bounded, so  $\xi_{2,\alpha}$  is bounded. Taking Laplace transforms on both sides of (3.8) yields

$$\hat{H}_\alpha(\theta) = \hat{\xi}_{1\alpha}(\theta) + \hat{\xi}_{2\alpha}(\theta) + \hat{H}_\alpha \cdot \hat{G}_\alpha(\theta),$$

which implies that

$$\tilde{H}_\alpha(\theta)(1 - \hat{G}_\alpha(\theta)) + (-\tilde{\xi}_{2\alpha}(\theta)) = \hat{\xi}_{1\alpha}(\theta).$$

Note that

$$\frac{f(1) - f(H(s, t - u))}{1 - H(s, t - u)} = f'(c) < f'(1) = m,$$

and, hence,  $\xi_2(t) = \int_0^t [1 - f(H(s, t - u)) - mH(t - u)] dG(u) < 0$ .

So, we have  $H_\alpha \geq 0$ ,  $\xi_{1\alpha} \geq 0$ ,  $\xi_{2\alpha} \leq 0$ , and  $G_\alpha \leq 1$ . Thus,  $\hat{H}_\alpha(\theta)(1 - \hat{G}_\alpha(\theta)) \geq 0$ ,  $-\hat{\xi}_{2\alpha}(\theta) \geq 0$ , and  $\hat{\xi}_{1\alpha}(\theta) \geq 0$ .

Moreover, by the monotone convergence theorem,

$$\lim_{\theta \downarrow 0} \hat{\xi}_{1\alpha}(\theta) = \lim_{\theta \downarrow 0} \int_0^\infty e^{-\theta t} \xi_{1\alpha}(t) dt = \int_0^\infty \xi_{1\alpha}(t) dt = \int_0^\infty e^{-\alpha t} \xi_1(t) dt < \infty,$$

and, hence,  $\lim_{\theta \downarrow 0} (-\hat{\xi}_{2\alpha}(\theta)) < \infty$  since  $\hat{H}_\alpha(\theta)(1 - \hat{G}_\alpha(\theta))$ ,  $-\hat{\xi}_{2\alpha}(\theta)$ , and  $\hat{\xi}_{1\alpha}(\theta)$  are of the same sign. Therefore, by the monotone convergence theorem again,

$$\begin{aligned} \int_0^\infty e^{-\alpha t} |\xi_2(t)| dt &= \int_0^\infty e^{-\alpha t} (-\xi_2(t)) dt \\ &= \int_0^\infty (-\xi_{2\alpha}(t)) dt \\ &= \lim_{\theta \downarrow 0} \int_0^\infty e^{-\theta t} (-\xi_{2\alpha}(t)) dt \\ &= \lim_{\theta \downarrow 0} (-\hat{\xi}_{2\alpha}(\theta)) \\ &< \infty. \end{aligned}$$

**Lemma 3.6.** *Let  $\xi_2$  be the function defined in (3.5). Then, under the hypotheses of Theorem 2.1,  $e^{-\alpha t} \xi_2(t)$  is directly Riemann integrable.*

*Proof.* Note that  $\alpha < 0$ . For  $n \leq t < n + 1$ , we have

$$\begin{aligned} e^{-\alpha t} |\xi_2(t)| &\leq e^{-\alpha(n+1)} \left| \int_0^t [1 - f(H(s, t - u)) - mH(t - u)] dG(u) \right| \\ &= e^{-\alpha(n+1)} \left| \int_0^n [1 - f(H(s, t - u)) - mH(t - u)] dG(u) \right. \\ &\quad \left. + \int_n^t [1 - f(H(s, t - u)) - mH(t - u)] dG(u) \right| \\ &\leq e^{-\alpha(n+1)} \left| \int_0^n [1 - f(H(s, n - u)) - mH(n - u)] dG(u) \right| \\ &\quad + e^{-\alpha(n+1)} \int_n^t |1 - f(H(s, t - u)) - mH(t - u)| dG(u) \\ &\leq e^{-\alpha} e^{-n} |\xi_2(n)| + e^{-\alpha} e^{-n} (G(1) - G(n)) \end{aligned}$$

since  $0 \leq f \leq 1$ ,  $0 \leq H \leq 1$ , and  $0 < m < 1$ . Then

$$|1 - f(H(s, t - u)) - mH(t - u)| \leq 1.$$

Moreover, by Lemma 3.4, we know that  $\int_0^\infty e^{-\alpha t} |\xi_2(t)| dt < \infty$  and, by the assumption, we also have  $\int_0^\infty e^{-\alpha t} (1 - G(t)) dt < \infty$ . So,

$$\sum_{n=0}^\infty e^{-\alpha n} |\xi_2(n)| < \infty \quad \text{and} \quad \sum_{n=0}^\infty e^{-\alpha n} (1 - G(n)) < \infty,$$

and, hence,

$$\sum_{n=0}^\infty \sup_{n \leq t < n+1} e^{-\alpha t} |\xi_2(t)| < \infty.$$

Since  $e^{-\alpha t} |\xi_2(t)|$  is continuous and bounded, by Remark 3.1(a),  $e^{-\alpha t} |\xi_2(t)|$  is directly Riemann integrable. Therefore, by Remark 3.1(c),  $e^{-\alpha t} \xi_2(t)$  is also directly Riemann integrable.

Now, we are ready to complete the rest of the proof.

By Lemma 3.4 and Lemma 3.6,

$$e^{-\alpha t} \xi_3(t) = e^{-\alpha t} \xi_1(t) + e^{-\alpha t} \xi_2(t) \quad \text{is directly Riemann integrable.}$$

Then, by Lemma 3.2, we know that the solution  $H$  of the integral equation

$$H(t) = \xi_3(t) + m \int_{[0,t]} H(t-u) dG(u)$$

satisfies

$$H(t) \sim c(s)e^{\alpha t} \quad \text{as } t \rightarrow \infty,$$

where

$$c(s) = \frac{\int_0^\infty e^{-\alpha u} \xi_3(u) du}{m \int_0^\infty u e^{-\alpha u} dG(u)}. \tag{3.9}$$

Recall that  $\xi_3(u)$  is a function of both  $s$  and  $u$  defined in (3.6). Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \middle| Z(t) > 0 \right) &= \lim_{t \rightarrow \infty} \left( 1 - \frac{1}{\mathbb{P}(Z(t) > 0)} (1 - H(s, t)) \right) \\ &= 1 - \lim_{t \rightarrow \infty} \frac{1}{e^{-\alpha t} \mathbb{P}(Z(t) > 0)} (1 - H(s, t)) e^{-\alpha t} \\ &= 1 - \lim_{t \rightarrow \infty} \frac{1}{e^{-\alpha t} \mathbb{P}(Z(t) > 0)} H(t) e^{-\alpha t} \\ &= 1 - \frac{c(s)}{Q(0)} \\ &\equiv \varphi(s), \end{aligned} \tag{3.10}$$

where  $Q$  is as defined in Theorem 1.1 and  $c(s)$  is as defined in (3.9).

Moreover, since, by the bounded convergence theorem,

$$\lim_{s \rightarrow 0+} H(s, t) = \lim_{s \rightarrow 0+} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \right) = 1,$$

we have

$$\lim_{s \rightarrow 0+} H(t) = \lim_{s \rightarrow 0+} 1 - H(s, t) = 0$$

and

$$\lim_{s \rightarrow 0+} \xi_1(t) = \lim_{s \rightarrow 0+} (1 - e^{-sh(t)})(1 - G(t)) = 0.$$

Again, by the bounded convergence theorem,

$$\lim_{s \rightarrow 0+} \xi_2(t) = \lim_{s \rightarrow 0+} \int_0^t (1 - f(H(s, t-u))) - mH(t-u) dG(u) = 0.$$

Hence,  $\lim_{s \rightarrow 0+} \xi_3(t) = \lim_{s \rightarrow 0+} (\xi_1(t) + \xi_2(t)) = 0$ .

Also, for any  $s \geq 0$ ,  $|e^{-\alpha t} \xi_3(t)| \leq e^{-\alpha t} |\xi_1(t)| + e^{-\alpha t} |\xi_2(t)|$ , where  $e^{-\alpha t} |\xi_1(t)|$  and  $e^{-\alpha t} |\xi_2(t)|$  are integrable. Then, by the dominated convergence theorem,

$$\lim_{s \rightarrow 0+} \int_0^\infty e^{-\alpha t} \xi_3(t) dt = \int_0^\infty \lim_{s \rightarrow 0+} e^{-\alpha t} \xi_3(t) dt = 0,$$

and, hence,

$$\lim_{s \rightarrow 0^+} \varphi(s) = \lim_{s \rightarrow 0^+} 1 - \frac{c(s)}{Q(0)} = 1 - \lim_{s \rightarrow 0^+} \frac{1}{Q(0)} \frac{\int_0^\infty e^{-\alpha u} \xi_3(u) du}{m \int_0^\infty u e^{-\alpha u} dG(u)} = 1 - 0 = 1.$$

Therefore,  $\varphi$  is a Laplace functional of a point process (see [4, pp. 429–434]).

Since, for any  $s \geq 0$ ,

$$\varphi(s) = \lim_{t \rightarrow \infty} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(a_{t,i}) \right] \middle| Z(t) > 0 \right)$$

and, by Theorem 1.2,

$$Z(t) \mid Z(t) > 0 \xrightarrow{D} Y \quad \text{as } t \rightarrow \infty,$$

there exists a point process  $\tilde{A} \equiv \{\tilde{a}_i : 1 \leq i \leq Y\}$  such that

$$\varphi(s) = \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^Y h(\tilde{a}_i) \right] \right)$$

for any  $s \geq 0$ , and, as  $t \rightarrow \infty$ ,

$$A(t) \mid Z(t) > 0 \xrightarrow{D} \tilde{A}.$$

This completes the proof of Theorem 2.1(a).

**Remark 3.2.** A more detailed study of  $\tilde{A}$  is an interesting open problem.

**3.2. Proof of Theorem 2.1(b)**

Let  $h : [0, \infty) \rightarrow [0, \infty)$  be a continuous function. Let

$$\tilde{K}(s, t) \equiv \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(r_{t,i}) \right] \middle| Z(t) > 0 \right).$$

Then

$$\tilde{K}(s, t) = \mathbb{E} \left( \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(r_{t,i}) \right] \middle| A(t), Z(t) > 0 \right) \middle| Z(t) > 0 \right).$$

Now, for  $t > 0$ ,

$$\mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(r_{t,i}) \right] \middle| A(t) \right) = \prod_{i=1}^{Z(t)} \psi(s, a_{t,i}),$$

where

$$\psi(s, x) \equiv \mathbb{E}(e^{-sh(r_{t,i})} \mid a_{t,i} = x) = \int_{[0, \infty)} e^{-sh(y)} \frac{dG(x+y)}{1-G(x)}$$

for  $0 < x < \infty$  and  $0 \leq s < \infty$ . So,

$$\mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(r_{t,i}) \right] \middle| A(t) \right) = \exp \left[ - \sum_{i=1}^{Z(t)} (-\log \psi(s, a_{t,i})) \right].$$

Since  $-\log \psi(s, x)$  is a positive continuous function of  $x$ , by Theorem 2.1(a), we have, as  $t \rightarrow \infty$ ,

$$\begin{aligned} \tilde{K}(s, t) &= \mathbb{E} \left( \exp \left[ - \sum_{i=1}^{Z(t)} (-\log \psi(s, a_{t,i})) \right] \mid Z(t) > 0 \right) \\ &\rightarrow \mathbb{E} \left( \exp \left[ - \sum_{i=1}^Y (-\log \psi(s, \tilde{a}_i)) \right] \right). \end{aligned}$$

Also, by the bounded convergence theorem,

$$\lim_{s \rightarrow 0^+} \mathbb{E} \left( \exp \left[ - \sum_{i=1}^Y (-\log \psi(s, \tilde{a}_i)) \right] \right) = 1,$$

and, hence,

$$\mathbb{E} \left( \exp \left[ - \sum_{i=1}^Y (-\log \psi(s, \tilde{a}_i)) \right] \right)$$

is a Laplace functional of a point process. Therefore, there exists a point process  $\tilde{R} \equiv \{\tilde{r}_i : 1 \leq i \leq Y\}$  such that

$$\mathbb{E} \left( \exp \left[ -s \sum_{i=1}^Y h(\tilde{r}_i) \right] \right) = \lim_{t \rightarrow \infty} \mathbb{E} \left( \exp \left[ -s \sum_{i=1}^{Z(t)} h(r_{t,i}) \right] \mid Z(t) > 0 \right)$$

for any  $s \geq 0$ . That is, as  $t \rightarrow \infty$ ,

$$R(t) \mid Z(t) > 0 \xrightarrow{D} \tilde{R}.$$

This completes the proof of Theorem 2.1(b).

### 3.3. Proof of Theorem 2.2

Let  $\{Z_{t,i}(u) : u \geq 0\}$  be the branching process initiated by the  $i$ th individual alive at time  $t$ . So,

$$Z(t) = \sum_{i=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u). \tag{3.11}$$

For any  $u \leq t$ ,

$$\begin{aligned} &\mathbb{P}(t - D_2(t) \geq u \mid Z(t) \geq 2) \\ &= \mathbb{P}(D_2(t) \leq t - u \mid Z(t) \geq 2) \\ &= \mathbb{E} \left( \frac{\sum_{i \neq j=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) Z_{t-u,j}(a_{t-u,j} + u)}{Z(t)(Z(t) - 1)} \mid Z(t) \geq 2 \right) \\ &= \frac{1}{\mathbb{P}(Z(t) \geq 2)} \mathbb{E} \left( \frac{\sum_{i \neq j=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) Z_{t-u,j}(a_{t-u,j} + u)}{Z(t)(Z(t) - 1)} \mathbf{1}_{\{Z(t) \geq 2\}} \right) \\ &= \frac{1}{\mathbb{P}(Z(t) \geq 2, Z(t) > 0)} \\ &\quad \times \mathbb{E} \left( \frac{\sum_{i \neq j=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) Z_{t-u,j}(a_{t-u,j} + u)}{Z(t)(Z(t) - 1)} \mathbf{1}_{\{Z(t) \geq 2\}} \mathbf{1}_{\{Z(t-u) > 0\}} \right). \end{aligned}$$

By (3.11) and the definition of the conditional probability, we have

$$\begin{aligned}
 & \mathbb{P}(t - D_2(t) \geq u \mid Z(t) \geq 2) \\
 &= \frac{\mathbb{P}(Z(t - u) > 0)}{\mathbb{P}(Z(t) \geq 2 \mid Z(t) > 0)\mathbb{P}(Z(t) > 0)} \\
 & \times \mathbb{E} \left( \frac{\sum_{i \neq j=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) Z_{t-u,j}(a_{t-u,j} + u)}{(\sum_{i=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u))(\sum_{i=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) - 1)} \right. \\
 & \quad \left. \times \mathbf{1}_{\{\sum_{i=1}^{Z(t-u)} Z_{t-u,i}(a_{t-u,i} + u) \geq 2\}} \mid Z(t - u) > 0 \right) \\
 &= \frac{\mathbb{P}(Z(t - u) > 0)}{\mathbb{P}(Z(t) \geq 2 \mid Z(t) > 0)\mathbb{P}(Z(t) > 0)} \\
 & \times \mathbb{E} \left( \frac{\sum_{i \neq j=1}^{Z(t-u)} \tilde{Z}_i(a_{t-u,i} + u) \tilde{Z}_j(a_{t-u,j} + u)}{(\sum_{i=1}^{Z(t-u)} \tilde{Z}_i(a_{t-u,i} + u))(\sum_{i=1}^{Z(t-u)} \tilde{Z}_i(a_{t-u,i} + u) - 1)} \right. \\
 & \quad \left. \times \mathbf{1}_{\{\sum_{i=1}^{Z(t-u)} \tilde{Z}_i(a_{t-u,i} + u) \geq 2\}} \mid Z(t - u) > 0 \right) \\
 &= \frac{1}{\mathbb{P}(Z(t) \geq 2 \mid Z(t) > 0)} \frac{\mathbb{P}(Z(t - u) > 0)}{\mathbb{P}(Z(t) > 0)} \\
 & \times \mathbb{E}(\phi(A(t - u), u) \mid A(t - u) = (a_{t-u,1}, a_{t-u,2}, \dots, a_{t-u,Z(t-u)})),
 \end{aligned}$$

where  $\{\tilde{Z}_i(t)\}_{i \geq 1}$  are i.i.d. copies of  $Z(t)$  and

$$\phi((a_1, a_2, \dots, a_k), u) = \mathbb{E} \left( \frac{\sum_{i \neq j=1}^k \tilde{Z}_i(a_i + u) \tilde{Z}_j(a_j + u)}{(\sum_{i=1}^k \tilde{Z}_i(a_i + u))(\sum_{i=1}^k \tilde{Z}_i(a_i + u) - 1)} \mathbf{1}_{\{\sum_{i=1}^k \tilde{Z}_i(a_i + u) \geq 2\}} \right)$$

for any positive integer  $k$  and any positive real numbers  $a_1, a_2, \dots, a_k$ .

Since, for any fixed  $u$ ,  $\phi(\cdot, u)$  is bounded and continuous and by Theorem 2.1 (see also [6, pp. 14–15]),

$$\mathbb{E}(\phi(A(t - u), u) \mid Z(t - u) > 0) \rightarrow \mathbb{E}(\phi(\tilde{A}, u))$$

as  $t \rightarrow \infty$ , where  $\tilde{A}$  is as in (2.1).

Moreover, by Theorem 1.2(b), i.e.  $\mathbb{P}(Z(t) > 0) \sim c(s)e^{-\alpha t}$ , we have

$$\begin{aligned}
 & \lim_{t \rightarrow \infty} \mathbb{P}(t - D_2(t) > u \mid Z(t) \geq 2) \\
 &= \lim_{t \rightarrow \infty} \frac{1}{\mathbb{P}(Z(t) \geq 2 \mid Z(t) > 0)} \frac{c(s)e^{\alpha(t-u)}}{c(s)e^{\alpha t}} \mathbb{E}(\phi_2(A(t - u), u) \mid Z(t - u) > 0) \\
 &= \frac{1}{\mathbb{P}(Y \geq 2)} e^{-\alpha u} \mathbb{E}(\phi(\tilde{A}, u)) \\
 &\equiv 1 - H_2(u).
 \end{aligned}$$

It remains to show that  $H_2$  is a proper probability distribution, i.e.  $H_2(u) \rightarrow 1$  as  $u \rightarrow \infty$ .

It suffices to prove that

$$\lim_{u \rightarrow \infty} e^{-\alpha u} \mathbb{E}(\phi(\tilde{A}, u)) = 0.$$

First, we have

$$\begin{aligned} \mathbb{E}(\phi(\tilde{A}, u)) &= \mathbb{E}\left(\frac{\sum_{i \neq j=1}^Y \tilde{Z}_i(\tilde{a}_i + u)\tilde{Z}_j(\tilde{a}_j + u)}{(\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u))(\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u) - 1)} \mathbf{1}_{\{\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u) \geq 2\}}\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\frac{\sum_{i \neq j=1}^Y \tilde{Z}_i(\tilde{a}_i + u)\tilde{Z}_j(\tilde{a}_j + u)}{(\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u))(\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u) - 1)} \mathbf{1}_{\{\sum_{i=1}^Y \tilde{Z}_i(\tilde{a}_i + u) \geq 2\}} \middle| \tilde{A}\right)\right) \\ &\leq \mathbb{E}(\mathbb{P}(\text{there exist } 1 \leq i, j \leq Y \text{ such that } i \neq j, \tilde{Z}_i(\tilde{a}_i + u) > 0, \\ &\quad \text{and } \tilde{Z}_j(\tilde{a}_j + u) > 0 \mid \tilde{A})) \\ &\leq \mathbb{E}(1 - \mathbb{P}(\tilde{Z}_i(\tilde{a}_i + u) = 0 \text{ for all } i = 1, 2, \dots, Y \mid \tilde{A}) \\ &\quad - \mathbb{P}(\tilde{Z}_i(\tilde{a}_i + u) > 0 \text{ for some } i \text{ and } \tilde{Z}_j(\tilde{a}_j + u) = 0 \text{ for all } j \neq i \mid \tilde{A})). \end{aligned}$$

For any  $0 \leq s \leq 1$  and  $t \geq 0$ , let

$$F(s, t) = \sum_{j=0}^{\infty} \mathbb{P}(Z(t) = j) s^j,$$

and, by Theorem 1.1, we have

$$\lim_{t \rightarrow \infty} e^{-\alpha t} (1 - F(s, t)) \equiv Q(s) \quad \text{exists for } 0 \leq s \leq 1.$$

So,

$$\begin{aligned} &e^{-\alpha u} \mathbb{E}(\phi(\tilde{A}, u)) \\ &\leq e^{-\alpha u} \mathbb{E}\left(1 - \prod_{i=1}^Y F(0, \tilde{a}_i + u) - \sum_{i=1}^Y (1 - F(0, \tilde{a}_i + u)) \prod_{j \neq i} F(0, \tilde{a}_j + u)\right). \end{aligned}$$

Note that the assumption of  $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$  implies that  $0 < \mathbb{E}Y < \infty$  and, hence,  $\mathbb{P}(0 < Y < \infty) = 1$ .

Now, conditioned on the limit age chart  $\tilde{A}$ , we have

$$\begin{aligned} &\lim_{u \rightarrow \infty} e^{-\alpha u} \left(1 - \prod_{i=1}^Y F(0, \tilde{a}_i + u)\right) \\ &= \lim_{u \rightarrow \infty} e^{-\alpha u} \left(1 - \prod_{i=1}^Y (1 - Q(0)e^{\alpha(\tilde{a}_i + u)})\right) \\ &= \lim_{u \rightarrow \infty} \frac{1 - \prod_{i=1}^Y (1 - Q(0)e^{\alpha(\tilde{a}_i + u)})}{e^{\alpha u}} \\ &= \lim_{u \rightarrow \infty} \frac{-\sum_{i=1}^Y (-Q(0)e^{\alpha \tilde{a}_i} \alpha e^{\alpha u}) \prod_{j \neq i} (1 - Q(0)e^{\alpha(\tilde{a}_j + u)})}{\alpha e^{\alpha u}} \\ &= \lim_{u \rightarrow \infty} \sum_{i=1}^Y Q(0)e^{\alpha \tilde{a}_i} \prod_{j \neq i} (1 - Q(0)e^{\alpha(\tilde{a}_j + u)}) \\ &= Q(0) \sum_{i=1}^Y e^{\alpha \tilde{a}_i} \end{aligned}$$

and

$$\begin{aligned}
 & \lim_{u \rightarrow \infty} e^{-\alpha u} \left( \sum_{i=1}^Y (1 - F(0, \tilde{a}_i + u)) \prod_{j \neq i} F(0, \tilde{a}_j + u) \right) \\
 &= \lim_{u \rightarrow \infty} e^{-\alpha u} \left( \sum_{i=1}^Y Q(0) e^{\alpha(\tilde{a}_i + u)} \prod_{j \neq i} (1 - Q(0) e^{\alpha(\tilde{a}_j + u)}) \right) \\
 &\geq \lim_{u \rightarrow \infty} e^{-\alpha u} \left( \sum_{i=1}^Y Q(0) e^{\alpha(\tilde{a}_i + u)} \prod_{j \neq i} (1 - Q(0) e^{\alpha u}) \right) \\
 &= \lim_{u \rightarrow \infty} e^{-\alpha u} \left( \sum_{i=1}^Y Q(0) e^{\alpha(\tilde{a}_i + u)} (1 - Q(0) e^{\alpha u})^{Y-1} \right) \\
 &= \lim_{u \rightarrow \infty} \sum_{i=1}^Y Q(0) e^{\alpha \tilde{a}_i} (1 - Q(0) e^{\alpha u})^{Y-1} \\
 &= Q(0) \sum_{i=1}^Y e^{\alpha \tilde{a}_i}.
 \end{aligned}$$

Hence, conditioned on  $\tilde{A}$ ,

$$\begin{aligned}
 0 &\leq \lim_{u \rightarrow \infty} e^{-\alpha u} \left( 1 - \prod_{i=1}^Y F(0, \tilde{a}_i + u) - \sum_{i=1}^Y (1 - F(0, \tilde{a}_i + u)) \prod_{j \neq i} F(0, \tilde{a}_j + u) \right) \\
 &= \lim_{u \rightarrow \infty} e^{-\alpha u} \left( 1 - \prod_{i=1}^Y F(0, \tilde{a}_i + u) \right) \\
 &\quad - \lim_{u \rightarrow \infty} e^{-\alpha u} \left( \sum_{i=1}^Y (1 - F(0, \tilde{a}_i + u)) \prod_{j \neq i} F(0, \tilde{a}_j + u) \right) \\
 &\leq Q(0) \sum_{i=1}^Y e^{\alpha \tilde{a}_i} - Q(0) \sum_{i=1}^Y e^{\alpha \tilde{a}_i} \\
 &= 0 \quad \text{with probability 1.}
 \end{aligned}$$

Therefore, by the bounded convergence theorem,

$$\begin{aligned}
 & \lim_{u \rightarrow \infty} e^{-\alpha u} \mathbb{E}(\phi(\tilde{A}, u)) \\
 &= \lim_{u \rightarrow \infty} e^{-\alpha u} \mathbb{E} \left( 1 - \prod_{i=1}^Y F(0, \tilde{a}_i + u) - \sum_{i=1}^Y (1 - F(0, \tilde{a}_i + u)) \prod_{j \neq i} F(0, \tilde{a}_j + u) \right) \\
 &= 0.
 \end{aligned}$$

This completes the proof.

### References

[1] ATHREYA, K. B. (2012). Coalescence in the recent past in rapidly growing populations. *Stoch. Process. Appl.* **122**, 3757–3766.  
 [2] ATHREYA, K. B. (2012). Coalescence in critical and subcritical Galton–Watson branching processes. *J. Appl. Prob.* **49**, 627–638.

- [3] ATHREYA, K. B. AND NEY, P. E. (2004). *Branching Processes*. Dover Publications, Mineola, NY.
- [4] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edn. John Wiley, New York.
- [5] HONG, J. (2011). Coalescence in Bellman–Harris and multi-type branching processes. Doctoral Thesis, Iowa State University.
- [6] KALLENBERG, O. (1986). *Random Measures*, 4th edn. Akademie, Berlin.
- [7] LAMBERT, A. (2003). Coalescence times for the branching process. *Adv. Appl. Prob.* **35**, 1071–1089.